Fractional Brownian Motion and Generalized KMO-Langevin Equation

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Abstract: From a standpoint of the stochastic Ito-Volterra equation and the canonical representation of Gaussian processes ([13] and [2]), we investigate self-similar processes derived from fractional Brownian motions ([10]). In particular, we generalize a key property of T-positivity that was assumed in Okabe's theory for stationary Gaussian processes ([15]～[17]).

Key words: Fractional Brownian motion, Fractional calculus, Canonical representation, Stochastic Ito-Volterra equation, T-positivity, Multiple Markov Gaussian process, KMO-Langevin equation.

§1. Introduction

The present paper is, in some sense, a continuation of the author's previous papers [11]～[14]. As was mentioned in Introduction of [13], a fractional Brownian motion $B_h(t)$ with exponent $h \neq 1$, $0 < h < 2$, has these important properties:
(i) Gaussian in distribution; (ii) self-similarity; (iii) stationarity of the increments; (*iv*) the fractional nature $(dt)^h$ of the variance of the infinitesimal increment $dB_h(t) = B_h(t+dt) - B_h(t)$. This last property refuses one to apply a familiar stochastic Ito-Volterra equation of the form

$$dY(t) = \{\int_0^t k(t,u)dY(u)\}dt + \sigma dB_0(t), \tag{1-1}$$

where $B_0(t)$ is a standard Brownian motion that expresses the innovation process of a Gaussian process $Y(t)$, $t \geqq 0$.

Keeping such a fractional nature of $B_h(t)$ in mind, we here deal with two self-similar Gaussian processes $X_{h,1}(t)$ and $X_{h,2}(t)$, which express the odd and even parts of the fractional Brownian motion, respectively. Their canonical representations were obtained in

[10]:

$$X_{h,i}(t)=C_h\int_0^t f_{h,i}(t/u)u^{(h-1)/2}dB_0(u), \tag{1-2}$$

where we see that the kernel $f_{h,i}(s)\sim(s^2-1)^{(h-1)/2}$, as $s=t/u\downarrow 1$. In order to grasp the main term $\sigma dB_0(t)$ in a stochastic equation like (1-1) for such processes, we are naturally led to operate the fractional derivative D_α of order $\alpha=(h-1)/2$, to study new processes $Y_{h,i}(t)=D_\alpha X_{h,i}(t)$ in § 2. Then we get

$$Y_{h,i}(t)=C_h\int_0^t g_{h,i}(t/u)dB_0(u). \tag{1-3}$$

with a finite positive value $g_{h,i}(1)$.

This canonical representation (1-3) of the transformed process $Y_{h,i}(t)$ enables us to write

$$dY_{h,i}(t)=\{-\int_0^t j_{h,i}(t/u)u^{-1}dB_0(u)\}dt+\sigma_h dB_0(t),\quad t>0, \tag{1-4}$$

with positive constant $\sigma_h=C_h g_{h,i}(1)$ and $j_{h,i}(s)=-C_h\frac{d}{ds}g_{h,i}(s)$ (Proposition 1). Associated with (1-4), we need consider the resolvent equation for the Volterra kernel $r(t,u)=j_{h,i}(t/u)(\sigma_h u)^{-1}$,

$$k(t,u)+r(t,u)=\int_u^t r(t,v)k(v,u)dv,\quad 0<u<t. \tag{1-5}$$

Whenever we find a solution $k(t,u)$ of this integral equation, we can reach our goal in § 2, i.e., the following stochastic Ito-Volterra equation that is shown to be equivalent to (1-4) above (cf. [2], Chapter 6):

$$dY_{h,i}(t)=\{\int_0^t k(t,u)dY_{h,i}(u)\}dt+\sigma_h dB_0(t),\quad t>0. \tag{1-1'}$$

Indeed, we are going to prove the existence of such a solution $k(t,u)$ of (1-5) in the simplest case $0<h<1$ and $i=1$ (Theorem 2). Since the kernel $r(t,u)$ does not satisfy a familiar L^2-integrability condition

$$\int_0^T\{\int_0^t r^2(t,u)du\}dt<\infty\quad\text{for any } T>0,$$

we follow our approach taken in [11]. Namely, we show the following inequality instead:

$$\int_0^t |r(t,u)|du<1\quad\text{for any } t>0. \tag{1-6}$$

Then we easily find a desired solution $k(t,u)$ in the L^∞-space, by means of a well-known expansion

$$k(t,u)=-\sum_{n=1}^{\infty}r^{(n)}(t,u), \tag{1-7}$$

$r^{(n)}(t,u)$, $n=1, 2, \cdots$, being the iterated kernels of $r(t,u)$ ([19]).

In § 3, we take up another approach toward a KMO-Langevin equation describing the fractional Brownian motion. By a well-known change of parameter, any self-similar Gaussian process can be transformed into a stationary Gaussian process. Stationary processes $Z_{h,i}(x)$, $x \in \mathrm{R}$, thus transformed from $X_{h,i}(t)$, $t \geqq 0$, turn out to have a notable structure of their spectral density functions. With this observation in mind, we are going to discuss a generalization of T-positivity (=reflection positivity=OS positivity, also used as a terminology).

