Some Multiple Markov Gaussian Processes with Stationary Increments

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Some Multiple Markov Gaussian Processes with Stationary Increments

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Abstract: As a continuation of the author's previous paper [10], we discuss a certain multiple Markov Gaussian process $X(t)$, $t \geqq 0$, with stationary increments, and give a detailed description of the canonical representation as well as of the stochastic Ito-Volterra equation for such a process.

§1. Introduction

Let $X(t)$ be a centered Gaussian process with stationary increments, which is characterized by the structure function (the variance of the increment)

(1) $$\phi(|t-s|) = \mathsf{E}[(X(t)-X(s))^2], \qquad t, s \in \mathsf{R},$$

(cf. [14]). In this paper we take a rather special form of $\phi(t)$ given by

(2) $$\phi(t) = c_0 t + 2\sum_{i=1}^{m} c_i(1-\exp[-\lambda_i t])/\lambda_i, \qquad t \geqq 0,$$

where $c_0 \geqq 0$, $c_i > 0$ $(1 \leqq i \leqq m)$ and $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_m$, and investigate the time evolution of $X(t)$, $t \geqq 0$, starting at $X(0)=0$. The covariance function $\Gamma(t, s)$ of $X(t)$ is seen to have the following expression of Goursat type of order $m+1$:

(3) $$\Gamma(t, s) = \{\phi(t) + \phi(s) - \phi(t-s)\}/2 = \sum_{i=0}^{m} f_i(t) h_i(s),$$

for $0 \leqq s \leqq t$, where we put

(4) $$\begin{cases} f_i(t) = \exp[-\lambda_i t] \qquad (0 \leqq i \leqq m) \text{ and} \\ h_0(s) = \sum_{i=0}^{m} c_i \int_0^s f_i(u)du, \quad h_i(s) = c_i \int_0^s (1/f_i(u))du \qquad (1 \leqq i \leqq m). \end{cases}$$

For such a Gaussian process $X(t)$, we describe its time evolution by means of two kinds of stochastic equations involving Brownian motion $B(t)$. One is the canonical representation ([1],[2])

(5) $$X(t) = \int_0^t F(t, u)\sigma(u)dB(u) \quad \text{with} \quad F(t, t) = 1;$$

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the other is the stochastic Ito-Volterra equation (cf. [4])

(6) $$X(t)=\int_0^t K(t,u)dX(u)+\int_0^t \sigma(u)dB(u) \quad \text{with} \quad K(t,t)=0.$$

In terms of the differential form, the above equations are settled within the well-known framework of Ito's stochastic calculus:

(5′) $$dX(t)=dt\int_0^t \frac{\partial}{\partial t}F(t,u)\,\sigma(u)dB(u)+\sigma(t)dB(t)$$

and

(6′) $$dX(t)=dt\int_0^t \frac{\partial}{\partial t}K(t,u)dX(u)+\sigma(t)dB(t).$$

Here the innovation $dB(t)$ is independent of the past σ-field $\mathsf{F}_t(X)=\sigma\{X(u);u\leqq t\}$, which, by virtue of the canonical property of (5), is equal to $\mathsf{F}_t(B)=\sigma\{B(u);u\leqq t\}$ for every $t>0$. Since the stochastic variation $dX(t)$ observed on the infinitesimal interval $(t,t+dt)$ has a natural decomposition

(7) $$dX(t)=\mathsf{E}[dX(t)\mid \mathsf{F}_t(X)]+\sigma(t)dB(t),$$

the key point of the equations (5′) and (6′) lies in the fact that the conditional expectation in (7) is expressible as $\mu(t)dt$, with $\mu(t)$ admitting two equivalent expressions

(8) $$\mu(t)=\int_0^t \frac{\partial}{\partial t}F(t,u)\,\sigma(u)dB(u)=\int_0^t \frac{\partial}{\partial t}K(t,u)dX(u).$$

This paper deals with multiple Markov Gaussian processes having Goursat kernels (3) as covariance functions, and hence can be thought of as a continuation of the author's previous paper [10], in which double Markov processes were discussed in detail. Under the present restriction of the $(m+1)$-ple Markov property ([1], [12]), the canonical kernel $F(t,u)$ takes the Goursat form of order $(m+1)$:

(9) $$F(t,u)=\sum_{i=0}^{m} f_i(t)g_i(u), \quad g_0(u)=1-\sum_{i=1}^{m} f_i(u)g_i(u).$$

In addition, the Volterra kernel $K(t,u)$ in (6) has an analogous expression

(10) $$K(t,u)=\sum_{i=1}^{m}\int_u^t f_i'(s)v_i(s,u)ds, \qquad 0\leqq u\leqq t,$$

where $v_i(t,u)$, $1\leqq i\leqq m$, are combined with $g_i(u)$, $1\leqq i\leqq m$, by the resolvent equation (see (15) in §2).

Now we come to state our main result; we are able to determine exactly those functions $g_i(u)$ and $v_i(t,u)$ in (9) and (10) from the given data $\{f_i(t),h_i(s)\}$ in (3) and (4). In the next section we will show a route of reducing this problem to a certain boundary value problem for a system of second order linear differential equations with constant coefficients (see Theorem 1). The system of linear differential equations arising in this process of reduction will be discussed in the final section (see Proposition 2).

It deserves mentioning that the present approach is also valid in *non-stationary* cases, although we will not go into details.

Before closing this section we would like to mention some reasons we focus our attention to the particular form (2) of $\phi(t)$.

