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# Method for Analysis of Robustness properties of Calcium channel on the excitatory cellular membrane by H2 control theory.

H, Hirayama, *Y. Okita., *T, Sugiura,
*M, Kimura and **T. Kazui
Department of Public Health Asahikawa medical college.
E mail , hirayama@asahikawa-med.ac.jp
*Shizuoka university. **Hamamatsu medical college.

We have proposed an H2 control principle for evaluating the noise filtering function of the Calcium ion channel on the excitable biological membrane. The multi phase channel properties were characterized by four identical subunits. Each subunit has a voltage sensitive molecule which acts as a gating regulator for channel opening and closing. By applying the H2 control theory, we obtained matrix differential equations including the observer and estimator and two Riccati equations for the observer and state. They minimized the 2 norm of the transfer function from the noise to the out put signal. The concentrations of any channel species per unit membrane area were significantly smoothed under the H2 control. The changes in weighting coefficients induced definite changes in the temporal changes in the species.

Calcium ion channel, Robust H2 control, Noise input, Riccati equation, 2 norm, Transfer function.

# 興奮性生体膜上のカルシウイオンムチャンネルのロバスト特性解析

平山博史、*沖田善光、*杉浦敏文、*木村元彦、**数井輝久

旭川医科大学、旭川市緑が丘東２条１丁目
*静岡大学.　**浜松医科大学

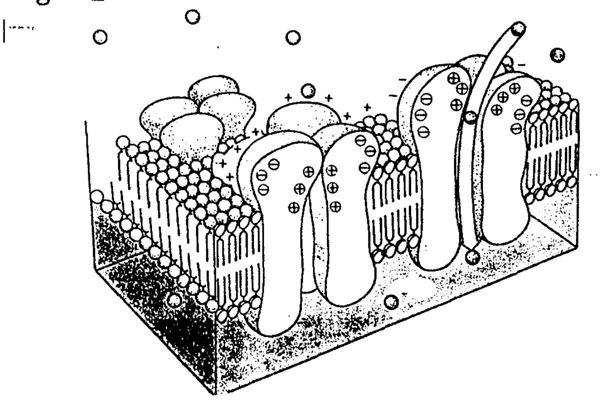
カルシウムイオンチャンネルの雑音フィルター機能を評価する目的でＨ２制御方策を提唱した。　カルシウムイオンチャンネルの多位相遷移は４つのサブユニットにそれぞれ存在す細胞膜電位感受性分子の位置によって開状態、閉状態それぞれ４とうりを設定した。Ｈ２制御理論により、制御器からの出力および制御器からの入力を含めたマトリックス方程式群を導いた。これに対して、雑音入力から信号出力までの閉回路の伝達関数の２ノルムを最少にするべくリッカチ方程式を状態変数と観測器の両者に対して導出した。チャンネルの種々の分子構造状態が単位膜面積に占める量の時間経過はＨ２制御下ではその変動が著しく低減された。システム方程式中のある特定の重み係数を減少させるとその重み係数が関与するチャンネル状態量の時間経過が大きく変化した。

カルシウムイオンチャンネル. ロバスト.　Ｈ２制御. 雑音入力. 信号出力. リッカチ方程式

## 1. Introduction.

Calcium ionic flow into cellular space is gated by Calcium ion specific channel. (Fig 1). It is composed of four identical subunits of protein helix. Each subunit is further consisted of six substructures, s1 ,s2,s3,s4,s5 and s6. s5 and s6 locate at the inner most part which face to the channel pore. s1 ,s2,s3 and s4 locate outer region of the subunit group. Among these six subunits, s4 has the

Fig 1



highest sensitivity to the electrical membrane potential gap across the cell ( Fig 2 ). It behaves as a voltage sensor to coordinate dynamical positional changes of the subunits. This regulates the opening and closing the channel pore.

For gating the calcium ions, there are a lot of bio physical molecules that compete the binding sites with calcium ions. Since they are non specific, they can be regarded as white Gausian noises. The calcium ion channel must filter these noises. Hence, the channel can be interpreted as H2 controller. In the present work, we introduce a method for modeling state transitions of Ca channel conformation and solving the H2 problem.

## 2. Modeling of the system.

Fig 3 shows the transitional positional changes of the voltage sensor s4 molecule denoted by +. Since there are four subunits, there are four possible configurations of the positionings of the s4. When two s2 molecules took the activated positions, there are six possible positions.

