Some Periodic Gaussian Processes and the Quasi-Markov Property on the Circle

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Abstract: Under a restriction of the quasi-Markov property on the circle S^1, we give detailed descriptions of stationary Gaussian processes $X(t)$, $t \in S^1$, and of Gaussian processes $Y(t)$ with stationary increments to observe the mutual dependence of two innovation processes arising from the forward and backward canonical representations.

§ 1. Introduction

With the theory of multiple Markov Gaussian processes in mind ([5], [12]), we will discuss some Gaussian processes $Z(t)$ indexed by $t \in S^1$ (the unit circle), for which the *quasi-Markov* property ([8], [9]) holds: For each open arc $I \subset S^1$, we have the interpolation formula

(1) $\quad E[Z(s) \mid Z(t);\ t \notin I] = E[Z(s) \mid Z(t);\ t \in \delta I]$

for every $s \in I$. (Note that the boundary δI consists of two end points of I.) This significant extension of the usual Markov property is also called *reciprocal* ([1],[2] and [17]) or *two-sided Markov* ([7]). In fact, we will solve innovation problems arising from the *forward* and *backward canonical representations*, for stationary quasi-Markov processes $X(t)$ in § 2 and for quasi-Markov processes $Y(t)$ with stationary increments in § 3.

We will start with studying a typical class of stationary Gaussian processes $X(t)$ which possess, in addition to the quasi-Markov property, the *reflection-positivity* on the circle. Parametrizing the point $t \in S^1$ by $-\pi \leqq t \leqq \pi$ (modulo 2π), the latter property (also called *T*- or *OS-positive* ([5], [6], [7] and [13]) can be stated as follows: For any $0 \leqq t_1 < t_2 < \cdots < t_n \leqq \pi$ and any $a_i \in R$, $i=1,2,\cdots,n$, we have the inequality

(2) $\quad E[\sum_{i=1}^{n} a_i X(-t_i))(\sum_{j=1}^{n} a_j X(t_j))] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j R(d(-t_i,t_j)) \geqq 0,$

where $R(t)$ denotes the covariance function of $X(t)$, i.e.,

(3) $\quad R(d(t,s)) = E[X(t)X(s)],$

$d(t,s) := |t-s| \wedge (2\pi - |t-s|)$ being the distance function on the circle. Such a covariance function $R(t)$ verifying (2) admits the following integral representation ([7]):

(4) $R(t) = \int_0^\infty \cosh m(\pi - t)\, d\mu(m),\ 0 \leqq t \leqq \pi.$

In particular, $R(t) = \cosh m(\pi - t)$ coming from the Dirac measure $\mu = \delta_{\{m\}}$ for some $m > 0$, is nothing but the one that we will deal with in § 2 and for which the quasi-Markov property (1) is valid.

§ 3 is devoted to a study of a wider class of Gaussian processes $Y(t)$ with stationary increments, which are expressible as

(5) $Y(t) = (X(t) - X(0))/\sqrt{2},\ t \in S^1,$

in terms of stationary quasi-Markov processes $X(t)$ with covariance functions $R(d(t,s))$. For such processes $Y(t)$, Molchan ([9]) determined all possible forms of structure functions

(6) $V(d(t,s)) := E[(Y(t) - Y(s))^2] = R(0) - R(d(t,s));$

In addition to the above-mentioned family a) $\cosh m(\pi - t)$ $(0 < m < \infty)$, we should take two other families of $R(t)$: b) $\sinh m|\pi - t|$ $(0 < m < \infty)$ and c) $\sin m|\pi - t|$ $(0 < m \leqq 1/2)$. For these structure functions $V(t)$ we will show the double Markov structure of $Y(t)$, which means that this paper can be thought of as a continuation of the author's previous papers [11] and [12] (see also Remark 1 in this connection).

The purpose of the present paper is to investigate both the forward and backward canonical representations and then observe the mutual dependence of their innovation processes. Namely, define

(7) $X_\pm(t) = X(\pm t) - E[X(\pm t) \mid X(0)],\ 0 \leqq t \leqq \pi,$

which are simple Markov, and hence their canonical representations easily follow:

(8) $X_\pm(t) = \int_0^t f_1(t) g_1(u)\, dB_\pm(u),$

$f_1(t)$ and $g_1(u)$ being given explicitly. For any process $Y(t)$ of another kind, starting with the expression

(9) $Y_\pm(t) := Y(\pm t) = \{X_\pm(t) + (R(t)/R(0) - 1) X(0)\}/\sqrt{2},\quad 0 \leqq t < \pi,$

we can go likewise to reach the desired canonical representation

($\tilde{8}$) $Y_\pm(t) = \int_0^t \{\tilde{f}_1(t)\tilde{g}_1(u) + \tilde{f}_2(t)\tilde{g}_2(u)\}\, d\tilde{B}_\pm(u)$

(see Theorem 4).