(i) The expression (2) admits an immediate generalization like this:

$$(2') \qquad \phi(t)=c_0 t+2\int_0^\infty(1-\exp[-\lambda t])\,\sigma(d\lambda)/\lambda, \qquad t\geqq 0,$$

where σ is a measure on $(0,\infty)$ such that $\int_0^\infty(1\wedge 1/\lambda)\,d\sigma(\lambda)<\infty$. The notation $a\wedge b=\min(a,b)$, $a\vee b=\max(a,b)$ was and will be used in what follows. Our observation about (2′) is now in order: A centered Gaussian process $X(t)$ ($X(0)=0$) having this type of structure function possesses the *reflection* (or *T-*)*positivity* ([1], [11]). Namely, for any $n\in\mathsf{N}$, any $t_1,\cdots,t_n\geqq 0$ and any $a_1,\cdots a_n\in\mathsf{R}$, we have the inequality

$$\begin{aligned}&\mathsf{E}\Big[\Big(\sum_{i=1}^n a_i X(-t_i)\Big)\Big(\sum_{j=1}^n a_j X(t_j)\Big)\Big]\\ &=\sum_{i,j=1}^n a_i a_j\{\phi(t_i)+\phi(t_j)-\phi(t_i+t_j)\}/2\\ &=\int_0^\infty\Big\{\sum_{i=1}^n a_i(1-\exp[-\lambda t_i])\Big\}^2\,\sigma(d\lambda)/\lambda\geqq 0.\end{aligned}$$

In order to confirm the multiple Markov property, we have imposed a finite support condition on σ and got the expression (2), which corresponds to the choice of $\sigma(d\lambda)=\sum_{i=1}^m c_i\,\delta_{\{\lambda_i\}}(d\lambda)$.

(ii) We are interested in *fractional Brownian motion* $B_h(t)$ having the structure function $\phi_h(t)=t^h$, $0<h<2$. The canonical representations of $B_h(t)$ and related processes are well known ([6], [7] and [8]). On the other hand, the stochastic Ito-Volterra equation is not an immediate consequence of the canonical representation; we must face difficulties that come from the fact that the stochastic variation $dB_h(t)$ is of the form $\xi_t(dt)^{h/2}$ (ξ_t has N(0,1)), and refuses us to apply the usual equation (6′) of Ito type.

When the exponent h moves in the range $1<h<2$, the derivative $\dot{B}_h(t)$, $t\in\mathsf{R}$, is a stationary generalized process and possesses the reflection positivity. Hence its time evolution can be well described by appealing to the theory of KMO-Langevin equations ([13]). This remark is due to A. Inoue.

In the other case $0<h<1$, the function $\phi_h(t)$ belongs to the family of (2′) ($c_0=0$ and $\sigma(d\lambda)=\{h/2\Gamma(1-h)\lambda^h\}d\lambda$). Since each function in the family can be approximated by functions of the form (2) as $m\uparrow\infty$ (cf. [11]), the present study for multiple Markov processes would be a necessary step toward a full theory of stochastic infinitesimal equations ([4]) for the class of all reflection positive Gaussian processes $X(t)$ with stationary increments, among which fractional Brownian motions $B_h(t)$ are important members that are characterized by self-similarity ([5]). Detailed discussions of $B_h(t)$

and their multi-parametrizations $B_h(x)$, $x \in \mathsf{R}^d$ (cf. [3],[8] and [9]) will be planned in a forthcoming paper.

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§2. The time evolution of multiple Markov Gaussian processes

Let $X(t)$, $t \geqq 0$, be a centered Gaussian process such that $X(0)=0$ and $\mathsf{E}[(X(t)-X(s))^2]=\phi(|t-s|)$, where the structure function $\phi(t)$ is assumed to take the form (2). The covariance function $\Gamma(t,s)=\mathsf{E}[X(t)X(s)]$, $0 \leqq s \leqq t$, is immediately computed and one can reach the expression (3) of Goursat type:

$$\begin{aligned}\Gamma(t,s) &= \{\phi(t)+\phi(s)-\phi(t-s)\}/2 \\ &= c_0 s + \sum_{i=1}^{m} c_i\{1-\exp[-\lambda_i t]-\exp[-\lambda_i s]+\exp[-\lambda_i(t-s)]\}/\lambda_i \\ &= \{c_0 s + \sum_{i=1}^{m} c_i(1-\exp[-\lambda_i s])/\lambda_i\} + \sum_{i=1}^{m} \exp[-\lambda_i t]c_i(\exp[\lambda_i s]-1)/\lambda_i \\ &= \sum_{i=0}^{m} f_i(t)h_i(s),\end{aligned}$$

where $f_i(t)$ and $h_i(s)$ were defined by (4).

It is our central task to analyse the stochastic variation $dX(t)=X(t+dt)-X(t)$, observed on the infinitesimal interval $(t, t+dt)$. Its variance $\mathsf{E}[(dX(t))^2]$ is easily seen to be of the form $\sigma^2(t)dt$, and we get

$$\begin{aligned}\sigma^2(t) &= \lim_{s\uparrow t}\left[\frac{\partial}{\partial s}\Gamma(t,s)-\frac{\partial}{\partial t}\Gamma(t,s)\right]=\sum_{i=0}^{m} c_i f_i(t)+\sum_{i=1}^{m} c_i(2-f_i(t)) \\ &= c_0+2\sum_{i=1}^{m} c_i,\end{aligned}$$

which means that $\sigma(t)$ is constant:

$$\sigma(t) \equiv \sqrt{c_0+2\sum_{i=1}^{m} c_i} := \alpha > 0. \tag{11}$$

Next, we have to compute the conditional expectation $\mathsf{E}[dX(t) \mid \mathsf{F}_t(X)]$ in (7) based on the past σ-field $\mathsf{F}_t(X)$.