Fig 2

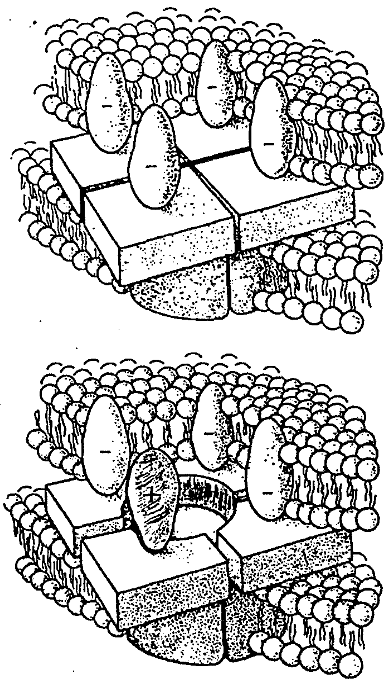
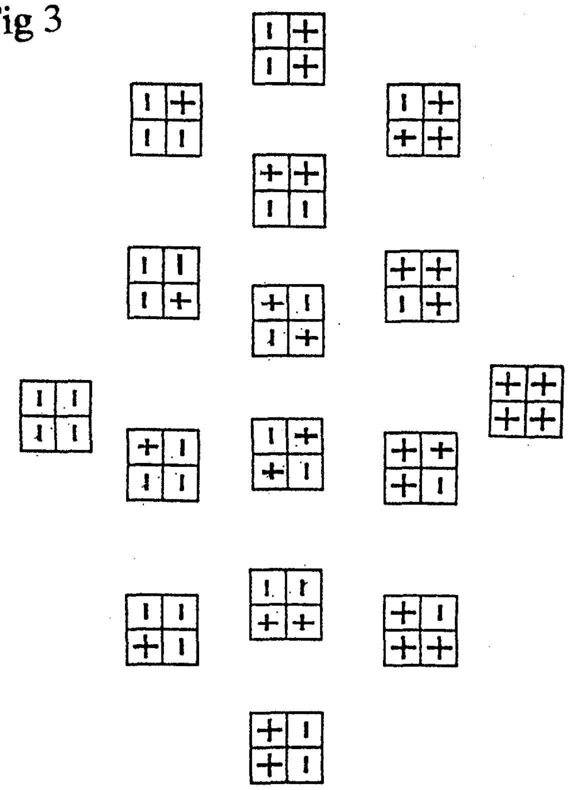


Fig 3



## 3. Mathematical description and H2 control.

In this section, we denote the amounts of closed and open channel states per unit membrane area by Cn and On. n denotes the number of the voltage sensor molecules that have taken the activating positions in the corresponding subunits. The temporal behaviors of calcium channel state is expressed by

$$\partial C_0/\partial t = k_{-c} C_1 + k_{-L}/f^4\, O_0 - (4k_c + k_L f^4) C_0 + p_9' u_3 + p_1' u_1 \quad \text{----------(1)}$$

$$\partial C_1/\partial t = 4k_c C_0 + 2k_{-c} C_2 + k_{-L}/f^3\, O_1 - (k_{-c} + 3k_c + k_L f^3) C_1 + p_1 u_1 + p_{10}' u_3 + p_2' u_1 \quad \text{-----------(2)}$$

$$\partial C_2/\partial t = 3k_c C_1 + 3k_{-c} C_3 + k_{-L}/f^2\, O_2 - (2k_{-c} + 2k_c + k_L f^2) C_2 + p_2 u_1 + p_{11}' u_3 + p_3' u_1 \quad \text{----------(3)}$$

$$\partial C_3/\partial t = 2k_c C_2 + 4k_{-c} C_4 + k_{-L}/f\, O_3 - (3k_{-c} + k_c + k_L f) C_3 + p_3 u_1 + p_{12}' u_3 + p_4' u_1 \quad \text{---------(4)}$$

$$\partial C_4/\partial t = k_c C_3 + k_{-L} O_4 - (4k_{-c} + k_L) C_4 + p_4 u_1 + p_{13}' u_3 \quad \text{---------(5)}$$

$$\partial O_0/\partial t = f^4 k_L C_0 + k_{-c} f O_1 - (k_{-L}/f^4 + 4k_c/f) O_0 + p_9 u_3 + p_5' u_2 \quad \text{--------(6)}$$

$$\partial O_1/\partial t = 4k_c/f\, O_0 + k_L f^3 C_1 + 2k_{-c} f^2 O_2 - (k_{-c} f + k_{-L}/f^3 + 3k_c/f^2) O_1 + p_{10} u_3 + p_6' u_2 \quad \text{-------(7)}$$

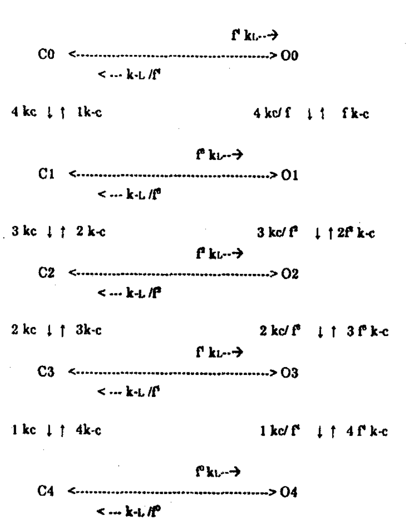
$$\partial O_2/\partial t = 3k_c/f^2\, O_1 + k_L f^2 C_2 + 3k_{-c} f^3 O_3 - (2f^2 k_{-c} + k_{-L}/f^2 + 2k_c/f^3) O_2 + p_{11} u_3 + p_7' u_2 \quad \text{------(8)}$$

$$\partial O_3/\partial t = 2k_c/f^3\, O_2 + k_L f C_3 + 4k_{-c} f^4 O_4 - (3k_{-c} f^3 + k_{-L}/f + k_c/f^4) O_3 + p_7 u_2 + p_{12} u_3 + p_8' u_2 \quad \text{-----(9)}$$

$$\partial O_4/\partial t = k_c/f^4\, O_3 + k_L C_4 - (4k_{-c} f^4 + k_{-L}) O_4 + p_8 u_2 + p_{13} u_3 \quad \text{-----(10)}$$

where $u_1$ is the control input acting on an individual transition from the closed state $C_0$ to $C_4$. The effect of $u_1$ is to convert the spatial position of an inactive voltage sensor from its resting to the activating position

Schema 1. Transition diagram for closed and open channel conformations.




