Some Double Markov Gaussian Processes

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Abstract : We discuss a stationary Gaussian process $U(t)$, $t \in \mathsf{R}$, having a correlation function of the form $c_1 e^{-\lambda_1 |t|} + c_2 e^{-\lambda_2 |t|}$, and give a complete description of the double Markov structure of $U(t)$ on the half-line $t \geqslant 0$. In fact, we obtain the exact form of KM_2O-Langevin equation as well as the canonical representation of $U(t)$, $t \geqslant 0$.

§1. Introduction

Let $U(t)$, $t \in \mathsf{R}$, be a stationary Gaussian process having a correlation function $r(t)$ of the form

(1) $$r(t) = c_1 e^{-\lambda_1 |t|} + c_2 e^{-\lambda_2 |t|},$$

where λ_1, $\lambda_2 > 0$ and c_1, $c_2 > 0$ with the normalizing condition $c_1 + c_2 = 1$. The time evolution of such a process $U(t)$ starting at the remote past $t_0 = -\infty$ was studied by Okabe within his framework of KMO-Langevin equations ([10], [12] and [13]).

The purpose of this paper is to investigate the time evolution of $U(t)$ when the initial time t_0 is finite (we take $t_0 = 0$ for notational simplicity) ; indeed, we establish a complete description of the variation $dU(t) = U(t+dt) - U(t)$ in the form of KM_2O-Langevin equation (see [8], [11] and [13]). At the same time, we obtain the canonical representation of $U(t)$ on the half-line $t \geqslant 0$ (see (5) and (7) below).

For our purpose, we define a Gaussian process

(2) $$X(t) = U(t) - \mathsf{E}[U(t) \mid U(0)] = U(t) - r(t)U(0), \; t \geqslant 0,$$

and calculate its covariance function

(3) $$\Gamma(t, s) = \mathsf{E}[X(t)X(s)] = r(t-s) - r(t)r(s) = \sum_{i=1}^{2} c_i e^{-\lambda_i t}(e^{\lambda_i s} - r(s)), \quad 0 \leqslant s \leqslant t.$$

This process is easily seen to be *double Markov* ; for the definition and basic facts about double Markov Gaussian processes, we refer to the book [2].

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We carry out detailed computations for this particular process to get the following stochastic differential equation of the Ito type:

(4) $$dX(t) = \alpha\, dB(t) + dt\ [-\beta(t)X(t) + \int_0^t \gamma(t, u)X(u)du].$$

This yields the desired KM$_2$O-Langevin equation

(5) $$dU(t) = \alpha\, dB(t) + dt\ [-\beta(t)U(t) + \int_0^t \gamma(t, u)U(u)du + \delta(t)U(0)],\ t > 0.$$

The canonical representations of $X(t)$ and of $U(t)$ then take the following forms, respectively:

(6) $$X(t) = \int_0^t \{\sum_{i=1}^{2} e^{-\lambda_i t} g_i(u)\}\ \alpha\, dB(u),$$

and

(7) $$U(t) = r(t)U(0) + \int_0^t \{\sum_{i=1}^{2} e^{-\lambda_i t} g_i(u)\}\ \alpha\, dB(u),\ t \geqslant 0.$$

In the above expressions, $B(t)$ is a standard Brownian motion expressing the innovation process of $X(t)$, and the initial value $U(0)$ is an N(0, 1)-random variable independent of the Brownian motion $B(t)$.

Both the coefficients $\{\alpha,\ \beta(t),\ \gamma(t, u),\ \delta(t)\}$ and the functions $\{g_i(u)\}_{i=1}^{2}1$ satisfying $\sum_{i=1}^{2} e^{-\lambda_i u} g_i(u) \equiv 1$, can be determined explicitly in terms of given data $\{\lambda_i,\ c_i\}_{i=1}^{2}$ of $r(t)$, which will be discussed in Sections 2 and 3. These formulae stated in Theorems 1～3 constitute the main result of this paper.