Since the $(m+1)\times(m+1)$ matrix $(f_i(\tau_j))_{0\leqq i,j\leqq m}$ is always non-singular for any $0<\tau_0<\tau_1<\cdots<\tau_m<\infty$, our process $X(t)$ is expected to possess the $(m+1)$-ple Markov property, which tells us that the canonical representation (5) must take the Goursat kernel (9) of order $m+1$ (cf.[1]). In the sequel we are going to determine these functions $g_i(u)(1\leqq i\leqq m)$, which should be linearly independent in $\mathsf{L}^2((0,T), du)$ for every $T<\infty$.

Now, the canonical representation (5) with $\sigma(t)\equiv\alpha$ leads us to write

(12) $$dX(t)=dt\sum_{i=1}^{m} f_i'(t)\int_0^t g_i(u)\,\alpha\, dB(u)+\alpha\, dB(t).$$

This implies another equivalent form

(13) $$dX(t)=dt\sum_{i=1}^{m} f_i'(t)\int_0^t v_i(t,u)dX(u)+\alpha\, dB(t),$$

as well as its integral version (6) with (10), which we have called a stochastic Ito-Volterra equation. In view of the canonical property $\mathsf{F}_t(X)=\mathsf{F}_t(B)$ for every $t>0$, it turns out that each random variable $\int_0^t g_i(u)\,\alpha\, dB(u)\in\mathsf{F}_t(B)$ is expressible as $\int_0^t v_i(t,u)$ $dX(u)\in\mathsf{F}_t(X)$, and hence we can write

(14) $$\begin{cases} \mathsf{E}[dX(t)\mid \mathsf{F}_t(X)]=\mu(t)dt,\\ \mu(t)=\sum_{i=1}^{m} f_i'(t)\int_0^t g_i(u)\,\alpha\, dB(u)=\sum_{i=1}^{m} f_i'(t)\int_0^t v_i(t,u)dX(u),\end{cases}$$

because the innovation $dB(t)$ in (12) and (13) is independent of $\mathsf{F}_t(X)$ (cf.(8)). It follows from the two equations (12) and (13) that the above kernels $v_i(t,u)$ must satisfy the following resolvent equation:

(15) $$v_i(t,u)=g_i(u)-\int_u^t g_i(s)\sum_{j=1}^{m} f_j'(s)v_j(s,u)ds,\qquad 1\leqq i\leqq m.$$

Introducing a matrix-valued function $A(t):=(g_i(t)f_j'(t))_{1\leqq i,j\leqq m}$, a vector-valued function $\mathsf{g}(t)={}^t(g_1(t),\cdots,g_m(t))$ and a vector-valued kernel $\mathsf{v}(t,u)={}^t(v_1(t,u),\cdots,$ $v_m(t,u))$, $0\leqq u\leqq t<\infty$, we get

(16) $$\begin{cases} \dfrac{\partial}{\partial t}\mathsf{v}(t,u)=-A(t)\mathsf{v}(t,u)\\ \mathsf{v}(t,t)=\mathsf{g}(t),\end{cases}$$

which yields

(17) $$\mathsf{v}(t,u)=\exp[-\int_u^t A(s)ds]\mathsf{g}(u).$$

We are now ready to derive an integral equation that determines the Volterra kernel $\mathsf{v}(t,u)$. To this end, we first note that

$$\mathsf{E}[dX(u)X(s)]=\{\frac{\partial}{\partial u}\Gamma(u\vee s,u\wedge s)\}du,$$

$$\frac{\partial}{\partial u}\Gamma(u\vee s,u\wedge s)=\begin{cases}\sum_{j=0}^{m} f_j(s)h_j'(u) & (0<u<s)\\ \sum_{j=1}^{m} f_j'(u)h_j(s) & (s<u).\end{cases}$$

Since the innovation $dB(t)$ is independent of $X(s)$, $0<s<t$, we obtain, as a consequence of (13),

$$\sum_{i=1}^{m} f_i'(t)\int_0^t v_i(t,u)\frac{\partial}{\partial u}\Gamma(u\vee s,u\wedge s)du=\frac{\partial}{\partial \mathrm{t}}\Gamma(\mathrm{t},\mathrm{s}),$$

which yields the required integral equation

(18) $$\sum_{i=1}^{m} f_i'(t)\{\sum_{j=0}^{m} f_j(s)\int_0^s v_i(t,u)h_j'(u)du+\sum_{j=1}^{m} h_j(s)\int_s^t v_i(t,u)f_j'(u)du\} =\sum_{i=1}^{m} f_i'(t)h_i(s)$$

for all $0<s<t<\infty$.