Schema 1. Associated diagram of the channel state transition.

within the subunit. $p_1$,$p_2$,$p_3$ and $p_4$ are the weighting coefficients to measure the relative amount of u1 operating on the transitions of C0 -→ C1, C1-→C2, C2-→ C3 and C3-→ C4 respectively. $p_4'$,$p_3'$,$p_2'$ and $p_1'$ are those from $C_4$ state to $C_0$ state. $u_2$ is the same control input as $u_1$ but operates on an individual transition from the open state $O_0$ to $O_4$. $p_5$, $p_6$,$p_7$ and $p_8$ are the weighting coefficients that determine the relative strength of $u_2$ acting on the transitions of O0 -→ O1, O1 -→ O2, O2-→ O3 and O3 -→ O4 respectively. $p_5'$,$p_6'$,$p_7'$ and $p_8'$ are those from $O_4$ state to $O_0$ state. $u_3$ is the control input acting for transitions from a closed to an open state. $p_9$,$p_{10}$,$p_{11}$,$p_{12}$ and $p_{13}$ are the weighting coefficients to characterize the relative magnitude of $u_3$ working on the transitions of C0-→ O0, C1 -→ O1, C2-→ O2, C3-→ O3 and C4 --- O4 respectively. $p_9'$ to $p_{13}'$ are those from the opens states to closed state. Vector form of the state equation is given by

$$\partial \mathbf{x}'(t)/\partial t = \mathbf{A}\ \mathbf{x} + \mathbf{B1}\ \mathbf{w} + \mathbf{B2}\ \mathbf{u} \qquad \text{-----------(11)}$$

Where **u** is the control input, **w** is noise. I supposed that the noise acts on an entire state variables. Vector form of the equation for the controlled out put **z** is

$$\mathbf{z} = \mathbf{C1}\ \mathbf{x} + \mathbf{D12}\ \mathbf{u} \qquad \text{--------------(12)}$$

Vector form of the equation for the input **y** to an observer is given by

$$\mathbf{y} = \mathbf{C2}\ \mathbf{x} + \mathbf{D21}\ \mathbf{w} \qquad \text{-----------------(13)}$$

For simplicity, all the elements in $\mathbf{B_1}$, $\mathbf{B_2}$ and $\mathbf{D_{21}}$ were set to unity. The elements in $\mathbf{C_1}$, $\mathbf{D_{12}}$ and $\mathbf{C_2}$ were expressed by non dimensional weighting parameters $q_1$ to $q_9$, $q_{10}$ to $q_{12}$ and $s_1$ to $s_9$ respectively. Elements $q_1$ to $q_9$ in the matrix $\mathbf{C_1}$ signify relative weights of the state variables on the controlled out put z. $q_{10}$ to $q_{12}$ in the matrix $\mathbf{D_{12}}$ signify relative weights of the control input on the out put **z**. $s_1$ to $s_9$ in $\mathbf{C_2}$ signify those of the state variables on the input for observer **y**. The vector form of the optimized control **u^** is given [5]by the product of matrices **B2, X** and **x^**

$$\mathbf{u}^\wedge = -\mathbf{B2}^T\ \mathbf{X}\ \mathbf{x}^\wedge \qquad \text{-----------------(14)}$$

where **x^** is the state vector of the observers and T denotes transpose.

$$\mathbf{x}^{\wedge T} = [\, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\,]^T \text{-----(15)}$$

which correspond to state variables. **X** is the solution of related algebraic Riccati equation [

$$\mathbf{A}^T\ \mathbf{X} + \mathbf{X}\ \mathbf{A} - \mathbf{X}\ \mathbf{B2}\ \mathbf{B2}^T\ \mathbf{X} + \mathbf{C1}^T\ \mathbf{C1} = 0 \quad (16)$$

The vector equation of observer **x^** is given by

$$\partial \mathbf{x}^\wedge / \partial t = \mathbf{A}\ \mathbf{x}^\wedge + \mathbf{B2}\ \mathbf{u} + \mathbf{Y}\ \mathbf{C2}^T\ (\,\mathbf{y} - \mathbf{C2}\ \mathbf{x}^\wedge\,) \text{-(17)}$$

where **Y** is the solution of adjoint algebraic Riccati equation

$$\mathbf{A}\ \mathbf{Y} + \mathbf{Y}\ \mathbf{A}^T - \mathbf{Y}\ \mathbf{C2}^T\ \mathbf{C2}\ \mathbf{Y} + \mathbf{B1}\ \mathbf{B1}^T = 0 \quad \text{--(18)}$$

