Some Representations of Fractional Brownian Motions and Wavelet Transforms

Some Representations of Fractional Brownian Motions and Wavelet Transforms

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Abstract: For fractional Brownian motions with exponent h ($0< h <2$), we give some representations of a new type which have a connection with the continuous wavelet transform.

Key words: Fractional Brownian motion, white noise (Gaussian random measure), continuous wavelet transform

§ 1. Introduction and results in the one-dimensional case

The purpose of this short note is to establish some representations of a new type for a fractional Brownian motion $\{B_h(x);\ x\in \mathsf{R}^d\}$, which has the covariance function

$$\Gamma_h(x,\ y)=\{|x|^h+|y|^h-|x-y|^h\}/2 \tag{1}$$

where $0<h<2$ and $|x-y|$ denotes the Euclidean distance between x and y. In this section we prove our result (Theorem 1) in the case $d=1$; in the next section, we are going to discuss an extension to the multi-dimensional cases $d\geqq 2$ (see Theorem 2).

A usual representation of a centered Gaussian system $\{X(t):\ t\in T\}$ with covariance function $\Gamma(t,\ s)$ is constructed as follows (cf. [3] and [6]):

i) First choose a suitable measure space $(S,\ \mu)$ to find a family of real-valued functions $\{f_t(u);\ t\in T\}$ in the Hilbert space $\mathsf{L}^2\ (S,\ \mu)$ such that

$$\Gamma(t,\ s)=\int_S f_t(u)f_s(u)d\mu(u) \quad \text{for any } t,\ s\in T. \tag{2}$$

Such an expression (2) was called a *model* of the covariance function in [6].

ii) Next take a white noise (Gaussian random measure) $\{W(A);\ A\subset S\}$ based on the measure space $(S,\ \mu)$. Then we get

$$X(t)=\int_S f_t(u)\, dW(u). \tag{3}$$

Now for $T=\mathsf{R}$, let $\{B_h(t);\ t\in\mathsf{R}\}$ be the fractional Brownian motion with exponent h. The Brownian motion corresponds to the case $h=1$, in which the increments $dB_1(t)=B_1(t+dt)-B_1(t)$, $t\in\mathsf{R}$, are mutually independent. On the other hand, the correlation function of $B_h(t)$ has the persistency when $h>1$ and the antipersistency when $h<1$ ([1]). As was discussed so far in the theory of Gaussian processes, we take the Lebesgue measure space (R, du) as a natural choice of (S, μ), and from the property of stationary increments we are led to set the following form:

$$f_t(u)=K(t-u)-K(-u). \tag{4}$$

It turns out that the scaling property of $B_h(t)$ imposes a strong condition on the above kernel function $K(t)$ (cf. [7], [9] and [13]):

$$K(ct)=c^{(h-1)/2}K(t), \text{ for every } c>0, \tag{5}$$

which yields the following well-known expression of $B_h(t)$:

$$B_h(t)=\int_{-\infty}^{\infty}\{K_h(t-u)-K_h(-u)\}dW(u), \tag{6}$$

$$K_h(t)=c_+|t|^{(h-1)/2}+c_-|t|^{(h-1)/2}\mathrm{sgn}(t), \tag{6'}$$

where (c_+, c_-) is a pair of normalizing positive constants and $\mathrm{sgn}(t)=+1$ for $t>0$, $=-1$ for $t<0$.

By virtue of a recent result ([2]) on noncanonical representations of Brownian motion, one can construct a class of isometries $J_{\{g_1,\cdots,g_N\}}$ defined on $L^2(\mathsf{R}, du)$ such that the range of $J_{\{g_1,\cdots,g_N\}}$ is equal to

$$[g_1, \cdots, g_N]^{\perp}=\{f\in L^2(\mathsf{R}, du);\ \int_{-\infty}^{\infty}f(u)g_i(u)du=0 \quad \textit{for all } i,\ 1\leq i\leq N\}.$$

Applying such an isometry to (6), we can derive a different representation

$$B_h(t)=\int_{-\infty}^{\infty}[J_{\{g_1,\cdots,g_N\}}(K_h(t-\bullet)-K_h(-\bullet))](u)\ dW(u). \tag{7}$$

We are now in a position to discuss our result on representations of $B_h(t)$ related to the continuous (also called integral) wavelet transform T_Ψ (see [4] and [5]). In contrast with the previous measure space (R, du), our present choice of (S, μ) becomes $S=(0, \infty)\times\mathsf{R}$, $d\mu(a, b)=a^{-2}da\, db$, where $a\in(0, \infty)$ and $b\in\mathsf{R}$ indicate dilation $x\to ax$ and translation $x\to x+b$ on R, respectively.