Now, new objects discussed in this paper are in order:

(10) $I(t,s) = E[B_+(t) B_-(s)],\ \ 0 \leqq t,s \leqq \pi,$

and

($\tilde{10}$) $\tilde{I}(t,s) = E[\tilde{B}_+(t) \tilde{B}_-(s)],\ \ 0 \leqq t,s \leqq \pi,$

where $B_{\pm}(t)$ (resp. $\tilde{B}_{\pm}(t)$) are the innovation processes of $X_{\pm}(t)$ (resp. $Y_{\pm}(t)$) given in (8) (resp. ($\tilde{8}$)). These quantities are important in Okabe's theory ([13],[14]) concerning the fluctuation-dissipation principle for stationary processes with discrete as well as continuous time parameter. Our main results then lie in the following formulae: Putting $\varepsilon = +1$ for a), $= -1$ for b) and c),

(11) $I(t,s) = \varepsilon\, p_1(t) p_1(s)$,

and

($\tilde{11}$) $\tilde{I}(t,s) = \varepsilon\, \tilde{p}_1(t)\tilde{p}_1(s) + \tilde{p}_2(t)\tilde{p}_2(s)$,

thereby calculating explicit expressions of these p-functions (see Theorems 2 and 5).

The present analysis of both pairs $(X_{\pm}(t), B_{\pm}(t))$ and $(Y_{\pm}(t), \tilde{B}_{\pm}(t))$ has a new aspect in contrast with our previous one of the time evolution structure for multiple Markov Gaussian processes on the line ([11], [12]). Indeed, we are going to investigate the following key expression

(∗) $Z(t) = \sum_{i=1}^{N} F_i(t)\xi_i + \int_0^t \sigma(u)dB(u), \quad 0 \leqq t \leqq \pi,$

where $\boldsymbol{\xi} = \{\xi_i\}_{i=1}^{N}$ is an i.i.d. sequence of N(0,1)-random variables and $B(t)$ is a standard Brownian motion. As for a required relation between two random elements $\boldsymbol{\xi}$ and $B(t)$, we consider the following two:

(+) $\boldsymbol{\xi}$ and $B(t)$ are assumed to be independent,

which occurs here for $N=1$ when we investigate (9);

(−) $\boldsymbol{\xi}$ and $B(t)$ are assumed to have a particular correlation expressed in the form $E[\xi_i(\int_0^t \sigma(u)dB(u))] = -F_i(t), \quad 1 \leqq i \leqq N,$

which yields the independence of $\boldsymbol{\xi}$ and $Z(t)$. The latter case (−) occurs for $N=1$ when we give a realization of two mutually dependent Brownian motions $B_{\pm}(t)$ in the framework of canonical representation theory (see Theorem 3 and also [10]).

Such a process $Z(t)$ often arises since one needs to add independent random elements ξ_i to a basic additive process $B_\sigma(t) := \int_0^t \sigma(u)dB(u)$; This goes to the first case (+). On the other hand, in the second case (−) the expression (∗) should be rewritten into the form of expanding $B_\sigma(t)$ into independent random terms:

(∗)' $B_\sigma(t) = Z(t) + \sum_{i=1}^{N} F_i(t)(-\xi_i).$

Then by extracting some building blocks ξ_i from this expansion, one finds a process $Z(t)$ having less randomness than the original process $B_\sigma(t)$. As a famous example of (∗)' with $\sigma(u) \equiv 1$, we mention the Paley-Wiener method to construct a standard Brownian motion ([4]).

The covariance function $\Gamma(t,s)$ of (∗) becomes

(12) $\Gamma(t,s)=\int_0^{t\wedge s}\sigma^2(u)du \pm \sum_{i=1}^{N} F_i(t)F_i(s),$

which tells us a special symmetric nature within a general class of Goursat kernels of order $N+1$ ([5]):

(12)' $\Gamma(t,s)=\sum_{i=0}^{N} F_i(t\vee s)H_i(t\wedge s).$

That is, $H_i(t)=\pm F_i(t)$ except $i=0$ (here the sign corresponding to the assumption $(\pm)$), and $F_0(t)\equiv 1$, $H_0(t)=\int_0^t \sigma^2(u)du$. This symmetric nature of (12) will lead us to derive a stochastic Ito-Volterra equation for (*), from which the canonical representation of $Z(t)$ follows via the resolvent equation (29) (see also Remark 4 where the case $N=2$ is briefly mentioned).

§ 2. Periodic stationary reflection-positive Gaussian processes

We begin with discussing a periodic stationary Gaussian process $X(t)$, $t\in \mathbf{R}^1$, with mean 0 and covariance function $R(|t-s|)$; the period is here taken to be 2π (i.e. $X(t+2\pi)=X(t)$ for all t), and $R(t)$, determined up to modulo 2π, is taken from the class a) mentioned in §1: For $0\leqq t\leqq 2\pi$,

(13) $R(t)=\cosh m(\pi-t), \qquad m>0.$

The parameter t can be thought of as the point moving on the circle $\mathbf{S}^1\sim[-\pi,\pi]$, and we would rather consider a stationary process $X(t)$ indexed by $t\in\mathbf{S}^1$ and having the covariance function

(13)' $R(d(t,s))=\cosh m(\pi-d(t,s)),$

where $d(t,s)$ denotes the distance function on $\mathbf{S}^1$.