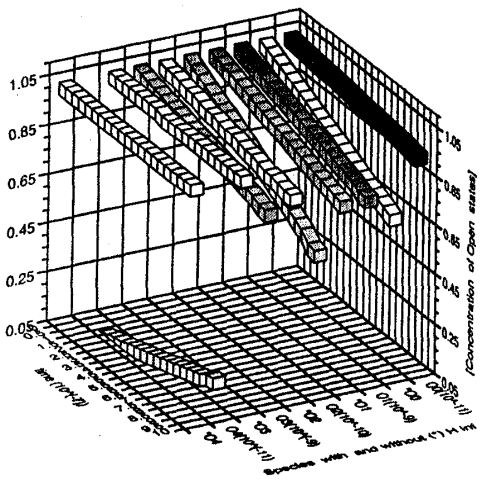
To close the feed back loop, **y** can be related to state variable **x** by

$$\mathbf{y} = \mathrm{xd} - \mathbf{C2}\ \mathbf{x} \qquad \text{-----------------(19)}$$

for simplicity. we set xd =0. The plant of the present system is

$$\mathbf{G} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B1} & \mathbf{B2} \\ \hline \mathbf{C1} & \mathbf{0} & \mathbf{D12} \\ \mathbf{C2} & \mathbf{D21} & \mathbf{0} \end{array}\right] \qquad \text{-----(20)}$$

Fig 4



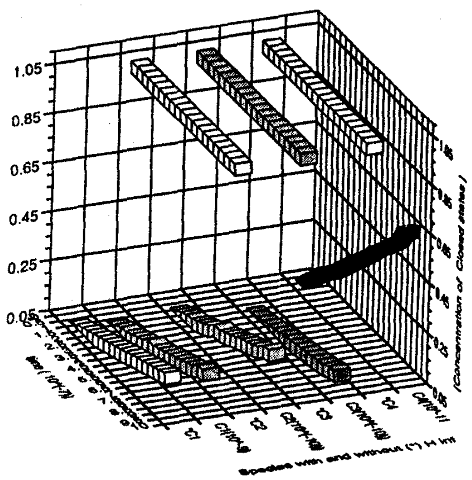
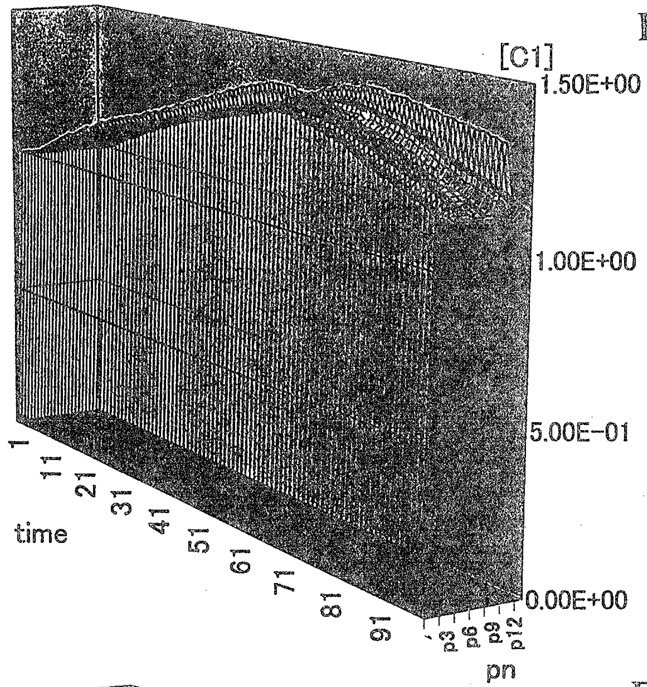



Fig 5



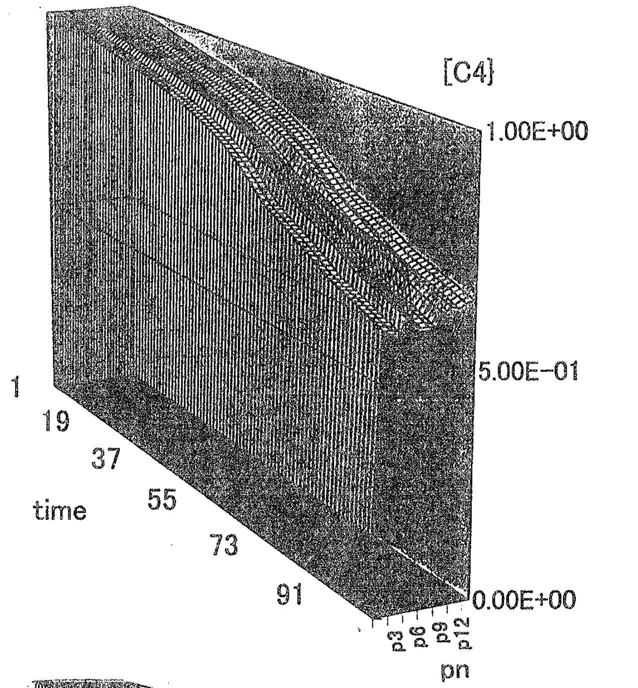
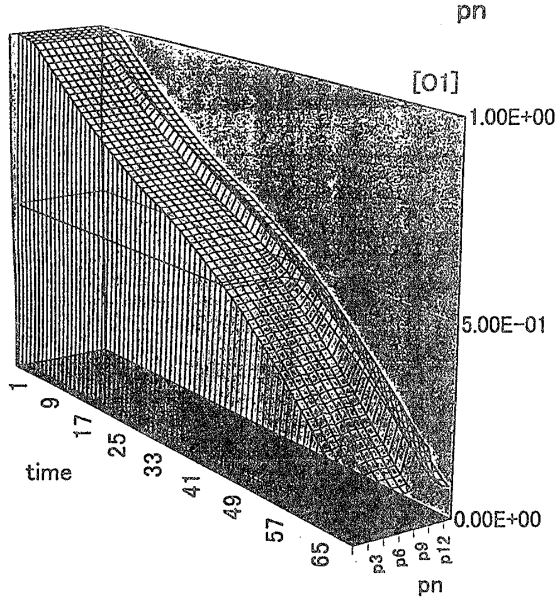