Now, let $\psi(x)\in L^2(\mathsf{R})$ be an arbitrary real-valued function satisfying

$$2\pi\int_0^{\infty}|\widetilde{\psi}(\xi)|^2|\xi|^{-1}d\xi=2\pi\int_{-\infty}^{0}|\widetilde{\psi}(\xi)|^2|\xi|^{-1}d\xi=C_\psi<\infty, \tag{8}$$

the Fourier transform of $\psi(x)$ being defined by

$$\tilde{\psi}(\xi)=\int_{-\infty}^{\infty} e^{-i\xi x}\psi(x)dx.$$

Such a mother wavelet $\psi(x)$ brings her daughter wavelets

(9) $$\psi_{a,b}(x)=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right),\ (a,\ b)\in(0,\ \infty)\times\mathsf{R},$$

by which an integral transform from $\mathsf{L}^2(\mathsf{R},\ du)$ into $\mathsf{L}^2(S,\ \mu)$ can be defined as follows:

(10) $$(T_\psi f)(a,\ b)=\int_{-\infty}^{\infty}\psi_{a,b}(x)f(x)dx.$$

The dual of this continuous wavelet transform T_ψ admits the expression

(11) $$(T_\psi^* h)(x)=\int_0^\infty\int_{-\infty}^{\infty}\psi_{a,b}(x)h(a,\ b)a^{-2}da\,db,\ h(a,\ b)\in\mathsf{L}^2(S,\ \mu).$$

Thanks to the strong admissibility condition (8), we have

(12) $$\int_0^\infty\int_{-\infty}^{\infty}(T_\psi f_1)(a,\ b)(T_\psi f_2)(a,\ b)a^{-2}da\,db=C_\psi\int_{-\infty}^{\infty}f_1(x)f_2(x)dx,$$

which implies a useful inversion formula $\frac{1}{C_\psi}T_\psi^*\cdot T_\psi=I$ (identity) (cf. [4], [5] and [14]).

We are ready to state our result on the fractional Brownian motion $B_h(t)$. Take a Gaussian random measure $\{W(A):\ A\subset(0,\ \infty)\times\mathsf{R}\}$ based on the measure $d\mu(a,\ b)=a^{-2}da\,db$. Then the following key assumption enables us to prove Theorem 1 below:

(13) $$\int_{-\infty}^{\infty}|\tilde{\psi}(\xi)|^2|\xi|^h d\xi=2\int_0^\infty|\tilde{\psi}(\xi)|^2\,\xi^h\,d\xi=D_\psi(h)<\infty.$$

Simple examples of such a ψ should be mentioned here:

(14) $$\psi(x)=\int_0^\infty\cos(x\xi)\xi^\alpha e^{-\beta\xi^2}\,d\xi\quad(\alpha,\ \beta>0).$$

Theorem 1. For any real-valued function $\psi(x)$ satisfying the conditions (8) and (13), the fractional Brownian motion $B_h(t)$ can be expressed in the form

(15) $$B_h(t)=k\int_0^\infty\int_{-\infty}^{\infty}\{\psi_{a,b}(t)-\psi_{a,b}(0)\}a^{(h+1)/2}dW(a,\ b),$$

where k is a normalizing constant given by

(15)' $$k=\{2\Gamma(h+1)\sin(\pi h/2)/D_\psi(h)\}^{1/2}.$$

Proof. Let us begin with computing the variance of $B_h(\pm1)$. Noting that

$$\int_0^\infty a^{-(h+1)}\sin^2 a\,da=\frac{1}{h}\int_0^\infty\frac{\sin 2a}{a^h}da=\frac{\pi 2^{h-2}}{\Gamma(h+1)\sin(\pi h/2)}\quad(0<h<2),$$

we see by (15) and (15)' that

$$E[B_h^2(\pm1)]=k^2\int_0^\infty\int_{-\infty}^{\infty}\left\{\psi\left(\frac{\pm1-b}{a}\right)-\psi\left(-\frac{b}{a}\right)\right\}^2a^{h-2}da\,db$$

$$=k^2\int_0^\infty a^{-(h+1)}da\int_{-\infty}^{\infty}\{\psi(\pm a+u)-\psi(u)\}^2du$$

$$=k^2\int_0^\infty a^{-(h+1)}da\left\{\frac{1}{2\pi}\int_{-\infty}^\infty |e^{\mp ia\xi}-1|^2\,|\tilde{\psi}(\xi)|^2 d\xi\right\}$$

$$=\frac{4k^2}{2\pi}\int_{-\infty}^\infty |\tilde{\psi}(\xi)|^2 d\xi\left\{\int_0^\infty a^{-(h+1)}\sin^2(a\xi/2)da\right\}$$

$$=\frac{2^{2-h}k^2}{2\pi}\int_{-\infty}^\infty |\xi|^h\,|\tilde{\psi}(\xi)|^2 d\xi\int_0^\infty a^{-(h+1)}\sin^2 a\,da$$