The stationary process $U(t)$ admits a simple representation of independent sum

(8) $$U(t) = U_1(t) + U_2(t) = \sum_{i=1}^{2} \sqrt{2 c_i \lambda_i} \int_{-\infty}^{t} e^{-\lambda_i (t-u)}\, dB_i(u),\quad t \in \mathrm{R},$$

where $U_i(t)$ is a familiar Ornstein-Uhlenbeck process corresponding to $r_i(t) = c_i e^{-\lambda_i |t|}$ $(i = 1, 2)$, and $\{B_i(t)\}_{i=1}^{2}$ are *two* independent Brownian motions. The present task stems from our problem of seeking the canonical representation ([2], [6] and [7]), and in view of the stationary property, $U(t)$ is expressible by means of *one* Brownian motion $B(t)$ as in (7) (cf. [4] and [14]).

In case c_2 is very small, $U(t)$ should be viewed as a perturbation of the Ornstein-Uhlenbeck process $U_1(t)$. By describing the curve $y = \log r(t)$ on a short interval $0 < t < \tau$ (τ depending on $\{\lambda_i, c_i\}$), one can observe an interesting behaviour of $r(t)$ similar to the one studied in [3]. So the stationary process $U(t)$ can be thought of as a good model for the phenomena described in [3], and thus our results (5) and (7) are expected to be valuable beyond the theory of stochastic processes.

In the final section, we will briefly discuss a *double Markov process* $Y(t)$ *in the restricted sense* that is connected with the process (2) via the equation

(9) $$X(t) = \frac{1}{\psi(t)} \frac{d}{dt} Y(t),\quad t > 0,$$

with a suitable positive function $\psi(t)$ on $[0, \infty)$.

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§2. The time evolution of $X(t)$

This section is devoted to the study of the Gaussian process $X(t)$ having the covariance function (3). In particular, we are going to establish the equation (4) and the canonical representation (6) by carrying out explicit calculations of all requisite quantities $-\{\alpha, \beta(t), \gamma(t, u)\}$ and $\{g_i(u)\}_{i=1}^{2}$.

Assuming that $\lambda_1 > \lambda_2$ without loss of generality, we begin with discussing the modified process

(10)
$$\widetilde{X}(t) = X(t)/c_1 e^{-\lambda_1 t},\ t \geqslant 0.$$

Its covariance function takes the form

(11)
$$\widetilde{\Gamma}(t, s) = \Gamma(t, s)/c_1^2 e^{-\lambda_1(t+s)} = h_1(s) + f(t)\, h_2(s),\ 0 \leqslant s \leqslant t,$$

where $f(t) := (c_2/c_1)\, e^{(\lambda_1 - \lambda_2)t}$, $h_1(s) := e^{2\lambda_1 s}/c_1 - 1 - f(s)$ and $h_2(s) := e^{(\lambda_1+\lambda_2)s}/c_1 - 1 - f(s)$. In view of this form (11), the canonical representation of $\widetilde{X}(t)$ is expected to be expressible as

(12)
$$\widetilde{X}(t) = \int_0^t \{1 + (f(t) - f(u))\, g(u)\}\, \sigma(u)\, dB(u),$$

which leads us to write

(13)
$$d\widetilde{X}(t) = dt\, [f'(t) \int_0^t g(u)\, \sigma(u)\, dB(u)] + \sigma(t)\, dB(t).$$

In fact, the positive function $\sigma(t)$ can be first determined by computing the variance of (13):

$$\mathsf{E}\,[(d\widetilde{X}(t))^2] = \sigma^2(t)\, dt + o(dt).$$

This yields

$$\begin{aligned}\sigma^2(t) &= \{\widetilde{\Gamma}(t+dt, t+dt) - 2\widetilde{\Gamma}(t+dt, t) + \widetilde{\Gamma}(t, t)\}/dt \\ &= \lim_{s \uparrow t} [\frac{\partial}{\partial s}\widetilde{\Gamma}(t, s) - \frac{\partial}{\partial t}\widetilde{\Gamma}(t, s)] = h_1'(t) + f(t)\, h_2'(t) - f'(t)\, h_2(t) \\ &= e^{2\lambda_1 t}\, 2\,(c_1\lambda_1 + c_2\lambda_2)/c_1^2.\end{aligned}$$

We thus get

(14)
$$\sigma(t) = (\alpha/c_1)\, e^{\lambda_1 t},\ \alpha := \sqrt{2\,(c_1\lambda_1 + c_2\lambda_2)} > 0.$$