Fig 6



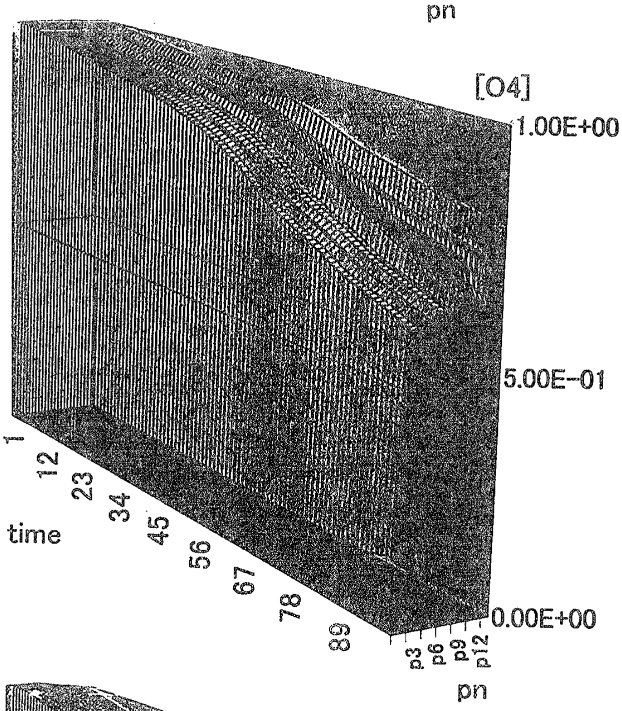
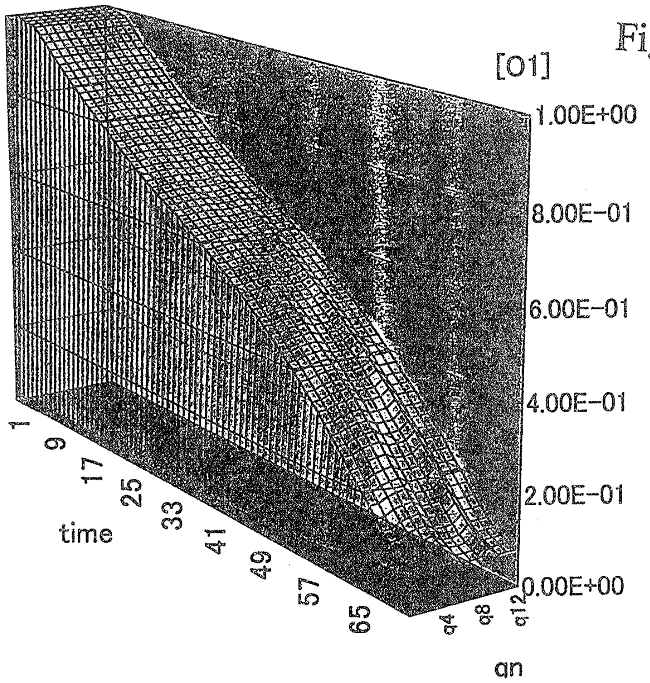
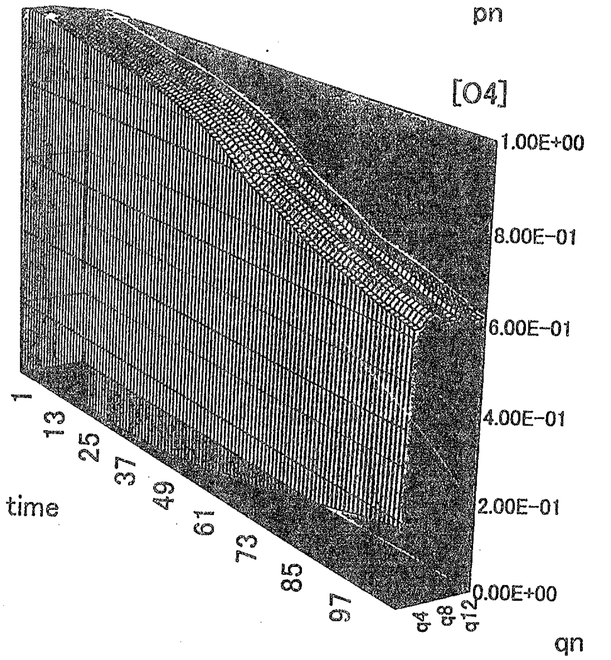