We are thus given a stationary process $X(t)$, $-\pi\leqq t\leqq\pi$; let us define the forward process

(14) $X_+(t):=X(t)-\mathrm{E}[X(t)\mid X(0)]=X(t)-R(t)X(0)/R(0), \quad 0\leqq t\leqq\pi,$

and the backward process

(14)' $X_-(t):=X(-t)-\mathrm{E}[X(-t)\mid X(0)]=X(-t)-R(t)X(0)/R(0), \quad 0\leqq t\leqq\pi,$

where the parameter t and $-t$ run the semicircles $\mathbf{S}_+\sim[0,\pi]$ and $\mathbf{S}_-\sim[-\pi,0]$, respectively. Then both processes have the same covariance function of the form

(15) $R_0(t,s)=\{R(t-s)R(0)-R(t)R(s)\}/R(0)$

$$=\{\cosh m(2\pi-t+s)-\cosh m(2\pi-t-s)\}/2\cosh m\pi$$

$$=\sinh m(2\pi-t)\sinh ms/\cosh m\pi, \quad 0\leqq s\leqq t\leqq\pi,$$

which shows the simple Markov property of $X_\pm(t)$, $0\leqq t\leqq\pi$.

We also need to see the similar form of $\mathrm{E}[X_+(t)X_-(s)]:=R_1(t,s)$:

(16) $R_1(t,s)=\{R(t+s)R(0)-R(t)R(s)\}/R(0)$

$=\{\cosh m(t+s)-\cosh m(t-s)\}/2\cosh m\pi$

$=\sinh mt \sinh ms/\cosh m\pi, \quad 0\leqq s\leqq t\leqq \pi,$

which implies the reflection-positivity (2) as follows:

$$\mathrm{E}\,[(\sum_{i=1}^{n} a_i X(-t_i))(\sum_{j=1}^{n} a_j X(t_j))]$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \{R_1(t_j,t_i)+R(t_i)R(t_j)/R(0)\}$$
$$=\{(\sum_{i=1}^{n} a_i \sinh mt_i)^2+(\sum_{i=1}^{n} a_i \cosh m(\pi-t_i))^2\}/\cosh m\pi \geqq 0.$$

It deserves mentioning that the quasi-Markov property of $X(t)$ is an immediate consequence of (15) and (16). That is, for each s and p, $0\leqq s\leqq p\leqq \pi$, the random variable

(17) $X(s)-\mathrm{E}\,[X(s)\mid X(0),X(p)]=X_+(s)-\mathrm{E}\,[X_+(s)\mid X_+(p)]$

$=X_+(s)-\sinh ms\,(\sinh mp)^{-1}X_+(p)=X(s)-\sinh ms\,(\sinh mp)^{-1}X(p)$

$-\{\sinh m(\pi+p-s)-\sinh m(\pi-p+s)\}\{\sinh m(\pi+p)-\sinh m(\pi-p)\}^{-1}X(0)$

is seen to be independent of all $X_+(t)$, $p\leqq t\leqq \pi$, and $X_-(u)$, $0\leqq u\leqq \pi$, which yields (1) for an open interval $I=(0,p)$.

Now, the simple Markov property of (15) enables us to form

$$X_\pm(t)=\sinh m(2\pi-t)\int_0^t g(u)dB_\pm(u), \quad 0\leqq t\leqq \pi,$$

with $g(u)$ satisfying $\int_0^s g^2(u)du=\sinh ms\{\cosh m\pi \ \sinh m(2\pi-s)\}^{-1}$. We thus get $g^2(u)=2m\sinh m\pi\ (\sinh m(2\pi-u))^{-2}$, to state the following

Proposition 1. *The forward and backward processes* $X_\pm(t)$ *have the canonical representations of the same form*:

(18) $X_\pm(t)=\sqrt{2m\sinh m\pi}\,\sinh m(2\pi-t)\int_0^t(\sinh m(2\pi-u))^{-1}dB_\pm(u).$

The innovation processes $B_\pm(t)$ *are then given by*

(19) $B_\pm(t)=\{X_\pm(t)+m\int_0^t \coth m(2\pi-u)X_\pm(u)du\}/\sqrt{2m\sinh m\pi}.$

Proof. We verify (19) as an easy consequence of (18):

$$B_\pm(t)=\{\int_0^t \sinh m(2\pi-u)\{X_\pm(u)/\sinh m(2\pi-u)\}'du\}/\sqrt{2m\sinh m\pi}$$
$$=\{X_\pm(t)-\int_0^t(\sinh m(2\pi-u))'(\sinh m(2\pi-u))^{-1}X_\pm(u)du\}/\sqrt{2m\sinh m\pi},$$

which is equal to the right-hand side of (19).

We now proceed to observe the mutual dependence of two innovation processes $B_+(t)$ and $B_-(s)$ derived above. By (16) and (19), we compute (10) this way:

$$
\begin{aligned}
I(t,s) &= \mathrm{E}\,[B_+(t)B_-(s)] \\
&= (2m \sinh m\pi)^{-1}\{R_1(t,s) + m\int_0^t \coth m(2\pi - u)R_1(u,s)du \\
&\quad + m\int_0^s \coth m(2\pi - v)R_1(t,v)dv \\
&\quad + m^2\int_0^t\int_0^s \coth m(2\pi - u) \coth m(2\pi - v)R_1(u,v)dudv\} \\
&= (m \sinh 2m\pi)^{-1}\{\sinh mt + m\int_0^t \coth m(2\pi - u) \sinh mu\, du\} \\
&\quad \{\sinh ms + m\int_0^s \coth m(2\pi - v) \sinh mv\, dv\}.
\end{aligned}
$$