In fact, we investigate a KMO-Langevin equation for a general multiple Markov stationary Gaussian process $Z(x)$ having a correlation function $\gamma(|x-y|)$ expressible as follows:

$$\gamma(t) = \sum_{i=1}^{N} p_i \exp[-a_i t], \quad t \geqq 0 , \tag{1-8}$$

with $0 < a_1 < a_2 < \cdots < a_N < \infty$, and $p_i \neq 0$ for every i. Let $\{U_i(x)\}_{i=1}^{N}$ be a sequence of mutually independent Ornstein-Uhlenbeck processes with correlation functions $\exp[-a_i|x-y|]$, $1 \leqq i \leqq N$. Then our process $Z(x)$, independent of all $U_i(x)$ such that $p_i < 0$, satisfies the following equality in distribution:

$$Z(x) + \sum_{i=1}^{N} \sqrt{\max\{-p_i, 0\}}\, U_i(x) = \sum_{i=1}^{N} \sqrt{\max\{p_i, 0\}}\, U_i(x). \tag{1-9}$$

In case $p_i > 0$ for all i, the process $Z(x) = \sum_{i=1}^{N} \sqrt{p_i}\, U_i(x)$ becomes T-positive, for which the author has discussed the KM_2O-Langevin equation in [12] and [13].

Now we are ready to state our assumptions. In addition to the usual condition

$$\gamma(0) = 1, \quad \frac{d}{dt}\gamma(0+) = -\sigma^2/2 < 0, \text{ and } \int_0^\infty \gamma(t)dt = \tau^2 > 0, \tag{1-8'}$$

we impose more general condition than that of T-positivity:

(SC) The number $S(p_0, p_1, \cdots, p_N, p_{N+1})$ of sign changes of the extended sequence $\{p_i\}_{i=0}^{N+1}$ is equal to 2,

where we always set $p_0 = p_{N+1} = -1$ in the present paper. A good example including negative p_i and satisfying (SC) comes from a suitable approximation to $Z_{h,i}(x)$, $1 < h < 2$, mentioned above. On the other hand, for $0 < h < 1$, each stationary process $Z_{h,i}(x)$ turns out to be T-positive (see Remark 2).

Under the present assumption of (SC), which contains the case of T-positivity, we are going to study the spectral density function

$$\phi(\lambda) = \int_{-\infty}^{\infty} \exp[i\lambda x]\gamma(|x|)dx/2\pi = \sum_{k=1}^{N} p_k a_k/\pi(\lambda^2 + a_k^2), \tag{1-10}$$

and find its $N-1$ distinct roots $\{ib_j\}_{j=1}^{N-1}$ with $0 < b_1 < b_2 < \cdots < b_{N-1} < \infty$. In view of the roots $\{ib_j\}_{j=1}^{N-1}$ and the poles $\{ia_k\}_{k=1}^{N}$ of $\phi(\lambda)$ distributed on the positive imaginary axis, the associated outer function $c(\lambda)$ is concluded to admit three different expressions:

(1-11)
$$\begin{aligned} c(\lambda) &= \sigma \prod_{j=1}^{N-1}(i\lambda+b_j) \Big/ \prod_{k=1}^{N}(i\lambda+a_k) = \sigma \sum_{k=1}^{N} r_k (i\lambda+a_k)^{-1} \\ &= \sigma\{(i\lambda+q_0)+i\lambda \sum_{j=1}^{N-1} q_j (i\lambda+b_j)^{-1}\}^{-1}. \end{aligned}$$

By these neat expressions (1-11), we are able to proceed on the lines of Okabe's theory ([15]~[17]), to establish the canonical representation of $Z(x)$ and the KMO-Langevin equation describing the time evolution of $Z(x)$ (Theorem 3).

The final topic (Proposition 4) is to discuss in detail the particular case $N=3$ without the above restriction (SC). Namely, when $p_1>0$, $p_2<0$ and $p_3>0$ (which means that $S(p_0, p_1, \cdots, p_4)=4$), we take up a sufficient condition that still guarantees the validity of (1-11). In the general N case, however, such an attempt would cause us difficulties.

§2. Fractional Brownian motion and fractional calculus

This section is devoted to a study of stochastic Ito-Volterra equations for self-similar Gaussian processes which are derived from the fractional Brownian motion $B_h(t)$ by using fractional calculus. In order to find a stochastic equation for $B_h(t)$, the main difficulty comes from that fractional nature $\mathsf{E}[(dB_h(t))^2]=(dt)^h$ mentioned in §1. We therefore use the fractional derivative D_α of order $\alpha=(h-1)/2$, and define a new process $Y(t)$ having this nice property:

(2-1)
$$\mathsf{E}[(dY(t))^2]=\sigma^2 dt+o(dt) \quad (\sigma>0),$$

which is expected to provide us with a stochastic Ito-Volterra equation of $Y(t)$ in terms of a standard Brownian motion $B_0(t)$.

Let $B_h(x)$, $-\infty<x<\infty$, denote a fractional Brownian motion with exponent $h\neq 1$, $0<h<2$ ([7] and [8]). That is, $B_h(x)$ is a centered Gaussian process such that its covariance function $\Gamma_h(x, y)$ is given by

(2-2)
$$\Gamma_h(x, y)=\{|x|^h+|y|^h-|x-y|^h\}/2, \quad -\infty<x,\ y<\infty.$$

We here study two self-similar Gaussian processes $X_{h,i}(t)$ on $[0, \infty)$, which are derived from $B_h(x)$ in the following way:

(2-3)
$$\begin{cases} X_{h,1}(t)=\{B_h(t)-B_h(-t)\}/\sqrt{2}, & t\geqq 0, \\ X_{h,2}(t)=\{B_h(t)+B_h(-t)\}/\sqrt{2}, & t\geqq 0. \end{cases}$$