Fig 7



## 4. Results.

Fig 4 compares the temporal changes in concentrations ( amounts or number of channel state per unit membrane area ) between those under the H2 controlled conditions and non H2 control for open ( left ) and closed ( right ) states. Note the difference of the order of the concentrations of the species. The changes in H2 controlled species were significantly reduced.

Fig 5 shows the concentration of C1 and C4 species under the H2 control when weighting coefficient pn for the control inputs un were reduced to 0.00001. We can observe definite oscillative changes in concentrations of species that correspond to reduction of pn. Fig 6 shows those for the open species O1 and O4. There were similar changes. Fig 7 shows those when the weighting coefficients qn in the matrix C1 were reduced to 0.00001. We could observe small but definite changes in the time courses of the both open species.

## 5. Discussion.

The present work has shown the method for modeling the multi phase Calcium channel gating under the H2 control. The smoothing effect of the H2 control ( Fig 4 ) were remarkable. Changes in weighting coefficients for the control pn and C1 revealed characteristic changes in the concentration of the corresponding species. This in turn indicates that when any change in time courses of concentrations of the species could be observed, we could speculate what kind of the weighting coefficients have been changes. The present method when extended, including the H infinite control, we can evaluate the noise filtering function of the calcium channel gate.

## 6. References.


[1]. Monod J., Wymann J. Changeaux P. On the nature of Allosteric transitions. J. Molecular. Biology. Vol 12. 88-118. 1965.

[2]. Rios E and Pizarro G. Voltage sensor of excitation-contraction coupling in skeletal muscle. Physilogical Reviews. Vol 71. No 3. 849-908. 1991.

[3]. Sanchez- Pena., R.S and Sznaier, M. Robust systems. Theory and applications. Hons Wiley $ Sons. 1998. NY.


## Appendix.

The entire form of the system equation is

x1' ←---9---→ ←----10----→ <-----3-----→ x1

↑ ↑ ↑

9 A 9 B1 9 B2

x9' ↓ ↓ ↓ x9

Z1 ←---9---→ <-----10-----→ <----3----→ w1

↑ ↑ ↑

12 C1 12 D11 12 D12

Z12 ↓ ↓ ↓ w10

y1 ←---9---→ <----10-----→ <----3----→ u1

↑ ↑ ↑

9 C2 9 D21 9 D22 u3

y9 ↓ ↓ ↓

Numbers denote those for variables of x, u and w.

## APPENDIX 2. Elelements in the Matrixes equations.

$$
\mathbf{A} = [\begin{matrix}
a11 & a12 & a13 & a14 & a15 & a16 & a17 & a18 & a19 \\
a21 & a22 & a23 & 0 & 0 & 0 & 0 & 0 & a29 \\
0 & a32 & a33 & a34 & 0 & 0 & 0 & a38 & 0 \\
0 & 0 & a43 & a44 & a45 & 0 & a47 & 0 & 0 \\
0 & 0 & 0 & a54 & a55 & a56 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a65 & a66 & a67 & 0 & 0 \\
0 & 0 & 0 & a74 & 0 & a76 & a77 & a78 & 0 \\
0 & 0 & a83 & 0 & 0 & 0 & a87 & a88 & a89 \\
a91 & a92 & a93 & a94 & a95 & a96 & a97 & a98 & a99
\end{matrix}];
$$

$$
\mathbf{B1} = [\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{matrix}];
$$

$$
\mathbf{B2} = [\begin{matrix}
0 & b1 & b2 \\
0 & b3 & b4 \\
0 & b5 & b6 \\
0 & b7 & b8 \\
0 & b9 & b10 \\
b11 & 0 & b12 \\
b13 & 0 & b14 \\
b15 & 0 & b16 \\
b17 & 0 & b18
\end{matrix}];
$$

$$
\mathbf{C1} = [\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}];
$$

$$
\mathbf{C2} = [\begin{matrix}
s1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & s2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & s5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & s6 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & s7 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s8 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s9
\end{matrix}];
$$

$$
\mathbf{D21} = [\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{matrix}];
$$

$$
\mathbf{D12} = [\begin{matrix}
q10 & 0 & 0\\
0 & q11 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & q12
\end{matrix}];
$$

$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$

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Classical linear Quadratic regulator Problem.

Given the LTI system

$\partial x(t)/\partial t = A\,x(t) + Bu(t)$

$z(t) = C\,x(t) + Du(t)$ ---5.7)

with (A,B) stabilizable, ---(5.8)

(C,A) detectable ---(5.9)

$C^T D = 0$ ---(5.10

$D^T D = I$ ···· (5.11)

Find an optimal control law u ? $L2(0,\infty)$.