For each $t>0$ fixed, it suffices to seek a solution $\mathsf{v}(t, s)$ to the equation

$$\int_0^s h_0'(u)\,\mathsf{v}(t,u)du+\int_0^s <\mathsf{f}(s), \mathsf{h}'(u)>\mathsf{v}(t,u)du + \int_s^t <\mathsf{h}(s), \mathsf{f}'(u)>\mathsf{v}(t,u)du=\mathsf{h}(s), \qquad 0<s<t, \tag{19}$$

where $\mathsf{h}(s)={}^t(h_1(s),\cdots,h_m(s))$ and $<\mathsf{f},\mathsf{h}>$ denotes the inner product. Differentiate both sides of (19) with respect to s and apply the formula of $\sigma^2(t)$. We thus arrive at the key equation

$$\alpha^2\mathsf{v}(t,s)+\int_0^t <\mathsf{f}'(s\vee u), \mathsf{h}'(s\wedge u)>\mathsf{v}(t,u)du=\mathsf{h}'(s), \qquad 0<s<t. \tag{20}$$

In order to derive a linear differential equation of the second order from this equation (20), we put

$$P_j(t):=-h_j'(t)/f_j'(t)=c_j\exp[2\lambda_j t]/\lambda_j>0,$$
$$p_j(t):=P_j'(t)=2\lambda_j P_j(t).$$

Then $<\mathsf{f}'(s\vee u),\mathsf{h}'(s\wedge u)>=-<\mathsf{f}'(s)\cdot\mathsf{P}(s\wedge u), \mathsf{f}'(u)>$ with multiplication $\mathsf{a}\cdot\mathsf{b}$ defined by $(\mathsf{a}\cdot\mathsf{b})_i=a_ib_i$, $1\leqq i\leqq m$. Applying the integration by parts formula and noting that

$$\frac{\partial}{\partial u}\exp[-\int_u^t A(s)ds]=(v_i(t,u)f_j'(u))_{1\leqq i,j\leqq m}, \tag{21}$$

we obtain

$$\int_0^t <\mathsf{f}'(s\vee u), \mathsf{h}'(s\wedge u)>\mathsf{v}(t,u)du$$
$$=-\int_0^t \{\frac{\partial}{\partial u}\exp[-\int_u^t A(v)dv]\}\mathsf{f}'(s)\cdot\mathsf{P}(s\wedge u)du$$
$$=\mathsf{h}'(s)+\exp[-\int_0^t A(v)dv]\mathsf{f}'(s)\cdot\mathsf{P}(0)+\int_0^s\exp[-\int_u^t A(v)dv]\mathsf{f}'(s)\cdot\mathsf{p}(u)du.$$

It follows from (20) that

$$\alpha^2\exp[-\int_s^t A(v)dv]\,\mathsf{g}(s)+\exp[-\int_0^t A(v)dv]\,\mathsf{f}'(s)\cdot\mathsf{P}(0) + \int_0^s\exp[-\int_u^t A(v)dv]\,\mathsf{f}'(s)\cdot\mathsf{p}(u)du=0, \qquad 0<s<t. \tag{22}$$

Now define

$$y_j{}^i(s;t)=\int_0^s(\exp[-\int_u^t A(v)dv])_{ij}\,p_j(u)du, \qquad 0\leqq s\leqq t, \tag{23}$$

to get

$$\alpha^2\frac{d}{ds}(\exp[-\int_s^t A(v)dv])_{ik}$$
$$=\alpha^2\sum_{j=1}^m(\exp[-\int_s^t A(v)dv])_{ij}\,g_j(s)f_k'(s)$$
$$=-\sum_{j=1}^m\{(\exp[-\int_0^t A(v)dv])_{ij}\,p_j(0)/2\lambda_j+y_j{}^i(s;t)\}f_j'(s)f_k'(s),$$

which shows that the system of functions $\{y_j{}^i(s;t)\}_{j=1}^m$, for each fixed i and t, satisfies the following linear differential equation of the second order:

$$\frac{d}{ds}\{(\alpha^2/p_k(s))\frac{d}{ds}y_k{}^i(s;t)\} + f_k'(s)\sum_{j=1}^m f_j'(s)\{\frac{d}{ds}y_j{}^i(0;t)/2\lambda_j+y_j{}^i(s;t)\}=0, \qquad 0<s<t, \tag{24}$$

with boundary conditions

(24′) $y_k^i(0;t)=0$ and $\frac{d}{ds}y_k^i(t;t)=\delta_k^i p_k(t), \quad 1\leqq k\leqq m,$

where δ_k^i is the Kronecker delta.

At the final stage, we make the transformation

(25) $$z_k^i(s;t)=\exp[-\lambda_k s]\{y_k^i(s;t)+\frac{d}{ds}y_k^i(0;t)/2\lambda_k\}.$$

Then the above equation (24) changes into a linear differential equation with constant coefficients for $\{z_k^i(s;t)\}_{k=1}^m$:

(26) $$\frac{d^2}{ds^2}z_k^i(s;t)-\lambda_k^2 z_k^i(s;t)+(2c_k\lambda_k/\alpha^2)\sum_{j=1}^m \lambda_j z_j^i(s;t)=0, \quad 0\leqq s\leqq t.$$

The boundary conditions (24′) become

(26′) $$\begin{cases} \lambda_k z_k^i(0;t)-\frac{d}{ds}z_k^i(0;t)=0 \quad \text{and} \\ \lambda_k z_k^i(t;t)+\frac{d}{ds}z_k^i(t;t)=2c_k\delta_k^i\exp[\lambda_k t], \quad 1\leqq t\leqq m. \end{cases}$$

We have thus proved that our task to determine $\{g_i(u)\}$ in (12) and $\{v_i(t,u)\}$ in (13) can be reduced to the one to solve the boundary value problem for the second order differential equation (26).