We have proved the following

Theorem 2. *We have* $I(t,s) = p_1(t)p_1(s)$ *with*

$$
\begin{aligned}
p_1(t) &= \sqrt{m \sinh 2m\pi}\int_0^t (\sinh m(2\pi - u))^{-1}du \\
&= (\sqrt{\sinh 2m\pi / m})\log[\tanh m\pi / \tanh m(\pi - t/2)].
\end{aligned}
$$

We are now in a position to give a realization of these mutually dependent innovation processes $B_\pm(t)$, $0 \leqq t \leqq \pi$, in terms of a standard Brownian motion $B_0(t)$, $-\pi \leqq t \leqq \pi$ (cf. [10]). First fix the backward Brownian motion $B_-(t) = B_0(-t)$, $0 \leqq t \leqq \pi$. Then the forward Brownian motion $B_+(t)$ admits a representation

$$(20) \quad B_+(t) = \int_0^\pi \phi(t,u)dB_-(u) + \int_0^t \psi(t,u)dB_0(u), \quad 0 \leqq t \leqq \pi,$$

where kernel functions ϕ and ψ must satisfy

$$\mathrm{E}\,[B_+(t)B_-(s)] = \int_0^s \phi(t,u)du = p_1(t)p_1(s),$$

and

$$\mathrm{E}\,[B_+(t)B_+(s)] = \int_0^\pi \phi(t,u)\,\phi(s,u)du + \int_0^{t\wedge s} \psi(t,u)\,\psi(s,u)du = t\wedge s.$$

Hence we get $\phi(t,u) = p_1(t)p_1'(u)$, and $\psi(t,u)$ should be determined by

$$(21) \quad \int_0^{t\wedge s} \psi(t,u)\,\psi(s,u)du = t\wedge s - k^2 p_1(t)p_1(s),$$

where $k := \| p_1'(u) \|_{L^2(0,\pi)} = \sqrt{\sinh 2m\pi\ [\coth m(2\pi - u)]_0^\pi} = 1$. We thus find (12) with minus sign by taking $N=1$, $\sigma(u) \equiv 1$ and $F_1(t) = p_1(t)$.

Now, a new Gaussian process defined by

$$
\begin{aligned}
(22) \quad Z(t) &:= B_+(t) - \mathrm{E}\,[B_+(t) \mid \mathsf{F}_\pi(B_-)] \\
&= B_+(t) - p_1(t)\int_0^\pi p_1'(u)dB_-(u) = \int_0^t \psi(t,s)dB_0(u), \quad 0 \leqq t < \pi,
\end{aligned}
$$

admits the key expression

$$(23) \quad Z(t) = p_1(t)\,\xi + B_+(t),$$

with $\xi := -\int_0^\pi p_1'(u)dB_-(u)$ and $\mathrm{E}\,[\,\xi B_+(t)] = -p_1(t)$. Here $\mathsf{F}_t(X) := \sigma\,\{X(u)\,;\,0 \leqq u \leqq t\}$ for any process $X(t)$, and we note that the filtering problem for (23) is easily solved:

$$(24) \quad \hat{\xi}(t) = \mathrm{E}\,[\,\xi \mid \mathsf{F}_t(B_+)] = -\int_0^t p_1'(u)dB_+(u),$$

and

$$
\begin{aligned}
(25) \quad \mathrm{E}\,[(\xi - \hat{\xi}(t))^2] &= 1 - \int_0^t (p_1'(u))^2 du \\
&= 1 + \cosh 2m\pi - \sinh 2m\pi \coth m(2\pi - t) > 0
\end{aligned}
$$

for every $t < \pi$.

We are ready to derive a stochastic Ito-Volterra equation for the process (23) having the covariance function (21). Namely, we can write

(26) $dZ(t)=dt\int_0^t L(t,s)dZ(s)+dB_0(t),$

to determine a Volterra kernel $L(t,s)$ by the following integral equation:

(27) $L(t,s)=-\{p_1'(t)-\int_0^t L(t,u)p_1'(u)du\}\ p_1'(s),\quad 0\leqq s<t<\pi,$

which is equivalent to the independence of $dB_0(t)$ and $\mathsf{F}_t(Z)$. As unique solution of this (27), we get

(28) $L(t,s)=-p_1'(t)\{1-\int_0^t (p_1'(u))^2du\}^{-1}p_1'(s).$

The resolvent equation

(29) $L(t,s)+K(t,s)=\int_s^t L(t,u)K(u,s)du$

then provides us with the resolvent kernel

(30) $K(t,s)=p_1'(t)p_1'(s)\{1-\int_0^s (p_1'(u))^2du\}^{-1}.$

In view of the known formula $\psi(t,u)=1-\int_u^t K(s,u)ds$ ([5]), we thus arrive at the desired canonical kernel $\psi(t,u)$ in (22), which is stated as the following

Theorem 3. *The Gaussian process* $Z(t)=B_+(t)-\mathrm{E}[B_+(t)\mid \mathsf{F}_\pi(B_-)]$, $0\leqq t<\pi$, *is a double Markov process to be represented canonically as follows*:

(31) $Z(t)=\int_0^t [1-p_1'(u)\int_u^t p_1'(s)ds\{1-\int_0^u (p_1'(v))^2dv\}^{-1}]dB_0(u),$

with $p_1'(t)=\sqrt{m\sinh 2m\pi}/\sinh m(2\pi-t).$

By virtue of the equation (26), the innovation process $B_0(t)$ is, in turn, expressible in terms of $Z(t)$ and hence in terms of ξ and $B_+(t)$; we obtain

$$\begin{aligned}(32)\quad B_0(t)&=Z(t)+\int_0^t p_1'(s)dZ(s)\int_s^t p_1'(u)\{1-\int_0^u (p_1'(v))^2dv\}^{-1}du\\ &=\xi\int_0^t p_1'(u)\{1-\int_0^u (p_1'(v))^2dv\}^{-1}du\\ &\quad +B_+(t)+\int_0^t p_1'(s)dB_+(s)\int_s^t p_1'(u)\{1-\int_0^u (p_1'(v))^2dv\}^{-1}du.\end{aligned}$$

Remark 1. As in the previous papers [11] and [12], we would like to study the following generalization of (13) treated in this section:

$(\tilde{4})$ $R(t)=\sum_{i=1}^n c_i\cosh m_i(\pi-t),$

where $0<m_1<m_2<\cdots<m_n$ and $c_i>0$, $i=1,2,\cdots,n$. Under the normalizing condition that $R(0)=\sum_{i=1}^n c_i\cosh m_i\pi=1$, we have, for $0\leqq s\leqq t\leqq\pi$,

$$\begin{aligned}(33)\quad R_0(t,s)&=\{\sum_{i=1}^n c_i\cosh m_i(\pi-t+s)\}\{\sum_{j=1}^n c_j\cosh m_j\pi\}\\ &\quad-\{\sum_{i=1}^n c_i\cosh m_i(\pi-t)\}\{\sum_{j=1}^n c_j\cosh m_j(\pi-s)\}\\ &=\sum_{i=1}^n c_i^2\sinh m_i(2\pi-t)\sinh m_is+\sum_{1\leqq i<j\leqq n} c_ic_j\{(\exp[m_i(\pi-t)+m_j\pi]\\ &\quad-\exp[-m_i\pi-m_j(\pi-t)])(\exp[m_is]-\exp[-m_js])\\ &\quad+(\exp[-m_i(\pi-t)-m_j\pi]-\exp[m_i\pi+m_j(\pi-t)])(\exp[-m_is]-\exp[m_js])\\ &\quad+(\exp[m_i(\pi-t)-m_j\pi]-\exp[-m_i\pi+m_j(\pi-t)])(\exp[m_is]-\exp[m_js])\end{aligned}$$

$+(\exp[-m_i(\pi-t)+m_j\pi]-\exp[m_i\pi-m_j(\pi-t)])$

$(\exp[-m_i s]-\exp[-m_j s])\}/4,$

which is a Goursat kernel of order $n(2n-1)$. We therefore apply the theory of multiple Markov Gaussian processes ([5] and [12]) to the processes $X_\pm(t)$ having this covariance function (33), although we cannot here enter into details.

§3. Periodic Gaussian processes with stationary increments

In this section we impose the condition of stationary increments instead of stationarity and study the analogous innovation problem for a wider class of quasi-Markov Gaussian processes.

Let $Y(t)$, $t \in S^1$, be a Gaussian process with mean 0 and structure function $V(d(t,s)) = E[(Y(t)-Y(s))^2]$. The quasi-Markov property of $Y(t)$ leads us to consider the particular form $V(t)=R(0)-R(t)$, $0 \leqq t \leqq \pi$, $R(t)$ being one of the following decreasing covariance functions:

a) $\cosh m(\pi-t)$ $(0<m<\infty)$; b) $\sinh m(\pi-t)$ $(0<m<\infty)$;

c) $\sin m(\pi-t)$ $(0<m\leqq 1/2)$.

This list is due to Molchan ([9]) (see also Remark 2 and [1]). Parametrizing the point on S^1 by $-\pi \leqq t \leqq \pi$ (modulo 2π) again, we see that such a process $Y(t)$ with $Y(0)=0$ is expressed in the form

$$Y(t)=\{X(t)-X(0)\}/\sqrt{2},$$

where $X(t)$ is a stationary process with covariance function $R(d(t,s))$.

Remark 2. i) In the preceding section we discussed $X(t)$ coming from the class a), which possesses the reflection-positivity on the circle. As an extremal case of this class, we mention the following limit for $V(t)$, not for $R(t)$ itself:

$$V_0(t)=\lim_{m\downarrow 0}\{\cosh m\pi-\cosh m(\pi-t)\}/m^2\pi = t(2\pi-t)/2\pi,$$

which is nothing but the structure function of a pinned Brownian motion. This simple Markov process is well-known ([4]), although its canonical representation follows from Theorem 4 below by the limiting procedure mentioned above.

ii) Let $\tilde{X}((x,y)), (x,y)\in R^2$, be a Lévy's Brownian motion and set $X(t)=\tilde{X}((\cos t, \sin t))$, $-\pi \leqq t \leqq \pi$. Then the covariance function of $X(t)$ is $R(t)=1-\sin(t/2)$ and hence $V(t)=\sin(t/2)$, which should be contrasted with the class c) with $m=1/2$. Si Si ([15], [16]) gave a nice account of this double Markov process by taking a different approach from ours (cf. [8]).