This pair of processes corresponds to the odd and even parts of the original process $B_h(x)$, and we see that $X_{h,1}(t)$ and $X_{h,2}(t)$ are mutually independent. The covariance function of $X_{h,i}(t)$ becomes

(2-4)
$$\begin{cases} \Gamma_{h,1}(t,\ s)=\{(t+s)^h-(t-s)^h\}/2, & 0\leqq s\leqq t, \\ \Gamma_{h,2}(t,\ s)=(t^h+s^h)-\{(t+s)^h+(t-s)^h\}/2, & 0\leqq s\leqq t, \end{cases}$$

and we calculate the variance of the infinitesimal increment $dX_{h,i}(t)$ for every $t>0$:

(2-5)
$$\begin{aligned} \mathsf{E}[(dX_{h,i}(t))^2] &= (dt)^h-(-1)^i2^{h-1}\{(t+dt)^h+t^h-2(t+dt/2)^h\} \\ &= (dt)^h+O((dt)^2), \quad i=1,\ 2. \end{aligned}$$

As was stated in §1, we know the canonical representation of $X_{h,i}(t)$:

(2-6)
$$X_{h,i}(t)=C_h\int_0^t f_{h,i}(t/u)u^{(h-1)/2}dB_0(u), \quad t\geqq 0,$$

where $C_h=(\sqrt{\pi}h/\Gamma((h+1)/2)\Gamma(1-h/2))^{1/2}$ and the kernel function $f_{h,i}(s)$, $s\geqq 1$, takes the following form ([10]):

(2-7)
$$\begin{cases} f_{h,1}(s)=(s^2-1)^{(h-1)/2}, \\ f_{h,2}(s)=(s^2-1)^{(h-1)/2}s^{-1}+\int_1^s(u^2-1)^{(h-1)/2}u^{-2}du. \end{cases}$$

Now we introduce the fractional derivative D_α of order $\alpha=(h-1)/2$, to define new Gaussian processes

(2-8)
$$Y_{h,i}(t)=D_\alpha X_{h,i}(t), \quad t>0.$$

The explicit form of D_α is given in terms of the fractional integral I_β of the Holmgren-Riemann-Liouville type ([5]):

(2-9)
$$(I_\beta\phi)(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-u)^{\beta-1}\phi(u)du, \quad t>0.$$

Indeed, we have, for $0<h<1$,

(2-10)
$$Y_{h,i}(t)=(I_{-\alpha}X_{h,i})(t)=\frac{1}{\Gamma((1-h)/2)}\int_0^t(t-u)^{-(h+1)/2}X_{h,i}(u)du,$$

and for $1<h<2$,

(2-10')
$$\begin{aligned} Y_{h,i}(t) &= \frac{d}{dt}(I_{1-\alpha}X_{h,i})(t) \\ &= \frac{1}{\Gamma((3-h)/2)}\frac{d}{dt}[\int_0^t(t-u)^{(1-h)/2}X_{h,i}(u)du]. \end{aligned}$$

By (2-6) and (2-10), for $0<h<1$ we can write

$$Y_{h,i}(t)=C_h\int_0^t\{\frac{v^{(h-1)/2}}{\Gamma((1-h)/2)}\int_v^t(t-u)^{-(h+1)/2}f_{h,i}(u/v)du\}dB_0(v).$$

Then, putting $p=(u-v)/(t-v)\in(0,\ 1)$ and $s=t/v\in(1,\ \infty)$, the above integral in

$\{\ \}$ becomes

$$\frac{v^{(h-1)/2}}{\Gamma((1-h)/2)}(t-v)^{(1-h)/2}\int_0^1(1-p)^{-(h+1)/2}f_{h,i}(1-p+pt/v)dp$$
$$=\frac{(s-1)^{(1-h)/2}}{\Gamma((1-h)/2)}\int_0^1(1-p)^{-(h+1)/2}f_{h,i}(1-p+ps)dp,$$

which we denote simply by $g_{h,i}(s)$.

By (2-7), we compute $g_{h,i}(s)$ as follows:

(2-11)
$$\begin{cases} g_{h,1}(s)=\int_0^1\{(2-p+ps)p\}^{(h-1)/2}(1-p)^{-(1+h)/2}dp/\Gamma((1-h)/2) \\ g_{h,2}(s)=\int_0^1\{(2-p+ps)p\}^{(h-1)/2}(1-p+ps)^{-1}(1-p)^{-(h+1)/2}dp/\Gamma((1-h)/2) \end{cases}$$
$$+(s-1)\int_0^1\{(2-p+ps)p\}^{(h-1)/2}(1-p+ps)^{-2}(1-p)^{(1-h)/2}dp/\Gamma((3-h)/2).$$

We thus arrive at the following canonical representation of $Y_{h,i}(t)$ in the first case $0<h<1$:

(2-12)
$$Y_{h,i}(t)=C_h\int_0^t g_{h,i}(t/u)dB_0(u).$$

Here, the kernal function $g_{h,i}(s)$ was given by (2-11).