The next task is to analyse the first term in (13), which is $\mathsf{F}_t(B)\ (=\mathsf{F}_t(\widetilde{X}))$-measurable and independent of the innovation $dB(t)$ occurred on the infinitesimal interval $(t, t+dt)$. Here we used the notation

$$\mathsf{F}_t(Y) = \text{the } \sigma\text{-field generated by } \{Y(s)\,;\, 0 \leqslant s \leqslant t\},$$

for $Y = B, \tilde{X}$. If we write

(15) $$\int_0^t g(u)\,\sigma(u)\,dB(u) = \int_0^t k(t, u)\,d\tilde{X}(u),$$

then we have

(13)′ $$\sigma(t)\,dB(t) = d\tilde{X}(t) - dt\,[f'(t)\int_0^t k(t, u)\,d\tilde{X}(u)].$$

A combination of (15) and (13)′ implies the resolvent equation

(16) $$k(t, u) = g(u) - \int_u^t f'(s)\,g(s)\,k(s, u)\,ds.$$

This equation is equivalent to

$$\frac{\partial}{\partial t} k(t, u) = -f'(t)\,g(t)\,k(t, u), \quad k(u, u) = g(u),$$

and further to the expression

(16)′ $$k(t, u) = g(u)\exp\,[-\int_u^t f'(s)\,g(s)\,ds].$$

Now, what one really wants to determine is this function:

(17) $$w(t) = \exp\,[\int_0^t f'(s)\,g(s)\,ds],$$

because unknown key functions can be expressed as follows:

(18) $$g(u) = (\log w(u))' / f'(u) = w'(u) / w(u)\,f'(u),$$

and

(19) $$k(t, u) = w(u)\,g(u) / w(t) = w'(u) / w(t)\,f'(u).$$

We are ready to derive a linear differential equation of the second order to give the exact form of $w(t)$ by using its solution. For that purpose, we start with the integral equation

$$\begin{aligned} &\mathsf{E}\,[(d\tilde{X}(t))\,\tilde{X}(s)] - dt\,\{f'(t)\int_0^t k(t, u)\,\mathsf{E}\,[(d\tilde{X}(u))\,\tilde{X}(s)]\} \\ &= \sigma(t)\,\mathsf{E}\,[(dB(t))\,\tilde{X}(s)] = 0, \quad 0 \leqslant s \leqslant t, \end{aligned}$$

which is expressible in terms of the covariance function $\tilde{\Gamma}$:

(20) $$\int_0^s k(t, u)\frac{\partial\,\tilde{\Gamma}(s, u)}{\partial u}\,du + \int_s^t k(t, u)\frac{\partial\,\tilde{\Gamma}(u, s)}{\partial u}\,du = \frac{\partial\,\tilde{\Gamma}(t, s)}{\partial t} / f'(t).$$

Differentiating this equation with respect to the variable s, we obtain

(20)′ $$\sigma^2(s)\,k(t, s) + \int_0^t k(t, u)\,f'(s \vee u)\,h_2'(s \wedge u)\,du = h_2'(s),\ 0 < s < t,$$

where $s \vee u = \max(s, u)$ and $s \wedge u = \min(s, u)$.

Note that

$$Q(s) := h_2'(s) / f'(s) = \frac{\lambda_1 + \lambda_2}{c_2\,(\lambda_1 - \lambda_2)}\,e^{2\lambda_2 s} - 1 > 0,$$

$$q(s) := Q'(s) = \frac{2\,\lambda_2\,(\lambda_1 + \lambda_2)}{c_2\,(\lambda_1 - \lambda_2)}\,e^{2\lambda_2 s} > 0,$$

and $\sigma(s)/f'(s) = \alpha e^{\lambda_2 s}/c_2(\lambda_1 - \lambda_2) = \sqrt{q(s)/a}$ with $a := c_2\lambda_2(\lambda_1^2 - \lambda_2^2)/(c_1\lambda_1 + c_2\lambda_2) > 0$. Then the integral term in (20)′ becomes

$$\int_0^t (w'(u)/w(t)) f'(s) Q(s \wedge u) du$$
$$= (f'(s)/w(t)) \{[w(u) Q(s \wedge u)]_0^t - \int_0^s w(u) q(u) du\}$$
$$= h_2'(s) - (f'(s)/w(t)) \{Q(0) + \int_0^s w(u) q(u) du\},$$

and hence

(21) $$q(s) w'(s) = a \{Q(0) + \int_0^s w(u) q(u) du\}$$

holds for all $0 < s < t < \infty$. It follows that $z(s) := q(s) w(s)$ is a solution of the following differential equation