$$=k^2D_\psi(h)/2\Gamma(h+1)\sin(\pi h/2)=1.$$

Then it easily follows that

$$E[(B_h(ct))^2]=k^2\int_0^\infty\int_{-\infty}^\infty\left\{\psi\left(\frac{ct-b}{a}\right)-\psi\left(-\frac{b}{a}\right)\right\}^2 a^{h-2}da\,db=c^hE[(B_h(t))^2]$$

for every $c>0$, which tells us that $E[(B_h(t))^2]=|t|^h$. Observing that the variance of the increment

$$B_h(t+s)-B_h(s)=k\int_0^\infty\int_{-\infty}^\infty\{\psi_{a,b}(t)-\psi_{a,b}(0)\}a^{(h+1)/2}dW(a,\ b+s)$$

does not depend on $s\in\mathbf{R}$, we have completed the proof of Theorem 1.

§ 2. An extension to the multi-dimensional cases

This section is devoted to an extension of Theorem 1 to the multi-dimensional case $x\in\mathbf{R}^d(d\geqq 2)$, in which the fractional Brownian motion $B_h(x)$ is a Gaussian random field with covariance function (1). Following Yor [15], we would like to introduce $d\times d$ matrix-valued functions and a corresponding Gaussian random measure.

Now set $S=(0,\ \infty)\times\mathbf{R}^d$, $d\mu(a,\ b)=a^{-(d+1)}da\,db$ with $a\in(0,\ \infty)$ and $b\in\mathbf{R}^d$ indicating dilation and translation on $\mathbf{R}^d$, respectively. We form a Hilbert space $L^2{}_d(S,\ \mu)$ consisting of all $d\times d$ matrix-valued functions $F(a,\ b)=(F_{ij}(a,\ b))_{i,j=1}^d$, each element $F_{ij}(a,\ b)$ being real-valued, such that

$$\|F(a,\ b)\|^2=\int_0^\infty\int_{R^d}\mathrm{Tr}(F(a,\ b)F^t(a,\ b))a^{-(d+1)}da\,db \tag{16}$$

$$=\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}F_{ij}^2(a,\ b)a^{-(d+1)}da\,db<\infty.$$

In order to form a Wiener integral of $F(a,\ b)$, we need a $d\times d$ matrix-valued white noise $\{W(A)=(W_{ij}(A))_{i,j=1}^d;\ A\subset S\}$ based on the measure space $(S,\ \mu)$. That is, each element $W_{ij}(A)$ is a Gaussian random measure with mean 0 and variance $\mu(A)$, and these random measures $W_{ij}(\bullet)(i,\ j=1,\ \cdots,\ d)$ are mutually independent. We then define a Wiener integral

$$\int_0^\infty\int_{R^d}F(a,\ b)dW(a,\ b)=\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}F_{ij}(a,\ b)dW_{ij}(a,\ b),$$

which turns out to be a Gaussian random variable with mean 0 and variance $\|F(a, b)\|^2$.

Let $\Psi(x)=(\Psi_{ij}(x))_{i,j=1}^d$ be an arbitrary function in $L^2_d(\mathbb{R}^d, dx)$ satisfying the following conditions:

(17) $$(2\pi)^d\sum_{i=1}^d\sum_{j=1}^d\int_{R^d}|\widetilde{\Psi}_{ij}(\xi)|^2|\xi|^{-d}d\xi=C_\Psi(d)<\infty,$$

(18) $$\sum_{i=1}^d\sum_{j=1}^d\int_{R^d}|\widetilde{\Psi}_{ij}(\xi)|^2|\xi_1|^h d\xi=D_\Psi(d, h)<\infty \quad (\xi=(\xi_1, \cdots, \xi_d)\in\mathbb{R}^d)$$

and

(19) $$\Psi(Qx)=Q\Psi(x)Q^t, \quad x\in\mathbb{R}^d,$$

for every orthogonal matrix $Q=(Q_{ij})_{i,j=1}^d$. Some examples of such a Ψ should be noted: Taking any real-valued even function $q(t)$ of the form (14), we have

i) $\Psi_1(x)=q(|x|)\,I$ with unit matrix $I=(\delta_{ij})_{i,j=1}^d$;

ii) $\Psi_2(x)=q(|x|)J(x)$ with $J(x)=\left(\frac{x_ix_j}{|x|^2}\right)_{i,j=1}^d$.

Here it deserves to mention that a particular $d\times d$ kernel function $K(x)$ of the form $|x|^{-p}\{I-pJ(x)\}$ $(p=(d-h)/2>0)$ was used in [15] and played a role similar to (6)'.