Conversely, let us assume that we find out a solution $\{z_k^i(s;t)\}$ to the differential equation (26) with boundary conditions (26′). Then a combination of (23) and (25) gives us the exprêssion

(27) $$(\exp[-\int_s^t A(v)dv])_{ik}=(1/p_k(s))\frac{d}{ds}\{\exp[\lambda_k s]z_k^i(s;t)\}.$$

So we define by (21)

$$v_i(t,u)=\frac{\partial}{\partial u}(\exp[-\int_u^t A(v)dv])_{ik}/f_k'(u)$$

for arbitrary k (we put $k=i$ for convenience), which is further calculated by applying (26):

(28) $$\begin{aligned} v_i(t,u)&=-(\exp[\lambda_i u]/2c_i\lambda_i)\frac{d}{du}\{\exp[-2\lambda_i u]\frac{d}{du}(\exp[\lambda_i u]z_i^i(u;t))\} \\ &=(1/2c_i\lambda_i)\{\lambda_i^2 z_i^i(u;t)-\frac{d^2}{du^2}z_i^i(u;t)\}=\sum_{j=1}^m \lambda_j z_j^i(u;t)/\alpha^2. \end{aligned}$$

By (16) we have

(28′) $$g_i(u)=v_i(u,u)=\sum_{j=1}^m \lambda_j z_j^i(u;u)/\alpha^2.$$

It is now an easy thing to check those integral equations (20), (19) and (18). Set

(13′) $$dB(t)=\{dX(t)-dt\sum_{i=1}^m f_i'(t)\int_0^t v_i(t,u)dX(u)\}/\alpha,$$

to find out a true innovation of $X(t)$. We are thus able to establish the expression (12) associated to (13) as well as the canonical representation (5) with (9) and (11), which is nothing but the integral form of (12) and with which we started the present discussions.

Summing up what we have discussed, we state

Theorem 1. *If we find a solution* $\{z_k^i(s;t)\}$ *to the differential equation* (26) *and* (26′), *then we obtain the canonical representation* (5) *with* (9) *and* (11) *as well as the stochastic Ito-Volterra equation* (6) *with* (10), *where Volterra kernels* $v_i(t, u)$ *and functions* $g_i(u)$ *are defined by* (28) *and* (28′), *respectively.*

Remark 1. Let $U(t)$, $t \in \mathrm{R}$, be a stationary Gaussian process with correlation function $r(|t-s|)$ of the form

(29) $$r(t) = \sum_{i=1}^{m} c_i \exp[-\lambda_i |t|],$$

where $c_i > 0$ $(1 \leqq i \leqq m)$, $\sum_{i=1}^{m} c_i = 1$ and $0 < \lambda_1 < \cdots < \lambda_m < \infty$. Note that $U(t)$ is expressible as a superposition of mutually independent Ornstein-Uhlenbeck processes $U(t; \lambda_i)$:

$$U(t) = \sum_{i=1}^{m} \sqrt{c_i}\, U(t; \lambda_i), \qquad t \in \mathrm{R}.$$

Having in mind the well-known framework of KM_2O-Langevin equations, we consider

(30) $$X(t) = U(t) - \mathrm{E}[U(t) \mid U(0)] = U(t) - r(t)U(0), \ t \geqq 0.$$

and calculate its covariance function

(31) $$\Gamma(t, s) = r(t-s) - r(t)r(s) = \sum_{i=1}^{m} f_i(t) h_i(s), \qquad 0 \leqq s \leqq t < \infty,$$

where

(32) $$f_i(t) = \exp[-\lambda_i t], \quad h_i(s) = c_i \{\exp[\lambda_i s] - r(s)\} \qquad (1 \leqq i \leqq m).$$

In his previous paper [10], the author gave a detailed account of the double Markov case $m=2$. As an extension to general cases $m \geqq 3$, we obtain a similar conclusion to Theorem 1 for the modified process

(30′) $$X^{\wedge}(t) = X(t)/f_m(t), \qquad t \geqq 0,$$

which has the m-ple Markov property. In other words, by the same method of solving the boundary value problem for a differential equation of the same character as (26), we can establish the canonical representation (5) and the stochastic Ito-Volterra equation (6) for $X^{\wedge}(t)$ and hence for $X(t)$. This time, the differential equation mentioned above takes the following form:

(33) $$\frac{d^2}{ds^2} z_k^i(s; t) - \lambda_k^2 z_k^i(s; t) - (2c_k \lambda_k(\lambda_k + \lambda_m)/\alpha^2) \sum_{j=1}^{m-1} (\lambda_m - \lambda_j) z_j^i(s; t) = 0, \qquad 0 < s < t,$$

where $\alpha = \sqrt{2 \sum_{i=1}^{m} c_i \lambda_i}$ comes from the variance of $dX(t)$, i.e. $\mathrm{E}[(dX(t))^2] = \alpha^2 dt$. The boundary conditions are written as follows:

(33′) $$\begin{cases} \{(2-c_m)\lambda_m + (1-c_k)\lambda_k\} \lambda_k z_k^i(0; t)/\{c_m \lambda_m + (c_k+1)\lambda_k\} \\ \quad - \dfrac{d}{ds} z_k^i(0; t) = 0 \qquad \text{and} \\ \lambda_k z_k^i(t; t) + \dfrac{d}{ds} z_k^i(t; t) \\ \quad = 2c_k \lambda_k (\lambda_k + \lambda_m) \delta_k^i \exp[\lambda_k t]/(\lambda_m - \lambda_k), \qquad 1 \leqq k \leqq m-1. \end{cases}$$