We can similarly proceed to the canonical representation of $Y_{h,i}(t)$ in the second case $1<h<2$. By (2-6) and (2-10'), we are led to set

$$g_{h,i}(s)=\frac{1}{\Gamma((3-h)/2)}\frac{d}{ds}[(s-1)^{(3-h)/2}\int_0^1(1-p)^{(1-h)/2}f_{h,i}(1-p+ps)dp],$$

and we obtain (2-12) also in this case. The exat form of the kernel $g_{h,i}(s)$ follows easily from (2-7):

(2-11')
$$\begin{cases} g_{h,1}(s)=\int_0^1\{(2-p+ps)p(1-p)^{-1}\}^{(h-1)/2}dp/\Gamma((3-h)/2) \\ \quad -(s-1)\int_0^1(2-p+ps)^{(h-3)/2}p^{(h+1)/2}(1-p)^{(1-h)/2}dp/\Gamma((1-h)/2), \\ g_{h,2}(s)=\int_0^1\{(1+p(s-1)^{(h-1)/2}\}\{(2-p+ps)p(1-p)^{-1}\}^{(h-1)/2} \end{cases}$$
$$(1-p+ps)^{-2}dp/\Gamma((3-h)/2)-(s-1)\int_0^1(2-p+ps)^{(h-3)/2}p^{(h+2)/2}$$
$$(1-p)^{(1-h)/2}(1-p+ps)^{-1}dp/\Gamma((1-h)/2).$$

Now, we are in a position to derive stochastic equations for our self-similar processes $Y_{h,i}(t)$, $i=1,2$. Noting that the two expressions (2-11) and (2-11') yield the same value $g_{h,1}(1)=g_{h,2}(1)=2^{(h-1)/2}\Gamma((h+1)/2)$, we can set

(2-13)
$$\sigma_h=C_h g_{h,i}(1)=(2^{h-1}\sqrt{\pi}\,h\Gamma((h+1)/2)/\Gamma(1-h/2))^{1/2}$$
$$=(2^h\sin(\pi h/2)\Gamma((h+1)/2)\Gamma(h/2+1)/\sqrt{\pi})^{1/2}>0.$$

Then the canonical representation (2-12) enables us to write

$$(2\text{-}14)\qquad dY_{h,i}(t)=\{-\int_0^t j_{h,i}(t/u)u^{-1}dB_0(u)\}dt+\sigma_h dB_0(t),\quad t>0,$$

where we put

$$j_{h,i}(s)=-C_h\frac{d}{ds}g_{h,i}(s),\quad s>1.$$

The explicit forms of $j_{h,i}(s)$ follow immediately from (2-11) and (2-11'), although we do not go into details here.

We can summerize the above discussions as follows:

Proposition 1. The self-similar Gaussian process $Y_{h,i}(t)$ defined by (2-8) admits the canonical representation (2-12) as well as the expression (2-14) for the infinitesimal increment $dY_{h,i}(t)$.

We are ready to face the main problem in this section. Under some condition that guarantees the existence of a solution of the resolvent equation

$$(2\text{-}15)\qquad k(t,u)+j_{h,i}(t/u)(\sigma_h u)^{-1}=\int_u^t j_{h,i}(t/v)(\sigma_h v)^{-1}k(v,u)dv,$$

for every $0<u<t$, we reach the following stochastic Ito-Volterra equation as an equivalent equation to (2-14) above:

$$(2\text{-}16)\qquad dY_{h,i}(t)=\{\int_0^t k(t,u)dY_{h,i}(u)\}dt+\sigma_h dB_0(t),\quad t>0,$$

where the kernel $k(t,u)$ is taken to be a soluion of (2-15).

In the present paper, we must be contented with finding a solution of (2-15) only in the case $0<h<1$ and $i=1$; The remaining cases as well as the multiparameter cases studied in [10] are planned to be discussed in a forthcoming paper.

In the case $0<h<1$, we obtain

$$j_{h,1}(s)=\{C_h(1-h)/2\Gamma((1-h)/2)\}\int_0^1(2-p+ps)^{(h-3)/2}\{p(1-p)^{-1}\}^{(h+1)/2}dp$$
$$=\sigma_h d_h\int_0^1(2-p+ps)^{(h-3)/2}\{p(1-p)^{-1}\}^{(h+1)/2}dp>0,$$

which is decreasing with $\lim_{s\to\infty} j_{h,1}(s)=0$. Here we simply set

$$d_h=[z2^z\sin\pi z]_{|z=(1-h)/2}/\pi.$$

In order to show a key estimate (1-6) in §1 for this continuous kernel

$$(2\text{-}17)\qquad r(t,u)=j_{h,1}(t/u)(\sigma_h u)^{-1}=-C_h(\sigma_h u)^{-1}\frac{d}{ds}g_{h,1}(s)_{|s=t/u},$$

we observe that $g_{h,1}(s)$ is positive and decreasing with $\lim_{s\to\infty} g_{h,1}(s)=0$. Then we have

$$\int_0^t |r(t, u)|du=\int_0^1 r(1, u)du=-C_h\sigma_h^{-1}\int_1^\infty \{\frac{d}{ds}g_{h,1}(s)\}s^{-1}ds$$
$$=1-(g_{h,1}(1))^{-1}\int_1^\infty g_{h,1}(s)s^{-2}ds<1,$$

which implies that an operator T_r defined by

$$(T_r\phi)(t)=\int_0^t r(t, u)\phi(u)du \quad \text{for any } \phi\in L^\infty((0, \infty)), \tag{2-18}$$

is a contraction operator on $L^\infty((0, \infty))$. Hence the resolvent equation (2-15) can be solved in a familiar manner (cf. [11] and [19]). Namely, we get the following for each (t, u) such that $0<u<t$:

$$k(t, u)=-\sum_{n=0}^{\infty}[(T_r)^n r(\cdot, u)](t)=-\sum_{n=1}^{\infty} r^{(n)}(t, u), \tag{2-19}$$

with $r^{(1)}=r(t, u)$ and for $n\geqq 2$,

$$r^{(n)}(t, u)=\int\cdots\int_{\{u<v_{n-1}<\cdots<v_1<t\}} r(t, v_1)r(v_1, v_2)\cdots r(v_{n-1}, u)dv_{n-1}\cdots dv_1. \tag{2-20}$$

We have thus proved the main result in this section.