(22) $$\frac{d}{ds} \{z'(s) - (q'(s)/q(s)) z(s)\} = a z(s), \quad s > 0,$$

under the initial condition

(23) $$z(0) = q(0) \text{ and } z'(0) = a Q(0) + 2\lambda_2 q(0).$$

Since $q'(s)/q(s) = 2\lambda_2$ (constant), (22) takes the simple form

(22)′ $$z''(s) - 2\lambda_2 z'(s) - a z(s) = 0,$$

which yields

(24) $$z(s) = e^{\lambda_2 s} \{b_1 \cosh \mu s + (b_2/\mu) \sinh \mu s\}, \quad s \geqslant 0,$$

with $\mu := \sqrt{\lambda_2^2 + a} = \sqrt{(c_2\lambda_1 + c_1\lambda_2)\lambda_1\lambda_2 / (c_1\lambda_1 + c_2\lambda_2)}$ $(\lambda_2 < \mu < \lambda_1)$. By virtue of (23), the coefficients b_i are given as follows:

$$b_1 = q(0) \text{ and } b_2 = a Q(0) + \lambda_2 q(0) = b_+ q(0),$$

where we put $b_{\underset{(-)}{+}} = \{c_2\lambda_1 + c_1\lambda_2 \underset{(-)}{\pm} \lambda_1\lambda_2/(c_1\lambda_1 + c_2\lambda_2)\} / 2$. Note that $\mu \leqslant b_+ < \lambda_1$, and another constant $b_- = c_1 c_2 (\lambda_1 - \lambda_2)^2 / 2(c_1\lambda_1 + c_2\lambda_2) > 0$ will be important in what follows. We thus arrive at the exact form of $w(s) = z(s)/q(s)$:

(25) $$w(s) = e^{-\lambda_2 s} \phi(s), \quad \phi(s) := \cosh \mu s + (b_+/\mu) \sinh \mu s.$$

Conversely, if we start with this function (25), then we define $g(u)$ by (18) and $k(t, u)$ by (19):

(26) $$g(u) = (c_1/c_2(\lambda_1 - \lambda_2)) e^{-(\lambda_1 - \lambda_2)u} (\phi'(u)/\phi(u) - \lambda_2)$$

and

(27) $$k(t, u) = (c_1/c_2(\lambda_1 - \lambda_2)) e^{\lambda_2 t - \lambda_1 u} (\phi'(u) - \lambda_2 \phi(u))/\phi(t).$$

Then the equation (20) as well as (20)′ holds for all $0 < s < t < \infty$, which tells us that the expression (13)′ defines a true innovation process. Taking (15) into account, we get the equation (13) equivalent to (13)′. Integrating this (13), we can reach the canonical representation (12) of $\tilde{X}(t)$. Thus we have found explicit expressions of all ingredients in the equations (12), (13) and (13)′.

We are now in a position to state our result on the Gaussian process $X(t)$.

Theorem 1. *We have the canonical representation* (6) *of* $X(t)$ *with* $g_1(u) = e^{\lambda_1 u}\{1-(\phi'(u)/\phi(u)-\lambda_2)/(\lambda_1-\lambda_2)\}$ *and* $g_2(u) = e^{\lambda_2 u}(\phi'(u)/\phi(u)-\lambda_2)/(\lambda_1-\lambda_2)$.

This is an immediate consequence of (10) and (12).

Theorem 2. *The process* $X(t)$ *satisfies the stochastic differential equation* (4) *of the Ito type with* $\beta(t) = \lambda_1+\lambda_2-\phi'(t)/\phi(t)$ *and* $\gamma(t, u) = (b_-/\phi(t))(\lambda_1+\lambda_2)\{\cosh\mu u+(\mu/(c_2\lambda_1+c_1\lambda_2))\sinh\mu u\}$.