That minimizes $\|z\|2^2$.

**Lemma 5.1.** Consider the following stable system

$U(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Where the pair (A,B) is controllable.

Let $X = X^T \geqq 0$ denotes the solution of Lyapunov equation

$A^T X + XA + C^T C = 0$ --(5.13)

Then, U(s) is an inner system if and only if

$D^T C + B^T X = 0$

$D^T D = I$

Proof. -->. Using the state-space formula

$$
U^{\sim}(s)\,U(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -C^TC & -A^T & -C^TD \\ \hline D^TC & B^T & D^TD \end{array}\right] \quad \text{---(5.15)}
$$

Applying the similarity transformation

$$
T = \left[\begin{matrix} I & 0 \\ X & I \end{matrix}\right]
$$

To the realization (5.15) and using (5.13) and (5.14)

$$
U^{\wedge}U = \left[\begin{array}{cc|c} A & 0 & B \\ -XA \;\; -A^TX - C^TC & -A^T & -XB - C^TD \\ \hline D^TC + B^TX & B^T & D^TD \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^T & 0 \\ \hline 0 & B^T & I \end{array}\right]
$$

$= I$

&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&

**Lemma 5.2.** Given the system (5.7) denote by $X \geqq 0$, the solution to the following algebraic Riccati equation

$A^T X + XA + C^T C - XBB^T X = 0$ --(5.17)

Consider the change of variable $v = u - F x$ where

$F = -B^T X$ -----(5.18)

And let U denote the system in terms of this new input variable v that is

$U(s) = [\begin{matrix} AF & B \\ CF & D \end{matrix}]$ $AF = A + BF$, $CF = C + DF$

Then, U is inner.

**Proof.** Using the definitions of AF and CF and the orthogonality condition $C^T D = 0$

(5.17) is equivalent to

$(AF - BF)^T X + X(AF - BF)$

$+ (CF - DF)^T (CF - DF) - XBB^T X = 0$

Setting by

$F = -B^T X$

$(AF - B(-B^T))^T X + X(AF - B(-B^T X))$

$+ (CF - DF)^T (CF - DF) - XBB^T X = 0$

Thus, we have

$AF^T X + X\,AF + CF^T CF = 0.$

Moreovre from the definition of F, using again the orthogonality condition, $D^T CF + B^T X = 0$. Thus, from

**Lemma 5.3** Consider again the feed back gain $F = -B^T X$ and denote Gc(s) by

$$
Gc(s) = \left[\begin{matrix} AF & I \\ CF & 0 \end{matrix}\right] \text{----(5.21)}
$$

Then, $U^{\sim}(s)\,Gc(s) \in RH2^{\perp}$

**Proof.** Since Gc(s) is strictly proper, we only have to show that $U^{\sim}Gc$ is strictly antistable.

Proceeding as in Lemma 5.1

$$
U^{\sim}\,Gc = \left[\begin{array}{cc|c} -AF^T & -CF^T CF & 0 \\ 0 & AF & I \\ \hline B^T & F & 0 \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} -AF^T & 0 & -X \\ 0 & AF & I \\ \hline B^T & & 0 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} -AF^T & -X \\ B^T & 0 \end{array}\right] \in RH2^{\perp}
$$

Which is antistable since AF is stable.

5.3 The standard H2 control problem.

Fig 5.1

w·→ G(s) ··→z

u·→ G(s) ·→y

u←· K(s) ←·y

A common control problem is to synthesize a controller that stabilizes the system and minimizes the size of the output to a given class of input w. The LQR problem in 5.2 is a special case where the inout is restricted to be the form of $w(t) = x_o\ \delta(t)$ where its is assumed that the states are available for feed back. Leading to the following state space realization for G(s)

$$GLQR = \left[\begin{array}{c|cc} A & I & B2 \\ \hline C1 & 0 & D12 \\ I & 0 & 0 \end{array}\right] \quad \cdots(5.28)$$

In the H2 control problem,

$$G = \left[\begin{array}{c|cc} A & B1 & B2 \\ \hline C1 & 0 & D12 \\ C2 & D21 & 0 \end{array}\right] \quad \cdots\cdots(5.29)$$

The object is to synthesize an internally stabilizing controller that minimizes the 2-norm of the closed loop transfer function from w to z $\|Tzw\|_2^2$. Its physical interpretation is to minimize the root mean square (RMS) value of the output z due to a Gaussian white noise input with unit covariance.

[ Standard H2 problem ]

< Given the plant G(s), find an internally stabilizing proper LTI controller that minimizes $\|Tzw\|_2$.

Assumptions

A1. (A,B2) is stabilizable and (C2,A) is detectable

A2. (A,B1) is stabilizable and (C1, A) is detectable

A3. $C1^T D12 = 0$ and $B1\ D21^T = 0$

A4. D12 has full column rank with $D12^T D12 = I$ and D21 has full row rank with $D21D21^T = 0$.