Theorem 2. The self-similar Gaussian process $Y_{h,1}(t)$ with exponent $0<h<1$, admits the stochastic Ito-Volterra equation (2-16) with kernel $k(t, u)=-\sum_{n=1}^{\infty} r^{(n)}(t, u)$, where $r^{(n)}(t, u)$ are the iterated kernels of (2-17).

§ 3. Generalization of T-positivity and KMO-Langevin equation

In this section we transform those self-similar processes $X_{h,i}(t)$, $t\geqq 0$, into stationary processes $Z_{h,i}(x)$, $x\in \mathrm{R}$, to see a notable structure of their spectral density functions, which leads us to investigate a generalization of T-positivity. The KMO-Langevin equation for each N-ple Markov stationary Gaussian process $Z(x)$ discussed here takes this form:

$$dZ(x)=\sigma dW(x)-dx[q_0Z(x)+\int_{-\infty}^{x}\{\sum_{j=1}^{N-1} q_j\exp[-b_j(x-y)]\}dZ(y)], \tag{3-1}$$

where σ, $q_0>0$, $0<b_1<b_2<\cdots<b_{N-1}<\infty$, $q_j\neq 0$, and $W(x)$, $x\in\mathrm{R}$, is a Wiener process (standard Brownian motion on R). Under the assumption of T-positivity (cf. [15]~[17], and also [2]), we have $q_j>0$ for all $j=1, 2,\cdots,N-1$. In the present paper, however, we would rather be interested in stationary processes that are not T-positive but satisfy the assumption (SC) stated in § 1, for which $q_j<0$ holds for some j. In the case $1<h<2$, the approximate processes $Z_{h,i}^{(N)}(x)$, $N=2, 3, \cdots$, discussed in Remark 2 constitute a good example of such processes.

Let us begin with studying a couple of stationary Gaussian processes that are related to the fractional Brownian motion. From the self-similar process $X_{h,i}(t)$ defined in the previous section, we change parameter to define a stationary Gaussian process

(3-2) $$Z_{h,i}(x)=e^{-hx}X_{h,i}(e^{2x})/\sqrt{\Gamma_{h,i}(1,\ 1)},\quad x\in \mathsf{R},$$

and calculate its correlation function

$$\gamma_{h,i}(|x|)=\mathsf{E}[Z_{h,i}(x)Z_{h,i}(0)]=e^{-hx}\,\Gamma_{h,i}(e^{2x},\ 1)/\Gamma_{h,i}(1,\ 1)$$

Explicitly we get, for $t\geqq 0$,

(3-3) $$\begin{aligned}\gamma_{h,1}(t)&=(\cosh t)^h-(\sinh t)^h\\ &=2^{1-h}\{h\exp[-(2-h)t]+\sum_{j=1}^{\infty}\tbinom{h}{2j+1}\exp[-(4j+2-h)t]\},\end{aligned}$$

and

(3-3') $$\begin{aligned}\gamma_{h,2}(t)&=(2^{2-h}-1)^{-1}\{2^{2-h}\cosh(ht)-(\cosh t)^h-(\sinh t)^h\}\\ &=(2-2^{h-1})^{-1}\{\exp[-ht]-\sum_{j=1}^{\infty}\tbinom{h}{2j}\exp[-(4j-h)t]\}.\end{aligned}$$

Note that in the above expansions, both $\binom{h}{2j+1}>0$ and $\binom{h}{2j}<0$ hold for $0<h<1$ and $j=1,\ 2,\ \cdots$, which tells us that the processes $Z_{h,1}(x)$ and $Z_{h,2}(x)$ are both T-positive when $0<h<1$. On the other hand, when $0<h<2$, we have the opposite inequalities for all j.

The spectral density function $\phi_{h,i}(\lambda)$ and the canonical representation of $Z_{h,i}(x)$ were obtained in [10]:

(3-4) $$\begin{aligned}\phi_{h,1}(\lambda)&=(2^{h-1}\pi)^{-1}\{h(2-h)(\lambda^2+(2-h)^2)^{-1}\\ &\qquad+\sum_{j=1}^{\infty}\tbinom{h}{2j+1}(4j+2-h)(\lambda^2+(4j+2-h)^2)^{-1}\}\\ &=h(2^{h+2}\pi)^{-1}|B((i\lambda+2-h)/4,\ (h+1)/2)|^2/B((h+1)/2,\ 1-h/2)\end{aligned}$$

and

(3-4') $$\begin{aligned}\phi_{h,2}(\lambda)&=\{(2-2^{h-1})\pi\}^{-1}\{h(\lambda^2+h^2)^{-1}-\sum_{j=1}^{\infty}\tbinom{h}{2j}(4j-h)(\lambda^2+(4j-h)^2)^{-1}\}\\ &=h\{(2^4-2^{h+2})\pi\}^{-1}(\lambda^2+(h+2)^2)(\lambda^2+h^2)^{-1}\\ &\qquad|B((i\lambda+4-h)/4,\ (h+1)/2)|^2/B((h+1)/2,\ 1-h/2),\end{aligned}$$