Proof. We have already shown the stochastic differential equation of the modified process (10):

$$(13)'' \quad d\widetilde{X}(t) = (e^{\lambda_1 t}/c_1)\,\alpha\, dB(t) + dt\,\{(f'(t)/w(t))\int_0^t (w'(u)/f'(u))\, d\widetilde{X}(u)\}.$$

With the exact form (25) in mind, we compute the above dt-part as follows:

$$\begin{aligned}
&(w'(t)/w(t))\,\widetilde{X}(t) - (f'(t)/w(t))\int_0^t (w'(u)/f'(u))'\,\widetilde{X}(u)\,du \\
&= (e^{\lambda_1 t}/c_1)\,\{(\log w(t))'X(t) - (e^{-\lambda_2 t}/w(t))\int_0^t (w'(u)\,e^{-(\lambda_1-\lambda_2)u})'\,e^{\lambda_1 u}X(u)\,du\} \\
&= (e^{\lambda_1 t}/c_1)\,\{(\phi'(t)/\phi(t)-\lambda_2)\,X(t) + (b_-/\phi(t))\int_0^t \psi(u)\,X(u)\,du\},
\end{aligned}$$

where we set

$$\begin{aligned}
(28) \quad \psi(u) &= -e^{\lambda_1 u}\frac{d}{du}\{e^{-\lambda_1 u}w'(u)\,e^{\lambda_2 u}\}/b_- \\
&= [\{(\lambda_1+\lambda_2)b_+-(\mu^2+\lambda_1\lambda_2)\}\cosh\mu u + \{(\lambda_1+\lambda_2)\mu - \\
&\qquad b_+(\mu^2+\lambda_1\lambda_2)/\mu\}\sinh\mu u]/b_- \\
&= (\lambda_1+\lambda_2)\{\cosh\mu u + (\mu/(c_2\lambda_1+c_1\lambda_2))\sinh\mu u\} > 0.
\end{aligned}$$

We thus have

$$\begin{aligned}
dX(t) &= c_1 e^{-\lambda_1 t}\, d\widetilde{X}(t) - \lambda_1 X(t)\,dt \\
&= \alpha\, dB(t) + dt\,\{-(\lambda_1+\lambda_2-\phi'(t)/\phi(t))\,X(t) + (b_-/\phi(t)) \\
&\qquad \int_0^t \psi(u)\,X(u)\,du\},
\end{aligned}$$

which completes the proof of Theorem 2.

Following Okabe [13], we prefer the present style of KMO-Langevin equation (4), although it admits another useful form similar to (13)′:

$$(4)' \qquad dX(t) = \alpha\, dB(t) - dt\int_0^t j(t, u)\, dX(u), \quad t > 0.$$

It is easy to see that the Volterra kernel in (4)′ is given by

$$(29) \qquad j(t, u) = \beta(t) - \int_u^t \gamma(t, s)\, ds.$$

Remark. The canonical representation (12) of $\widetilde{X}(t)$ can be derived by a different method due to Pitt [14] that includes solving a Riccati differential equation (cf. [15]). The method of [14] seems to be efficient only for double Markov Gaussian processes. On the other hand, the present approach via the linear differential equation (22) of the second order can be further extended to arbitrary *N-ple Markov* cases in which the correlation function of $U(t)$ can take the general form

$$(1)' \qquad r(t)=\sum_{i=1}^{N} c_i e^{-\lambda_i |t|} \quad (\sum_{i=1}^{N} c_i=1,\ N\geqslant 2).$$

The details will be published in a forthcoming paper.

§3. The time evolution of $U(t)$

Having analysed the probabilistic structure of $X(t)$ in the previous section, we now proceed to study the stationary process $U(t)$ itself. First note that the canonical representation (7) of $U(t)$ on the half-line $t\geqslant 0$ is an obvious consequence of Theorem 1, because $U(0)$ is an N(0,1)-random variable independent of the process $X(t)$.

We are ready to derive the KM_2O-Langevin equation (5) from Theorem 2, which gives us the following:

$$\begin{aligned} dU(t) &= r'(t)\, dt U(0) + dX(t) \\ &= \alpha\, dB(t) + dt\ [-\beta(t)\, U(t) + \int_0^t \gamma(t,u)\, U(u)\, du \\ &\qquad + \{r'(t) + r(t)\beta(t) - \int_0^t \gamma(t,u)\, r(u)\, du\}\ U(0)]. \end{aligned}$$

Hence the $\delta(t)$-function, important in the theory of KM_2O-Langevin equations ([13]), is equal to

$$(30) \qquad \delta(t) = r'(t) + r(t)\beta(t) - \int_0^t \gamma(t,u)\, r(u)\, du = \sum_{i=1}^{2} c_i\, \delta_i(t),$$

where we defined $\delta_i(t) = (e^{-\lambda_i t})' + e^{-\lambda_i t}\beta(t) - \int_0^t \gamma(t,u)\, e^{-\lambda_i u}\, du.$