Now observe that, for $0 \leqq s \leqq t \leqq \pi$,

(15)' $R_0(t,s)=\{R(t-s)R(0)-R(t)R(s)\}/R(0)=f(2\pi-t)f(s)/R(0)$,

where $f(t)=\sinh mt$ for a) and b), $=\sin mt$ for c). Note that $R(t)=f(\pi-t)$ for b) and c). Hence the forward and backward processes $X_\pm(t)$ are simple Markov and represented canonically by

(18)' $X_\pm(t)=\sqrt{2}\,Cf(2\pi-t)\int_0^t dB_\pm(u)/f(2\pi-u)$, $0\leqq t\leqq \pi$,

where we put $C^2:=mf(\pi)$ for a), $=f'(\pi)$ for b) and c). On the other hand, the correlation between $X_+(t)$ and $X_-(s)$ is expressed by

(16)' $R_1(t,s)=\{R(t+s)R(0)-R(t)R(s)\}/R(0)=\varepsilon f(t)f(s)/R(0)$,

for any $0\leqq t,s\leqq \pi$. Hence, in the cases b) and c), the sign $\varepsilon=-1$ violates the reflection-positivity (2), but the quasi-Markov property (1) still holds.

We are now going to investigate a new object $Y(t)$ of this section. Restricting its parameter t on the semicircles S_+ and S_-, respectively, we consider the forward and backward processes

(9)' $Y_\pm(t):=Y(\pm t)=V(t)\xi/\sqrt{2}R(0)+Cf(2\pi-t)\int_0^t dB_\pm(u)/f(2\pi-u)$, $0\leqq t\leqq\pi$,

where $\xi:=-X(0)/\sqrt{R(0)}$ is an N(0,1)-random variable independent of the innovation processes $B_\pm(t)$ both. We will see the double Markov property of $Y_\pm(t)$ to get the explicit canonical kernel of the form $(\tilde{8})$.

For that purpose, we begin with studying any process $Z(t)$ expressed in the form

(34) $Z(t)=\{\int_0^t h(u)\sigma(u)du\}\xi+\int_0^t\sigma(u)dB(u)$, $0\leqq t\leqq\pi$,

where an N(0,1)-random variable ξ and a Brownian motion $B(t)$ are mutually independent (i.e. in the case(+)), and where $\sigma(u)>0$ is continuous and $h\in \mathsf{L}^2(0,\pi)$. For such a process $Z(t)$, write the stochastic Ito-Volterra equation

(26)' $dZ(t)=dt\int_0^t L(t,s)dZ(s)+\sigma(t)d\tilde{B}(t)$,

where $\tilde{B}(t)$ is the innovation process of $Z(t)$ and a Volterra kernel $L(t,s)$ satisfies the integral equation

(27)' $\sigma(s)L(t,s)=\{\sigma(t)h(t)-\int_0^t L(t,u)h(u)\sigma(u)du\}h(s)$, $0<s<t\leqq\pi$.

As in the preceding section, unique solution to the above (27)' is obtained as follows:

(28)' $L(t,s)=\sigma(t)h(t)\{1+\int_0^t h^2(u)du\}^{-1}h(s)(\sigma(s))^{-1}$,

which yields the resolvent kernel $K(t,s)$ of $L(t,s)$:

(30)' $K(t,s)=-\sigma(t)h(t)h(s)\{\sigma(s)(1+\int_0^s h^2(u)du)\}^{-1}$.

We thus obtain the desired canonical representation

(31)' $Z(t)=\int_0^t[\sigma(u)+\{\int_u^t h(s)\sigma(s)ds\}h(u)\{1+\int_0^u h^2(v)dv\}^{-1}]d\tilde{B}(u)$,

which is a double Markov process. Furthermore, the Brownian motion $\tilde{B}(t)$ is expressible as

$$(32)' \quad \tilde{B}(t)=\int_0^t [1-h(s)\int_s^t h(u)\{1+\int_0^u h^2(v)dv)\}^{-1}du](\sigma(s))^{-1}dZ(s)$$
$$=\{\int_0^t [1-h(s)\int_s^t h(u)\{1+\int_0^u h^2(v)dv\}^{-1}du]h(s)ds\}\xi$$
$$+B(t)-\int_0^t\{\int_s^t h(u)\{1+\int_0^u h^2(v)dv\}^{-1}du\}h(s)dB(s).$$

Remark 3. As in (24) and (25), it is easy to solve the filtering problem for (34):

$$(35) \quad \hat{\xi}(t)=\mathrm{E}[\xi \mid \mathsf{F}_t(Z)]=\{1+\int_0^t h^2(v)dv)\}^{-1}\int_0^t h(u)(\sigma(u))^{-1}dZ(u),$$

and

$$(36) \quad e^2(t):=\mathrm{E}[(\xi-\hat{\xi}(t))^2]=\{1+\int_0^t h^2(v)dv)\}^{-1}.$$

Under the present condition that $h(u)\in \mathsf{L}^2(0,\pi)$, we have $e(t)>0$, which means that $\mathsf{F}_t(Z)=\mathsf{F}_t(\tilde{B})\subsetneqq\sigma(\xi)\vee\mathsf{F}_t(B)$ for every $t\in(0,\pi]$. On the other hand, if we had the first time t_1 such that $\int_0^{t_1}h^2(u)du=\infty$ for some $h(u)\in\mathsf{L}^1(0,\pi)$, then $e(t_1)=0$ and ξ would be measurable with respect to $\mathsf{F}_{t_1}(Z)=\mathsf{F}_{t_1}(\tilde{B})$; After this moment t_1 the increment $d\tilde{B}(t)$ of the innovation process would coincide with the original white noise $dB(t)$, $t>t_1$.