§3. Linear differential equations of the second order

The final section is devoted to an account of the differential equation (26) with (26′), which we encountered in the previous section. Indeed, taking into account the fact that the index i $(1 \leqq i \leqq m)$ and the end point t of the interval $[0, t]$ arise only in the boundary conditions, we are first going to discuss a general solution $z_k(s)$ defined on the whole interval $s \geqq 0$ to the following system of differential equations:

$$(34) \qquad z_k''(s) - \lambda_k^2 z_k(s) + (2 c_k \lambda_k / \alpha^2) \sum_{j=1}^{m} \lambda_j z_j(s) = 0, \qquad 1 \leqq k \leqq m.$$

Then the relevant boundary conditions

$$(35) \qquad \begin{cases} \lambda_k z_k(0) - z_k'(0) \quad \text{and} \\ \lambda_k z_k(t) + z_k'(t) = 2 c_k \delta_k{}^i \exp[\lambda_k t], \qquad 1 \leqq k \leqq m, \end{cases}$$

would determine our desired solution $z_k{}^i(s\,;t)$, and hence $v_i(t, u)$ and $g_i(u)$ by (28) and (28′).

Let us begin with the simplest case $m=1$. The above (34) is rewritten as follows:

$$(34') \qquad z''(s) = \mu^2 z(s), \quad \mu := \sqrt{c_0/(c_0+2c_1)}\,\lambda_1 \in [0, \lambda_1),$$

which implies a general form of the solution:

$$(36) \qquad z(s) = a\,\mathrm{e}(s\,;\mu) + b\,\mathrm{d}(s\,;\mu),$$

where we used the notation

$$\mathrm{e}(s\,;\mu) = \cosh \mu s \qquad \text{and} \qquad \mathrm{d}(s\,;\mu) = \sinh \mu s / \mu,$$

on the understanding that $\mathrm{e}(s\,;0)=1$, $\mathrm{d}(s\,;0)=s$ in the case $\mu=0$ (i.e. $c_0=0$).

The boundary conditions (35) for each fixed $t>0$, give us the values $a=a(t)$ and $b=b(t)$:

$$(37) \qquad \begin{cases} a = c_1 \exp[\lambda_1 t] / \{\lambda_1 \mathrm{e}(t\,;\mu) + ((c_0+c_1)/(c_0+2c_1))\,\lambda_1 \mathrm{d}(t\,;\mu)\}, \\ b = \lambda_1 a. \end{cases}$$

Hence we arrive at the following formula:

$$(38) \qquad z_1{}^1(s\,;t) = c_1 \exp[\lambda_1 t]\{\mathrm{e}(s\,;\mu) + \lambda_1 \mathrm{d}(s\,;\mu)\} / \\ / \{\lambda_1 \mathrm{e}(t\,;\mu) + ((c_0+c_1)/(c_0+2c_1))\,\lambda_1 \mathrm{d}(t\,;\mu)\}.$$

We now proceed to the general case $m \geqq 2$. We have to study the eigenvalue problem for the $m \times m$ matrix

$$R = (\delta_j{}^i \lambda_j^2 - c_i \lambda_i \lambda_j / \{(c_0/2) + \sum_{k=1}^{m} c_k\})_{1 \leqq i,j \leqq m},$$

by which (34) has a neat expression

$$(34'') \qquad \mathrm{z}''(s) = R\,\mathrm{z}(s), \quad \mathrm{z}(s) = {}^t(z_1(s), \cdots, z_m(s)).$$

Let us take a column vector $\mathrm{q}(\mu)$ having i-th component $q_i(\mu) = 2c_i \lambda_i / (\mu^2 - \lambda_i^2)$ and depending on a positive parameter μ. Then we see that

$$(R\,\mathrm{q}(\mu))_i=2c_i\lambda_i\{\lambda_i^2/(\mu^2-\lambda_i^2)-(\sum_{j=1}^m c_j\lambda_j^2/(\mu^2-\lambda_j^2))/((c_0/2)+\sum_{k=1}^m c_k)\}$$
$$=2c_i\lambda_i\{\lambda_i^2/(\mu^2-\lambda_i^2)+1\}=\mu^2 q_i(\mu),$$

if μ is a solution to the equation

$$\sum_{j=1}^m c_j\lambda_j^2/(\mu^2-\lambda_j^2)+c_0/2+\sum_{j=1}^m c_j=0,\qquad \mu>0. \tag{39}$$

It is easily shown that this equation has m distinct roots $\{\mu_j\}_{j=1}^m$ such that $\lambda_0=0\leqq\mu_1<\lambda_1<\cdots<\mu_m<\lambda_m$. When $c_0=0$, the first root μ_1 is equal to 0; otherwise $\mu_1>0$. We have thus diagonalized the matrix R:

$$R\,Q=Q(\delta_j{}^i\mu_j^2)_{1\leqq i,j\leqq m},\quad Q=(q_i(\mu_j))_{1\leqq i,j\leqq m}$$