In the proof of Theorem 2, we showed

$$\begin{aligned} \beta(t) &= \lambda_1 - (w'(t)/w(t)), \text{ and} \\ \gamma(t,u) &= -(e^{-\lambda_2 t}/w(t))\,(w'(t)\, e^{-(\lambda_1-\lambda_2)u})'\, e^{\lambda_1 u} \\ &= (b_-/\phi(t))\, \psi(u). \end{aligned}$$

We therefore have

$$\begin{aligned} \delta_1(t) &= -e^{-\lambda_1 t} w'(t)/w(t) + e^{-\lambda_2 t}\{w'(t)\, e^{-(\lambda_1-\lambda_2)t} - w'(0)\}/w(t) \\ &= (\lambda_2 - b_+)/\phi(t), \end{aligned}$$

and

$$\begin{aligned} \delta_2(t) &= e^{-\lambda_2 t}[\lambda_1-\lambda_2 - w'(t)/w(t) + \int_0^t (w'(u)e^{-(\lambda_1-\lambda_2)u})'\, e^{(\lambda_1-\lambda_2)u} du/w(t)] \\ &= e^{-\lambda_2 t}[\lambda_1-\lambda_2 - w'(t)/w(t) + \{w'(t) - w'(0) - (\lambda_1-\lambda_2)(w(t)-1)\}/w(t)] \\ &= (\lambda_1 - b_+)/\phi(t). \end{aligned}$$

It follows from (30) that

$$(31) \qquad \delta(t) = (c_2\lambda_1 + c_1\lambda_2 - b_+)/\phi(t) = b_-/\phi(t).$$

Using $\phi''(t) = \mu^2\phi(t)$ and the above formula (31), we can write

$$w(t) = b_- e^{-\lambda_2 t}/\delta(t),$$

as well as the following equalities:

$$(32)\quad \begin{cases} \beta(t)=\lambda_1+\lambda_2+\delta'(t)/\delta(t)=\psi(0)+(\log\delta(t))', \\ \gamma(t,u)=\delta(t)\psi(u), \\ \psi(u)=-e^{\lambda_1 u}\{(e^{-\lambda_2 u}/\delta(u))'e^{-(\lambda_1-\lambda_2)u}\}' \\ \qquad =(\lambda_1+\lambda_2)\{(1/\delta(u))'-\lambda_1\lambda_2/(c_1\lambda_1+c_2\lambda_2)\delta(u)\}. \end{cases}$$

Summing up the above discussions, we state the following

Theorem 3. *The process* $U(t)$, $t\geqslant 0$, *satisfies the* KM_2O*-Langevin equation* (5) *with* $\delta(t)$ *given by* (31).

In addition to (31) and (32) mentioned above, we prove some relations found by Okabe within his framework of stationary Gaussian processes (cf. his unpublished note and [11]).

Proposition 4. *We have* (i) $\beta'(t)=\delta^2(t)$; (ii) $\gamma(t,0)=\beta(t)\delta(t)-\delta'(t)$; (iii) $\frac{\partial}{\partial u}\gamma(t+u,u)+\delta(t+u)\gamma(t+u,t)=0$.

Proof. (i) is immediate; since $b_-^2+\mu^2=b_+^2$, we easily see that

$$\beta'(t)=-\begin{vmatrix}\phi(t) & \phi'(t)\\ \phi'(t) & \phi''(t)\end{vmatrix}/\phi^2(t)=\delta^2(t)\{(\phi'(t))^2-\mu^2\phi^2(t)\}/b_-^2$$

$$=\delta^2(t)(b_+^2-\mu^2)(\cosh^2\mu t-\sinh^2\mu t)/b_-^2=\delta^2(t).$$

(ii) We have only to recall the formula (32):

$$\beta(t)\delta(t)-\delta'(t)=\psi(0)\delta(t)=\gamma(t,0).$$

(iii) Note that

$$\frac{\partial}{\partial u}\gamma(t+u,u)=\delta'(t+u)\psi(u)+\delta(t+u)\psi'(u)$$