Now, we are ready to give a complete description of our processes $Y_\pm(t)$. By (9)', we set

$$(34)' \quad Z_\pm(t)=Y_\pm(t)/Cf(2\pi-t)=V(t)\{\sqrt{mf(2\pi)}f(2\pi-t)\}^{-1}\xi$$
$$+\int_0^t dB_\pm(u)/f(2\pi-u),$$

and therefore in the present concrete situation, we have

$$(37) \quad \begin{cases} \sigma(t)=1/f(2\pi-t) \quad \text{and} \\ h(t)=\{\sqrt{mf(2\pi)}\}^{-1}f(2\pi-t)\{V(t)/f(2\pi-t)\}' \end{cases}$$
$$=\{\sqrt{2m\phi(m\pi)}f(2\pi-t)\}^{-1}\{f'(2\pi-t)-\varepsilon m\}$$

where we put $\phi(t)=\tanh t$ for a), $=\coth t$ for b), $=\cot t$ for c). The two integrals appeared in the canonical kernel of (31)' are shown to take simpler forms like these: First, for a) and b), we have

$$(38) \quad \int_u^t h(s)\sigma(s)ds=\int_{m(2\pi-t)}^{m(2\pi-u)}(\cosh v-\varepsilon)(\sinh v)^{-2}dv/\sqrt{2m\phi(m\pi)}$$
$$=\int_{m(2\pi-t)}^{m(2\pi-u)}(\cosh v+\varepsilon)^{-1}dv/\sqrt{2m\phi(m\pi)}$$
$$=-\varepsilon\{\phi(m(\pi-t/2))-\phi(m(\pi-u/2))\}/\sqrt{2m\phi(m\pi)};$$

Similarly, for c),

$$(38)' \quad \int_u^t h(s)\sigma(s)ds=\int_{m(2\pi-t)}^{m(2\pi-u)}(1-\cos v)^{-1}dv/\sqrt{2m\phi(m\pi)}$$
$$=-\varepsilon\{\phi(m(\pi-t/2))-\phi(m(\pi-u/2))\}/\sqrt{2m\phi(m\pi)}.$$

Next, we see that

$$(39) \quad 1+\int_0^t h^2(u)du=\{\delta mt/2+\phi(m(\pi-t/2))\}/\phi(m\pi),$$

where $\delta=+1$ for a) and b), $=-1$ for c).

In view of the formulae (31)' and (32)' in a general situation, some calculations based on (37)～(39) lead us to the following result.

Theorem 4. *We have the canonical representation*

(40) $Y_{\pm}(t)=\int_0^t Cf(2\pi-t)[1-\varepsilon\{\phi(\mathrm{m}(\pi-\mathrm{t}/2))-\phi(m(\pi-u/2))\}$
$\{f'(2\pi-u)-\varepsilon m\}/2m\{\delta mu/2+\phi(m(\pi-u/2))\}]d\tilde{B}_{\pm}(u)/f(2\pi-u).$

Furthermore, the innovation processes $\tilde{B}_{\pm}(t)$ *are written in terms of* $Y(t)$ *as follows:*

(41) $\tilde{B}_{\pm}(t)=C^{-1}\int_0^t[1-\{(f'(2\pi-s)-\varepsilon m)/2mf(2\pi-s)\}$
$\int_s^t(f'(2\pi-u)-\varepsilon m)\{f(2\pi-u)(\delta mu/2+\phi(m(\pi-u/2)))\}^{-1}du]$
$f(2\pi-s)\,d[Y(\pm s)/f(2\pi-s)]$
$=C^{-1}\{Y(\pm t)+\int_0^t j(t,s)Y(\pm s)ds\},$

where we set

(42) $j(t,s):=f'(2\pi-s)/f(2\pi-s)-\{(f'(2\pi-s)-\varepsilon m)/f(2\pi-s)\}^2$
$\{2m(\delta ms/2+\phi(m(\pi-s/2)))\}^{-1}-(\delta m/2)$
$\int_s^t\{(f'(2\pi-u)-\varepsilon m)/f(2\pi-u)\}\{\delta mu/2+\phi(m(\pi-u/2))\}^{-1}du,$

which is a Goursat kernel of order 2.

We are now in a position to calculate the mutual dependence of the two innovation processes $\tilde{B}_{\pm}(t)$ derived above. For that purpose we note the following Goursat form of order 2 for the covariance function $R_2(t,s):=\mathrm{E}[Y(t)Y(-s)]$, $0\leqq t,s\leqq\pi$: By (9)' and (16)' we have

(43) $R_2(t,s)=\{\varepsilon f(t)f(s)+V(t)V(s)\}/2R(0).$

It follows from the sign of ε that the reflection-positivity for $Y(t)$ holds only in the case a).