We are now ready to obtain a general solution to (34″). Put $\mathrm{w}(s)=Q^{-1}\mathrm{z}(s)$ to get

$$w_i''(s)=(Q^{-1}\mathrm{z}''(s))_i=(Q^{-1}R\,Q\,\mathrm{w}(s))_i=\mu_i^2 w_i(s),$$

which implies

$$w_i(s)=a_i\mathrm{e}(s;\mu_i)+b_i\mathrm{d}(s;\mu_i),\qquad 1\leqq i\leqq m,$$

with m pairs of coefficients (a_i, b_i). Thus we have a required expression

$$z_i(s)=(Q\mathrm{w}(s))_i=2c_i\lambda_i\sum_{j=1}^m\{a_j\mathrm{e}(s;\mu_j)+b_j\mathrm{d}(s;\mu_j)\}/(\mu_j^2-\lambda_i^2). \tag{40}$$

By virtue of (40), the boundary conditions (35), for each i $(1\leqq i\leqq m)$ and $t>0$ fixed, change into the following system of linear equations:

$$\begin{cases}\sum_{j=1}^m(\lambda_k a_j-b_j)/(\mu_j^2-\lambda_k^2)=0 \quad\text{and}\\ \sum_{j=1}^m\{(\lambda_k\mathrm{e}(t;\mu_j)+\mu_j^2\,\mathrm{d}(t;\mu_j))a_j+(\mathrm{e}(t;\mu_j)+\lambda_k\mathrm{d}(t;\mu_j))b_j\}/\\ \qquad /(\mu_j^2-\lambda_k^2)=\delta_k{}^i\exp[\lambda_k t]/\lambda_k,\qquad 1\leqq k\leqq m.\end{cases} \tag{41}$$

The well-known formula

$$|\,1/(\mu_j^2-\lambda_k^2)\,|=(-1)^{m(m-1)/2}\prod_{1\leqq k<j\leqq m}(\mu_j^2-\mu_k^2)(\lambda_j^2-\lambda_k^2)/\prod_{j,k=1}^m(\mu_j^2-\lambda_k^2)$$

enables us to take the inverse $(1/(\mu_j^2-\lambda_k^2))^{-1}=(\rho_{kj})$, and from the first equality of (41) we write

$$b_j=\sum_{k,h=1}^m\rho_{jk}\lambda_k a_h/(\mu_h^2-\lambda_k^2)=\sum_{h=1}^m\tau_{jh}a_h, \tag{42}$$

with $\tau_{jh}:=\sum_{k=1}^m\rho_{jk}\lambda_k/(\mu_h^2-\lambda_k^2)$. In the case $m=2$, we have

$$\tau_{11}=\{\lambda_1(\lambda_2^2-\mu_1^2)(\mu_2^2-\lambda_1^2)+\lambda_2(\lambda_1^2-\mu_1^2)(\lambda_2^2-\mu_2^2)/(\mu_2^2-\mu_1^2)(\lambda_2^2-\lambda_1^2),$$

$$\tau_{12}=(\lambda_2-\lambda_1)(\lambda_1^2-\mu_1^2)(\lambda_2^2-\mu_1^2)/(\mu_2^2-\mu_1^2)(\lambda_2^2-\lambda_1^2),$$

$$\tau_{21}=(\lambda_2-\lambda_1)(\mu_2^2-\lambda_1^2)(\lambda_2^2-\mu_2^2)/(\mu_2^2-\mu_1^2)(\lambda_2^2-\lambda_1^2),\quad\text{and}$$

$$\tau_{22}=\{\lambda_1(\lambda_1^2-\mu_1^2)(\lambda_2^2-\mu_2^2)+\lambda_2(\lambda_2^2-\mu_1^2)(\mu_2^2-\lambda_1^2)\}/(\mu_2^2-\mu_1^2)(\lambda_2^2-\lambda_1^2).$$

Substituting (42) into the second equality of (41), we now get

$$\begin{aligned}&\sum_{j=1}^m\{(\lambda_k\mathrm{e}(t;\mu_j)+\mu_j^2\mathrm{d}(t;\mu_j))/(\mu_j^2-\lambda_k^2)\\ &\qquad+\sum_{h=1}^m(\mathrm{e}(t;\mu_h)+\lambda_k\mathrm{d}(t;\mu_h))\tau_{hj}/(\mu_h^2-\lambda_k^2)\}a_j\\ &=\delta_k^i\exp[\lambda_k t]/\lambda_k,\qquad 1\leqq k\leqq m,\end{aligned} \tag{43}$$

which should determine the desired values $a_j=a_j^i(t)$, and hence $b_j=b_j^i(t)$, $1\leqq j\leqq m$.

We set

$$(44)\qquad \Delta_m(t) := \left| \begin{array}{l} (\lambda_k e(t;\mu_j) + \mu_j^2 d(t;\mu_j))/(\mu_j^2 - \lambda_k^2) \\ \quad + \sum_{h=1}^{m} (e(t;\mu_h) + \lambda_k d(t;\mu_h))\tau_{hj}/(\mu_h^2 - \lambda_k^2) \end{array} \right|$$

to see that $\lim_{t \downarrow 0} \Delta_m(t) = |2\lambda_k/(\mu_j^2 - \lambda_k^2)| \neq 0$. If we define

$$(45)\qquad T_m := \inf\{t>0;\ \Delta_m(t)=0\},$$

we have $0 < T_m \leqq \infty$ and $(-1)^m \Delta_m(t) > 0$ for all $t < T_m$. That is, the equations (42) and (43) give us unique solution $\{a_j, b_j\}$ for each $t \in (0, T_m)$.

Conjecture. $T_m = \infty$ for every m, i.e. $(-1)^m \Delta_m(t) > 0$ for all $t > 0$.