$$=\frac{\delta^2(t+u)}{b_-}\begin{vmatrix}\phi(t+u) & \psi(u)\\ \phi'(t+u) & \psi'(u)\end{vmatrix},$$

and

$$\frac{d}{du}\begin{vmatrix}\phi(t+u) & \psi(u)\\ \phi'(t+u) & \psi'(u)\end{vmatrix}=\begin{vmatrix}\phi(t+u) & \psi(u)\\ \phi''(t+u) & \psi''(u)\end{vmatrix}=\mu^2\begin{vmatrix}\phi(t+u) & \psi(u)\\ \phi(t+u) & \psi(u)\end{vmatrix}\equiv 0.$$

Hence

$$\begin{vmatrix}\phi(t+u) & \psi(u)\\ \phi'(t+u) & \psi'(u)\end{vmatrix}=\begin{vmatrix}\phi(t) & \psi(0)\\ \phi'(t) & \psi'(0)\end{vmatrix}=\psi(0)\begin{vmatrix}\phi(t) & 1\\ \phi'(t) & \mu^2/(c_2\lambda_1+c_1\lambda_2)\end{vmatrix}$$

$$=(\lambda_1+\lambda_2)\{(\mu^2/(c_2\lambda_1+c_1\lambda_2)-b_+)\cosh\mu t$$

$$+(b_+/(c_2\lambda_1+c_1\lambda_2)-1)\mu\sinh\mu t\}=-b_-\psi(t),$$

which completes the proof of (iii).

Proposition 5. (i) $\lim_{t\to\infty} \beta(t) = \lambda_1 + \lambda_2 - \mu := \beta_\infty$;

(ii) $\lim_{s\to\infty} \gamma(t+s, u+s) = (\lambda_1 - \mu)(\mu - \lambda_2) e^{-\mu(t-u)} := \gamma_\infty(t, u)$ *for every* $0 \leqslant u \leqslant t$;

(iii) $\lim_{t\to\infty} \delta(t) = 0$.

The proof is obvious, so is omitted.

Before closing this section, we would like to mention that the KM_2O-Langevin equation of $U(t)$ for $t \geqslant t_0$ tends to the right KMO-Langevin equation of $U(t)$, $t \in \mathrm{R}$, as $t_0 \to -\infty$ ([11]). Namely, the stationarity of $U(t)$ enables us to write

$$(5)_0 \qquad dU(t) = \alpha\, dB(t) + dt\,[-\beta(t-t_0)\,U(t) + \int_{t_0}^{t} \gamma(t-t_0, u-t_0)\,U(u)\,du + \delta(t-t_0)\,U(t_0)],\ t > t_0.$$

Letting the initial time t_0 go to $-\infty$, it follows from Proposition 5 that the above equation tends to

$$(5)_\infty \qquad dU(t) = \alpha\, dB(t) + dt[-\beta_\infty U(t) + \int_{-\infty}^{t} \gamma_\infty(t, u)\,U(u)\,du],$$

which coincides with the result of Okabe [10], [12].

§4. Double Markov process in the restricted sense

In this final section, we find a double Markov process $Y(t)$ in the restricted sense that is derived from the process $X(t)$. Indeed, define

$$(33) \qquad Y(t) = \int_0^t \psi(u)\,X(u)\,du,\ t \geqslant 0.$$

Then we get $X(t) = Y'(t) / \psi(t)$ and

$$(4)'' \qquad dX(t) = \alpha\, dB(t) + dt\,[-\beta(t)\,X(t) + \delta(t)\,Y(t)],$$

which shows the simple Markov structure of the pair $(X(t), Y(t))$.

We are going to show the double Markov structure of $Y(t)$ in the restricted sense (cf. [2] and [9]). That is, we have to compute three positive functions $v_i(t)$ $(i = 0, 1, 2)$ such that

$$(34) \qquad Y(t) = v_2(t) \int_0^t \{\int_u^t v_1(s)\,ds\}\, v_0(u)\,\alpha\, dB(u),$$

which means that

$$(34)' \qquad \frac{1}{v_0(t)} \frac{d}{dt} \frac{1}{v_1(t)} \frac{d}{dt} (\frac{1}{v_2(t)} Y(t)) = \alpha \dot{B}(t) \quad \text{(white noise)}.$$