Now putting

(20) $$\Psi_{a,b}(x)=a^{-d/2}\Psi\left(\frac{x-b}{a}\right)=a^{-d/2}\left(\Psi_{ij}\left(\frac{x-b}{a}\right)\right)_{i,j=1}^d,$$

we are in a position to prove our main result on a representation of $B_h(x)$, $x\in\mathbb{R}^d$.

<u>Theorem 2.</u> For any $d\times d$ matrix-valued function $\Psi(x)=(\Psi_{ij}(x))_{i,j=1}^d$ on $\mathbb{R}^d$ satisfying the conditions (17), (18) and (19) mentioned above, the fractional Brownian motion $B_h(x)$ can be expressed in the following form:

(21) $$B_h(x)=k(d)\int_0^\infty\int_{R^d}\{\Psi_{a,b}(x)-\Psi_{a,b}(0)\}a^{(h+d)/2}dW(a, b)$$
$$=k(d)\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}\{(\Psi_{ij})_{a,b}(x)-(\Psi_{ij})_{a,b}(0)\}a^{(h+d)/2}dW_{ij}(a, b),$$

where $k(d)$ is a normalizing constant given by

(21)' $$k(d)=\{2^d\pi^{d-1}\Gamma(h+1)\sin(\pi h/2)/D_\Psi(d, h)\}^{1/2}.$$

<u>Proof.</u> We first compute the variance of $B_h(e_1)$ for $e_1=(1, 0,\cdots, 0)\in\mathbb{R}^d$:

$$E[B_h^2(e_1)]=k^2(d)\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}\left\{\Psi_{ij}\left(\frac{e_1-b}{a}\right)-\Psi_{ij}\left(-\frac{b}{a}\right)\right\}^2a^h\frac{da\,db}{a^{d+1}}$$
$$=k^2(d)\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}\{\Psi_{ij}(ae_1+u)-\Psi_{ij}(u)\}^2a^{-(h+1)}da\,du$$
$$=k^2(d)\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty a^{-(h+1)}da\left\{\frac{1}{(2\pi)^d}\int_{R^d}|e^{-ia(e_1,\xi)}-1|^2\,|\widetilde{\Psi}_{ij}(\xi)|^2d\xi\right\}$$
$$=k^2(d)\frac{4}{(2\pi)^d}\sum_{i=1}^d\sum_{j=1}^d\int_{R^d}|\widetilde{\Psi}_{ij}(\xi)|^2\,d\xi\int_0^\infty\sin^2(a\xi_1/2)a^{-(h+1)}da$$

$$=k^2(d)\frac{2^{2-h}}{(2\pi)^d}D_\Psi(d,\ h)\int_0^\infty a^{-(h+1)}\sin^2 a\,da=1.$$

The scaling property $E[B_h^2(cx)]=c^h E[B_h^2(x)]$ for every $c>0$, as well as the property of stationary increments immediately follows from the expression (21).

Our final task is to show the orthogonal invariance of (21) by making use of the key condition (19). Namely, for any orthogonal matrix $Q=(Q_{ij})_{i,j=1}^d$, we can write

$$E[B_h^2(Qx)]=k^2(d)\sum_{i=1}^d\sum_{j=1}^d\int_0^\infty\int_{R^d}\left\{\Psi_{i,j}\left(\frac{Qx-b'}{a}\right)-\Psi_{i,j}\left(-\frac{b'}{a}\right)\right\}^2 a^{h-d-1}da\,db'$$

$$=k^2(d)\int_0^\infty\int_{R^d}\mathrm{Tr}\left(\left\{\Psi\left(Q\left(\frac{x-b}{a}\right)\right)-\Psi\left(Q\left(-\frac{b}{a}\right)\right)\right\}\left\{\Psi\left(Q\left(\frac{x-b}{a}\right)\right)-\Psi\left(Q\left(-\frac{b}{a}\right)\right)\right\}^t\right)a^{h-d-1}da\,db$$

$$=k^2(d)\int_0^\infty\int_{R^d}\mathrm{Tr}\left(Q\left\{\Psi\left(\frac{x-b}{a}\right)-\Psi\left(-\frac{b}{a}\right)\right\}Q^t Q\left\{\Psi\left(\frac{x-b}{a}\right)-\Psi\left(-\frac{b}{a}\right)\right\}^t Q^t\right)a^{h-d-1}da\,db$$

$$=k^2(d)\int_0^\infty\int_{R^d}\mathrm{Tr}\left(\left\{\Psi\left(\frac{x-b}{a}\right)-\Psi\left(-\frac{b}{a}\right)\right\}\left\{\Psi\left(\frac{x-b}{a}\right)-\Psi\left(-\frac{b}{a}\right)\right\}^t\right)a^{h-d-1}da\,db=E[B_h^2(x)].$$

We thus get the conclusion that $E[(B_h(x+y)-B_h(y))^2]=|x|^h$, which completes the proof of Theorem 2.

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