Our conjecture is true for $m=1$. For $m \geqq 2$, however, the problem of determining the exact value of T_m is beyond the author's abilities, even in the simplest case $m=2$:

$$\begin{aligned} \Delta_2(t) = {} & [\{(\lambda_1 + \tau_{11}) e(t;\mu_1) + (\mu_1^2 + \lambda_1 \tau_{11}) d(t;\mu_1)\}/(\mu_1^2 - \lambda_1^2) \\ & + \tau_{21}(e(t;\mu_2) + \lambda_1 d(t;\mu_2))/((\mu_1^2 - \lambda_1^2)] \times \\ & \times [\tau_{12}(e(t;\mu_1) + \lambda_2 d(t;\mu_1))/((\mu_1^2 - \lambda_2^2) \\ & + \{(\lambda_2 + \tau_{22}) e(t;\mu_2) + (\mu_2^2 + \lambda_2 \tau_{22}) d(t;\mu_2)\}/(\mu_2^2 - \lambda_2^2)] \\ & - [\tau_{12}(e(t;\mu_1) + \lambda_1 d(t;\mu_1))/((\mu_1^2 - \lambda_1^2) \\ & + \{(\lambda_1 + \tau_{22}) e(t;\mu_2) + (\mu_2^2 + \lambda_1 \tau_{22}) d(t;\mu_2)\}/(\mu_2^2 - \lambda_1^2)] \times \\ & \times [\{(\lambda_2 + \tau_{11}) e(t;\mu_1) + (\mu_1^2 + \lambda_2 \tau_{11}) d(t;\mu_1)\}/(\mu_1^2 - \lambda_2^2) \\ & + \tau_{21}(e(t;\mu_2) + \lambda_2 d(t;\mu_2))/(\mu_2^2 - \lambda_2^2)]. \end{aligned}$$

What we have proved so far is summarized as follows:

Proposition 2. *The boundary value problem (35) for the differential equation (34) has, for every i $(1 \leqq i \leqq m)$ and t $(0 < t < T_m)$, unique solution $z_k{}^i(s;t)$ of the form (40) with coefficients (a_j, b_j) determined by (42) and (43).*

Remark 2. For the differential equation (33) in Remark 1, we can go on the lines of the above discussions to reach a similar conclusion to Proposition 2. Indeed, the $(m-1) \times (m-1)$ matrix

$$R^{\wedge} = (\delta_k{}^j \lambda_k^2 + c_k \lambda_k(\lambda_k + \lambda_m)(\lambda_m - \lambda_j)/(\sum_{h=1}^{m} c_h \lambda_h))$$

can be diagonalized by $Q^{\wedge} = (c_k \lambda_k(\lambda_k + \lambda_m)/(\mu_j^2 - \lambda_k^2))$, where the μ_j with $\lambda_1 < \mu_1 < \cdots < \mu_{m-1} < \lambda_m$ are taken to be roots of the equation

$$\sum_{j=1}^{m-1} c_j \lambda_j (\lambda_m^2 - \lambda_j^2)/(\mu^2 - \lambda_j^2) = \sum_{h=1}^{m} c_h \lambda_h, \qquad \mu > 0,$$

(cf. [11],[12]). So we have a general solution of the form

(46) $$z_k(s)=c_k\lambda_k(\lambda_k+\lambda_m)\sum_{j=1}^{m-1}\{a_j e(s;\mu_j)+b_j d(s;\mu_j)\}/(\mu_j^2-\lambda_k^2),$$

and the boundary conditions (33′) become the following linear equations:

$$\sum_{j=1}^{m-1}(\hat{\lambda}_k a_j-b_j)/(\mu_j^2-\lambda_k^2)=0$$

$$(\hat{\lambda}_k := \lambda_k\{(2-c_m)\lambda_m+(1-c_k)\lambda_k\}/\{c_m\lambda_m+(1+c_k)\lambda_k\}) \quad \text{and}$$

$$\sum_{j=1}^{m-1}\{(\lambda_k e(t;\mu_j)+\mu_j^2 d(t;\mu_j))a_j+(e(t;\mu_j)+\lambda_k d(t;\mu_j))b_j\}/(\mu_j^2-\lambda_k^2)$$
$$=2\delta_k^i\exp[\lambda_k t]/(\lambda_m-\lambda_k), \qquad 1\leqq k\leqq m-1.$$

This system of linear equations determines the required coefficients $a_j=a_j^i(t)$, $b_j=b_j^i(t)$ in (46) to obtain the solution $z_k^i(s;t)$ for each i $(1\leqq i\leqq m-1)$ and $t\in(0,\hat{T}_{m-1})$, where we put

$$\hat{\Delta}_{m-1}(t):=\left|\begin{array}{l}(\lambda_k e(t;\mu_j)+\mu_j^2 d(t;\mu_j))/(\mu_j^2-\lambda_k^2)\\ \quad+\sum_{h=1}^{m-1}(e(t;\mu_h)+\lambda_k d(t;\mu_h))\hat{\tau}_{hj}/(\mu_h^2-\lambda_k^2)\end{array}\right|$$

with $\hat{\tau}_{hj}:=\sum_{k=1}^{m-1}\rho_{hk}\hat{\lambda}_k/(\mu_j^2-\lambda_k^2)$, and

$$\hat{T}_{m-1}:=\inf\{t>0;\ \hat{\Delta}_{m-1}(t)=0\}\ >\ 0.$$

It is also conjectured that $\hat{T}_{m-1}=\infty$ for every m, which is true for $m=2$, the double Markov case treated in the author's previous paper [10].

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