Now, substitute the canonical representation (6) into (33). Then we have

$$Y(t) = \int_0^t F(t, u)\,\alpha\, dB(u),\ F(t, u) := \sum_{i=1}^{2} g_i(u) \int_u^t e^{-\lambda_i s} \psi(s)\,ds.$$

It follows from (31) and (32) that

$$b_- \int_u^t e^{-\lambda_1 s}\,\psi(s)\,ds = -[(e^{-\lambda_2 s}\,\phi(s))'\,e^{-(\lambda_1 - \lambda_2)s}]_u^t$$
$$= -e^{-\lambda_1 t}(\phi'(t) - \lambda_2 \phi(t)) + e^{-\lambda_1 u}(\phi'(u) - \lambda_2 \phi(u)),$$

and

$$b_-\int_u^t e^{-\lambda_2 s}\psi(s)\,ds = -[(e^{-\lambda_2 s}\phi(s))']_u^t + (\lambda_1-\lambda_2)\int_u^t (e^{-\lambda_2 s}\phi(s))'\,ds$$
$$= -e^{-\lambda_2 t}(\phi'(t)-\lambda_2\phi(t)) + e^{-\lambda_2 u}(\phi'(u)-\lambda_2\phi(u)) + (\lambda_1-\lambda_2)(e^{-\lambda_2 t}\phi(t)-e^{-\lambda_2 u}\phi(u)).$$

Hence by Theorem 1, we obtain

$$b_-F(t,u) = [e^{-\lambda_1(t-u)}(\phi'(t)-\lambda_2\phi(t))(\phi'(u)-\lambda_1\phi(u)) - e^{-\lambda_2(t-u)}(\phi'(t)-\lambda_1\phi(t))(\phi'(u)-\lambda_2\phi(u))]/(\lambda_1-\lambda_2)\phi(u).$$

Noting that $\lambda_1\phi(t)-\phi'(t)>0$, we can rewrite the preceding expression as follows:

$$F(t,u) = e^{-\lambda_2 t}(\lambda_1\phi(t)-\phi'(t))\{e^{-(\lambda_1-\lambda_2)t}\frac{\phi'(t)-\lambda_2\phi(t)}{\phi'(t)-\lambda_1\phi(t)} - e^{-(\lambda_1-\lambda_2)u}\frac{\phi'(u)-\lambda_2\phi(u)}{\phi'(u)-\lambda_1\phi(u)}\}\, e^{\lambda_1 u}(\lambda_1-\phi'(u)/\phi(u))/b_-(\lambda_1-\lambda_2),$$

which coincides with the specified form of (34) if we choose

$$\text{(35)}\quad \begin{cases} v_2(t) = e^{-\lambda_2 t}\phi(t)(\lambda_1-\phi'(t)/\phi(t)) > 0, \\ v_1(t) = (e^{-(\lambda_1-\lambda_2)t}\frac{\phi'(t)-\lambda_2\phi(t)}{\phi'(t)-\lambda_1\phi(t)})'/b_-(\lambda_1-\lambda_2), \\ v_0(t) = e^{\lambda_1 t}(\lambda_1-\phi'(t)/\phi(t)) > 0. \end{cases}$$

It is easy to check the positivity of $v_1(t)$:

$$v_1(t) = \frac{e^{-(\lambda_1-\lambda_2)t}}{b_-(\lambda_1-\lambda_2)}\{(\frac{\phi'(t)-\lambda_2\phi(t)}{\phi'(t)-\lambda_1\phi(t)})' - (\lambda_1-\lambda_2)\frac{\phi'(t)-\lambda_2\phi(t)}{\phi'(t)-\lambda_1\phi(t)}\}$$
$$= e^{-(\lambda_1-\lambda_2)t}(\lambda_1+\lambda_2)\phi(t)\{\phi'(t)-\lambda_1\lambda_2\phi(t)/(c_1\lambda_1+c_2\lambda_2)\}/b_-(\phi'(t)-\lambda_1\phi(t))^2$$
$$= e^{-(\lambda_1-\lambda_2)t}\phi(t)\psi(t)/(\phi'(t)-\lambda_1\phi(t))^2 > 0,$$

where we used again the relations (31) and (32).

We thus arrive at the following

Theorem 6. *The process $Y(t)$ is double Markov in the restricted sense and has the canonical representation (34) with positive functions $v_i(t)$ defined by (35).*

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