$B(p,\ q)$ being the beta function. The canonical representation is then written in the form

(3-5) $$Z_{h,i}(x)=C_h(\Gamma_{h,i}(1,\ 1))^{-1/2}\int_{-\infty}^{x}F_{h,i}(x-y)\,dW(y),$$

and the canonical kernel $F_{h,i}(t)$, $t>0$, is expressible as follows:

(3-6) $$\begin{aligned}F_{h,1}(t)&=\exp[-(2-h)t]\{1-\exp[-4t]\}^{(h-1)/2}\\ &=\sum_{k=0}^{\infty}\tbinom{(h-1)/2}{k}(-1)^k\exp[-(4k+2-h)t],\end{aligned}$$

and

(3-6')
$$F_{h,2}(t) = \exp[-(4-h)t]\{1-\exp[-4t]\}^{(h-1)/2} + 2\exp[-ht]\int_0^t \exp[-(4-2h)u]\{1-\exp[-4u]\}^{(h-1)/2}du.$$

Remark 1. It deserves to mention that the self-similar process

(3-7)
$$X_{h,0}(t) = \{X_{h,1}(t) + X_{h,2}(t)\}/\sqrt{2}, \quad t \geqq 0$$

and the associated stationary process

(3-2')
$$Z_{h,0}(x) = e^{-hx}X_{h,0}(e^{2x}), \quad x \in \mathrm{R}$$

have the same characters as $X_{h,i}(t)$ and $Z_{h,i}(x)$ $(i=1, 2)$, respectively. This process $X_{h,0}(t)$ is nothing but the restriction of parameter of $B_h(x)$, and its canonical representation was studied in [9] (see also [11]). Here it should be noted that the correlation function $\gamma_{h,0}(|x-y|)$ of $Z_{h,0}(x)$ is given by

(3-3")
$$\gamma_{h,0}(t) = (\cosh ht) - 2^{h-1}(\sinh t)^h = 2^{-1}\{\exp[-ht] - \sum_{j=1}^{\infty}\binom{h}{j}(-1)^j\exp[-(2j-h)t]\},$$

which is T-positive when $0<h<1$, but not so when $1<h<2$. In addition, we have

(3-5')
$$Z_{h,0}(x) = 2^{(h-2)/2}|h-1|C_h\int_{-\infty}^{x} F_{h,0}(x-y)dW(y),$$

and the canonical kernel $F_{h,0}(x)$ takes the analogous form to (3-6): for $0<h<1$,

$$F_{h,0}(t) = e^{-ht}\{\int_0^{e^{-2t}} u^{-h}(1-u)^{(h-3)/2}du + B(1-h, (h+1)/2)\}$$
$$= B(1-h, (h+1)/2)\exp[-ht] - \sum_{j=1}^{\infty}\binom{(h-3)/2}{j-1}(-1)^j(j-h)^{-1}\exp[-(2j-h)t],$$

and for $1<h<2$,

$$F_{h,0}(t) = e^{-ht}\int_{e^{-2t}}^{1} u^{-h}(1-u)^{(h-3)/2}du.$$

Inspired by these results on the processes $Z_{h,i}(x)$ $(i=0, 1, 2)$ that are transformed from a single fractional Brownian motion $B_h(x)$, we are now going to consider a general stationary Gaussian process $Z(x)$ having a correlation function $\gamma(|x|)$ of the form

(3-8)
$$\gamma(t) = \sum_{i=1}^{N} p_i\exp[-a_i t], \quad t \geqq 0,$$

where $0<a_1<a_2<\cdots<a_N<\infty$ and $p_i \neq 0$ for any i. If all p_i are positive, the process $Z(x)$ becomes T-positive. Our processes $Z_{h,i}(x)$ $(i=1, 2$ and $0)$ having those correlation functions (3-3), (3-3') and (3-3"), actually correspond to the infinite case $N=\infty$, but a full discussion would need a lot of space. Hence in the present paper, we restrict ourselves to the finite case $N<\infty$ in which $Z(x)$ turns out to be an N-ple Markov Gaussian process.

The purpose of the discussions below is to obtain a stochastic Ito-Volterra equation (3-1) of the KMO-Langevin type beyond that familiar framework of T-positivity. We impose a normalization condition like this:

(3-8') $$\sum_{i=1}^{N} p_i=1,\ \sum_{i=1}^{N} p_i a_i=\sigma^2/2>0, \text{ and } \sum_{i=1}^{N} p_i/a_i=\tau^2>0,$$

which yields $\gamma(0)=1$, $\frac{d}{dt}\gamma(0+)=-\sigma^2/2<0$, and $\int_0^\infty \gamma(t)dt=\tau^2>0$. As a direct consequence of (3-8'), we should observe that

$$\mathsf{E}[(dZ(x))^2]=2(\gamma(0)-\gamma(dx))=\sigma^2 dx+o(dx).$$

The spectral density function $\phi(\lambda)$ of $Z(x)$ then becomes

(3-9) $$\phi(\lambda)=\pi^{-1}\sum_{i=1}^{N} p_i a_i(\lambda^2+a_i^2)^{-1},$$

which is expressible as

(3-9') $$\phi(\lambda)=\sigma^2 h(\lambda^2)/2\pi\prod_{i=1}^{N}(\lambda^2+a_i^2),$$

with

$$h(x)=2\sigma^{-2}\sum_{i=1}^{N} p_i a_i \prod_{\substack{j=1\\ j\neq i}}^{N}(x+a_j^2)=x^{N-1}+c_1x^{N-2}+\cdots+c_{N-1}$$

being a polynomial of degree $N-1$.