We make use of (41)～(43) to get

$$\begin{aligned}\tilde{I}(t,s)&=\mathrm{E}[\tilde{B}_+(t)\tilde{B}_-(s)]\\&=C^{-2}\{R_2(t,s)+\int_0^t j(t,u)R_2(u,s)du+\int_0^s j(s,v)R_2(t,v)dv\\&\quad+\int_0^t\int_0^s j(t,u)j(s,v)R_2(u,v)dudv\}\\&=(2R(0)C^2)^{-1}\{\varepsilon(f(t)+\int_0^t j(t,u)f(u)du)(f(s)+\int_0^s j(s,v)f(v)dv)\\&\quad+(V(t)+\int_0^t j(t,u)V(u)du)(V(s)+\int_0^s j(s,v)V(v)dv)\}.\end{aligned}$$

We thus reach the desired formula $(\tilde{11})$ by setting

(44) $\tilde{p}_1(t):=(f(t)+\int_0^t j(t,u)f(u)du)/\sqrt{mf(2\pi)}$
$=\int_0^t[1-\{(f'(2\pi-s)-\varepsilon m)/2mf(2\pi-s)\}\int_s^t(f'(2\pi-u)-\varepsilon m)$
$\{f(2\pi-u)(\delta mu/2+\phi(m(\pi-u/2)))\}^{-1}du]\{\sqrt{mf(2\pi)}/f(2\pi-s)\}ds$
$=\int_0^t(\sqrt{mf(2\pi)}/f(2\pi-s))[1+(\varepsilon/2m)(f'(2\pi-s)-\varepsilon m)$
$\{\phi(m(\pi-s/2))-\phi(m\pi)\}\{\delta ms/2+\phi(m(\pi-s/2))\}^{-1}]ds,$

and

(45) $\tilde{p}_2(t):=(V(t)+\int_0^t j(t,u)V(u)du)/\sqrt{mf(2\pi)}$

$=\int_0^t[1-\{(f'(2\pi-s)-\varepsilon m)/2mf(2\pi-s)\}\int_s^t(f'(2\pi-u)-\varepsilon m)$

$\{f(2\pi-u)(\delta mu/2+\phi(m(\pi-u/2)))\}^{-1}du]$

$\{(f'(2\pi-s)-\varepsilon m)/\sqrt{2m\phi(m\pi)}\ f(2\pi-s)\}ds$

$=\int_0^t(\sqrt{\phi(m\pi)/2m}\ (f'(2\pi-s)-\varepsilon m)$

$\{f(2\pi-s)(\delta ms/2+\phi(m(\pi-s/2)))\}^{-1}ds.$

What we have proved is the following

Theorem 5. *We have* $\tilde{I}(t,s)=\varepsilon\tilde{p}_1(t)\tilde{p}_1(s)+\tilde{p}_2(t)\tilde{p}_2(s),\quad 0\leqq t,s\leqq\pi,$ *with* $\tilde{p}$*-functions defined by* (44) *and* (45).

Remark 4. As in Theorem 3, consider a Gaussian process

(46) $\tilde{Z}(t):=\tilde{B}_+(t)-E[\tilde{B}_+(t)\mid F_\pi(\tilde{B}_-)]$

$=\tilde{B}_+(t)-\int_0^\pi\{\varepsilon\tilde{p}_1(t)\tilde{p}_1'(u)+\tilde{p}_2(t)\tilde{p}_2'(u)\}d\tilde{B}_-(u)$

$=\tilde{B}_+(t)+\{\varepsilon k_1\tilde{p}_1(t)\xi_1+k_2\tilde{p}_2(t)\xi_2\},\qquad 0\leqq t<\pi,$

where we put

$k_1:=\|\tilde{p}_1'(u)\|_{L^2(0,\pi)}=[1-(f(2\pi)/2)\{\phi(m\pi)-\phi(m\pi/2)\}^2$

$\{\delta m\pi/2+\phi(m\pi/2)\}^{-1}]^{1/2}<1,$

$k_2:=\|\tilde{p}_2'(u)\|_{L^2(0,\pi)}=[1-\phi(m\pi)/(\delta m\pi/2+\phi(m\pi))]^{1/2}<1,$

and $\xi_i:=-\int_0^\pi\tilde{p}_i'(u)d\tilde{B}_-(u)/k_i\ (i=1,2)$, which are both $N(0,1)$-random variables and independent of $\tilde{Z}(t)$. Defining a new $N(0,1)$-random variable $\tilde{\xi}_2$ independent of ξ_1 by

$\tilde{\xi}_2:=(\xi_2-\rho\,\xi_1)/\sqrt{1-\rho^2},$

$\rho:=E[\xi_1\xi_2]=(\tilde{p}_1'(u),\tilde{p}_2'(u))_{L^2(0,\pi)}/k_1k_2$

$=C^2\varepsilon\{\phi(m\pi)-\phi(m\pi/2)\}/mk_1k_2\{\delta m\pi/2+\phi(m\pi/2)\}>0,$

we have

(46)' $\tilde{Z}(t)=\tilde{B}_+(t)+\{\varepsilon k_1\tilde{p}_1(t)+k_2\rho\tilde{p}_2(t)\}\xi_1+k_2\sqrt{1-\rho^2}\,\tilde{p}_2(t)\tilde{\xi}_2.$

We thus find again the key expression (∗) under the assumption (−) stated in §1. In this example, $N=2$, $\sigma(t)\equiv 1$ and $F_1(t)=\varepsilon k_1\tilde{p}_1(t)+k_2\rho\tilde{p}_2(t)$, $F_2(t)=k_2\sqrt{1-\rho^2}\tilde{p}_2(t)$. Because of the limited space of this paper, we cannot give a full account of (∗) for $N\geqq 2$; In a forthcoming paper, we plan to develop an extension of our results for $N=1$ to general N including the case $N=\infty$, as well as a generalization $(\tilde{4})$ mentioned in Remark 1.

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