Now, our task is to prove this assertion (see Theorem 3 below): if the sequence $\{p_i\}_{i=1}^N$ satisfies the assumption (SC) stated in §1, then the above polynomial $h(x)$ has $N-1$ distinct roots $-b_j^2$ with $0<b_1<b_2<\cdots<b_{N-1}<\infty$. This gives us another useful expression revealing the positiveness of $\phi(\lambda)$ for real λ:

(3-9") $$\phi(\lambda)=\sigma^2\prod_{j=1}^{N}(\lambda^2+b_j^2)/2\pi\prod_{i=1}^{N}(\lambda^2+a_i^2).$$

In addition, when a seuence $\{a_i\}_{i=1}^N$ is regarded as fixed, the following relation combining two sequences $\{b_j\}_{j=1}^{N-1}$ and $\{p_i\}_{i=1}^N$ with $\sum_{i=1}^{N} p_i=1$ is valid:

(3-10) $$\prod_{j=1}^{N-1}(b_j^2-a_i^2)=2\sigma^{-2}a_i p_i\prod_{\substack{k=1\\ k\neq i}}^{N}(a_k^2-a_i^2),\quad 1\leq i\leq N.$$

It follows from (3-9") that the corresponding outer function $c(\lambda)$ such that $|c(\lambda)|^2=2\pi\phi(\lambda)$, takes the following form:

(3-11) $$c(\lambda)=\sigma\prod_{j=1}^{N-1}(i\lambda+b_j)/\prod_{k=1}^{N}(i\lambda+a_k)$$

The decomposition (3-11) of a rational function $c(\lambda)$ in terms of its roots $\{ib_j\}_{j=1}^{N-1}$ and poles $\{ia_k\}_{k=1}^{N}$ then enables us to write

(3-11') $$c(\lambda)=\sigma\{(i\lambda+q_0)+i\lambda\sum_{j=1}^{N-1}q_j(i\lambda+b_j)^{-1}\}^{-1},$$

and

(3-11") $$c(\lambda)=\sigma\sum_{k=1}^{N}r_k(i\lambda+a_k)^{-1},$$

where we have

(3-12) $$q_0=\prod_{i=1}^{N}a_i\Big/\prod_{j=1}^{N-1}b_j,\ q_j=-\prod_{i=1}^{N}(a_i-b_j)\Big/b_j\prod_{\substack{k=1\\k\neq j}}^{N-1}(b_k-b_j),\ 1\leqq j\leqq N-1,$$

and

(3-12') $$r_k=\prod_{j=1}^{N-1}(b_j-a_k)\Big/\prod_{\substack{i=1\\i\neq k}}^{N}(a_i-a_k),\quad 1\leqq k\leqq N.$$

As is well known (see [15]~[17] and also [2], Chapter 5), such neat expressions (3-11)~(3-11") contain all information we need. In fact, the time evolution of $Z(x)$ can be described by the KMO-Langevin equation (3-1), and we obtain the canonical representation

(3-13) $$Z(x)=\int_{-\infty}^{x}\{\sum_{i=1}^{N}r_i\exp[-a_i(x-y)]\}\sigma dW(y).$$

Furthermore, this expression (3-13) yields a relation combining two sequences $\{r_i\}_{i=1}^{N}$ and $\{p_i\}_{i=1}^{N}$, which is given by

(3-14) $$\sum_{k=1}^{N}r_ir_k/(a_i+a_k)=p_i\sigma^{-2},\quad 1\leqq i\leqq N.$$

Now we are ready to state and prove our main result, which includes the above-mentioned assertion for the polynomial $h(x)$ in (3-9').

Theorem 3. (i) Suppose that a function $\gamma(t)$ of the form (3-8) satisfies the condition (SC) of sign changes of $\{p_i\}_{i=1}^{N}$ as well as the normalization condition (3-8'). Then $\gamma(|x-y|)$ is a correlation function of a certain stationary Gaussian process, which has a spectral density function of the form (3-9") with $0<b_1<b_2<\cdots<b_{N-1}<\infty$ given by (3-10).

(ii) The stationary Gaussian process $Z(x)$ arising in (i) admits the KMO-Langevin equation (3-1) and the canonical representation (3-13), where all coefficients required are given by (3-12) and (3-12').

Proof. It remains to prove (i). Putting $\lambda=i\sqrt{t}$, consider the function

$$\phi(t)=\pi\phi(i\sqrt{t})=\sum_{i=1}^{N}p_ia_i(-t+a_i^2)^{-1}=\sigma^2h(-t)/2\prod_{i=1}^{N}(-t+a_i^2),\quad 0\leqq t<\infty.$$

Then we observe that

$$\phi(0)=\sum_{i=1}^{N} p_i a_i^{-1}=\tau^2>0, \quad \phi(a_i^2\mp)=\pm(\text{sign } p_i)\infty, \quad i=1, \cdots, N,$$

and

$$\lim_{t\to\infty} t\phi(t)=-\lim_{t\to\infty}\sum_{i=1}^{N} p_i a_i\{1+a_i^2/(t-a_i^2)\}=-\sigma^2/2<0.$$

Here it should be noted that our convention $p_0=p_{N+1}=-1$ mentioned in §1 and corresponding to the ends $a_0=0$, $a_{N+1}=\infty$ of the interval $[0, \infty)$, turns out to be consistent with the above sign rule of $\phi(t)$ at every interior point a_i^2.

Now, taking into account the sign changes of $\phi(t)$ on $[0, \infty)$, we can find at least one root in the i-th subinterval (a_i^2, a_{i+1}^2) if $p_i p_{i+1}>0$, $i=0, 1, \cdots, N$. Our condition (SC) then tells us that the number $\#\{i;\ p_i p_{i+1}>0\}$ is just equal to $N-1$ (=the degree of the polynomial $h(x)$) which concludes that $\phi(t)$ has exactly $N-1$ distinct roots b_j^2, $1\leqq j\leqq N-1$, and hence the expression (3-9") follows. We have thus completed the proof of Theorem 3.

Remark 2. From the expansions (3-3)~(3-3") of $\gamma_{h,i}(t)$, those stationary Gaussian processes $Z_{h,i}(x)(i=1, 2, 0)$ naturally possess their approximate processes $Z_{h,i}^{(N)}(x)$, $N=2, 3, \cdots$. That is, the correlation function $\gamma_{h,i}^{(N)}(t)$ of each $Z_{h,i}^{(N)}(x)$ is defined in the form (3-8) with

$$(3\text{-}15)\quad \begin{cases} a_j=4j-2-h, \quad p_j=2^{1-h}\binom{h}{2j-1}, \quad 1\leqq j\leqq N, \quad \text{for } i=1, \\ a_1=h, \quad p_1=(2-2^{h-1})^{-1}, \quad a_j=4(j-1)-h, \quad p_j=-(2-2^{h-1})^{-1}\binom{h}{2j-2}, \quad 2\leqq j\leqq N, \\ \hfill \text{for } i=2, \\ a_1=h, \quad p_1=2^{-1}, \quad a_j=2(j-1)-h, \quad p_j=2^{-1}(-1)^j\binom{h}{j-1}, \quad 2\leqq j\leqq N, \quad \text{for } i=0, \end{cases}$$

which satisfies the assumptions required in Theorem 3. For these processes $Z_{h,i}^{(N)}(x)$, we can therefore write the KMO-Langevin equations of the form (3-1) by using these coefficients $\{\sigma_{h,i}^{(N)}, q_{0,h,i}^{(N)}, q_{j,h,i}^{(N)}, b_{j,h,i}^{(N)}, 1\leqq j\leqq N-1\}$. Then we wish to observe the asymptotic behavior of these coefficients as N goes to ∞, which seems to be difficult. Here we only note the following result on $\{b_{j,h,i}^{(N)}\}_{j=1}^{N-1}$:

(i) In the case $0<h<1$: $a_j<b_{j,h,i}^{(N)}<a_{j+1}$ for all $j=1, 2, \cdots$;

(ii) In the case $1<h<2$: $a_2<b_{1,h,i}^{(N)}<a_3$ $(i=1, 2)$, $a_1<b_{1,h,0}^{(N)}<a_2$,

and $a_{j+1}<b_{j,h,i}^{(N)}<a_{j+2}$ for all $j=2, 3, \cdots$.

Now, we come to the final topic. In the smallest case $N=2$, the condition (SC) covers all possibilities: 1) $p_1, p_2>0$, 2) $p_1<0$, $p_2>0$, and 3) $p_1>0$, $p_2<0$, which imply that, respectively, 1) $a_1 < b_1 < a_2$ with $q_1 > 0$, 2) $b_1 < a_1 < a_2$ with $q_1 < 0$, and 3) $a_1 < a_2 < b_1$ with $q_1 < 0$. In the next case $N=3$, however, there is one possibility ($p_1 > 0$, $p_2 < 0$ and $p_3 > 0$) that escapes from our restriction (SC).

Proposition 4. Suppose that a function $\gamma(t)$ of the form (3-8) with $N=3$, $p_1 > 0$, $p_3 > 0$ and $p_1 + p_3 > 1$ satisfies the following inequality

$$\rho = \sum_{i=1}^{3} a_i^2/2 - \sum_{i=1}^{3} p_i a_i^3/\sigma^2 > \sqrt{2}\, a_1 a_2 a_3 \tau/\sigma \tag{3-16}$$

as well as the condition (3-8'). Then $\gamma(|x-y|)$ is a correlation function having a spectral density function $\phi(\lambda)$ of the form (3-9"). Furthermore, we have

$$a_i < b_1 < b_2 < a_{i+1} \text{ if } a_i^2 < \rho < a_{i+1}^2, \quad i = 0,\ 1,\ 2,\ 3\ (a_0 = 0,\ a_4 = \infty).$$

Proof. The polynomial $h(x)$ appeared in (3-9') is easily calculated as follows:

$$h(x) = x^2 + 2\rho x + 2(a_1 a_2 a_3 \tau/\sigma)^2,$$

which has the discriminant $D = \rho^2 - 2(a_1 a_2 a_3 \tau/\sigma)^2 > 0$. We can therefore write

$$h(x) = (x + b_1^2)(x + b_2^2),\ b_j^2 = \rho + (-1)^j \sqrt{D}, \quad j = 1,\ 2.$$

Recalling the rule of sign changes described in the proof of Theorem 3, we see that the roots b_1^2, b_2^2 of $\phi(t)$, as well as their middle point ρ, lie in the same interval (a_i^2, a_{i+1}^2) under the present conditions. This completes the proof of Proposition 4.

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