The assumption D11 =0 guarantees that the closed loop transfer matrix is in H2. ( A real rational stable transfer matrix is in H2 if and only if it is strictly proper ).A1 is necessary for the system to be stabilzable via output feed back. A1 and A2 guarantee that the control and filtering Riccati equations associated with the H2 problem admit positive semidefinite stabilizing solutions. They can be relaxed to G12 and G21 not having invariant zeros on the imaginary axis. A3 is an orthogonality assumption. A4 is the rank assumption which guarantees that the H2 problem is non singular. The normalizing condition $D12^T D12 = I$, $D21D21^T = I$ do not entail any loss of generality since they can be met by redefining the input w and u.

Consider the Riccat equations

$$A^T X2 + X2\,A - X2B2B2^T X2 + C1^T C1 = 0 \quad \cdots(5.30)$$

$$AY2 + Y2A^T - Y2C2^T C2Y2 + B1B1^T = 0 \quad -(5.31)$$

These equations can be associated to the Hamiltonian matrices H2 and J2

$$H2 = \begin{array}{cc} A & -B2B2^T \\ -C1^T C1 & -A^T \end{array} \quad \cdots(5.32a)$$

$$J2 = \begin{array}{cc} A^T & -C2^T C2 \\ -B1B1^T & -A \end{array} \quad \cdots(5.32b)$$

Under the assumptions A2 and A3, these matrices satisfy H2, J2 ∈ dom(Ric). This in turn implies that 5.30 abd 5.31 have unique solutions $X2 = Ric(H2) \geqq 0$ and $Y2 = Ric(Y2) \geqq 0$. These solutions are stabilizing

$A - B2B2^T X2$ and $A - Y2C2^T C2$ are Hurwitz.

**Theorem 5.2** Under the assumptions A1 to A4. unique optimal H2 controller is given by

$$\begin{bmatrix} \hat{X} \\ U \end{bmatrix} \cdot Kopt(s) = \left[\begin{array}{c|c} AFL & L2 \\ \hline -F2 & 0 \end{array}\right] \begin{bmatrix} \hat{x} \\ y \end{bmatrix} \quad 5.33$$

$$\min_{K(s)\ \text{stabilizing}} \|Tzw(s)\|_2^2 = \|Gc(s)\,B1\|_2^2 + \|F2Gf(s)\|_2^2 \quad --(5.34)$$

$$Gc(s) = \left[\begin{array}{c|c} AF & I \\ \hline C1F & 0 \end{array}\right] \qquad : AF = A + B2\,F2 \qquad : C1F = C1 + D12\,F2$$

$$Gf(s) = \left[\begin{array}{c|c} AL & B1L \\ \hline I & 0 \end{array}\right] \qquad : AL = A + L2\,C2 \qquad : B1L = B1 + L2\,D21$$

$AFL = A + B2F2 + L2\,C2$

$F2 = -B2T\ X2,\ L2 = -Y2C2T$

$X2 = Ric\ (H2),\quad Y2 = Ric\ (J2).$

full information and out put estimation subproblems. To prove this, we decompose the output feed back problem into full information and output estimation sub Problem.

**5.3.1 Full Information problem.**

**Lemma 5.5** Consdier the FI case

$$G = \left[\begin{array}{c|cc} A & B1 & B2 \\ \hline C1 & 0 & D12 \\ \begin{pmatrix} I \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}\right] \quad --- (5.36)$$

Then, the following results hold

1. The optimal H2 controller $K_{FI} = [\ F2\ \ 0]$

2. The corresponding optimal value of the H2 cost is $\Gamma opt = \|Tzw\|_2^2 = \|Gc\ B1\|_2 = trace\ (B1TX2B1)^{1/2}$

3. Given $\gamma > \gamma opt$, the set of all internally stabilizing controllers such that $\|Tzw\|_2 \leqq \gamma$ is $\{ K(s) : K = [\ F2\quad Q(s)\ ],\ Q(s) \in RH2,\ \|Q\|_2^2 \leqq \gamma^2 - \|GcB1$

Proof. As in the LQR case, define a new control input $V = u(t) - F2\ x(t)$, obtaining

$$Z(s) = \left[\begin{array}{c|cc} AF & B1 & B2 \\ \hline C1F & 0 & D12 \end{array}\right] \cdot \begin{pmatrix} w(s) \\ v(s) \end{pmatrix} \quad \cdots 5.37.$$

Or equivalently

$$Tzw(s) = \left[\begin{array}{c|c} AF & B1 \\ \hline C1F & 0 \end{array}\right] + \left[\begin{array}{c|c} AF & B2 \\ \hline C1F & D12 \end{array}\right] Tvw(s)$$

$$= Gc(s)\ B1 + U(s)\ Tvw(s) \quad \cdots 5.38$$

Which produces with Lemmas 5.2 and 5.3

$$\|Tzw(s)\|_2^2 = \|Gc(s)\ B1\|_2^2 + \|Tvw(s)\|_2^2 \quad \cdots 5.39$$

This quantity is minimized by setting v=0 which results in Tvw =0. This is achieved by the unique static controller $K_{FI} = [\ F2\quad 0\ ]$ yielding

$$\min \|T_{zw}(s)\|_2^2 = \|G_c(s)\, B_1\|_2^2 = \mathrm{trace}\,(B_1^T X_2 B_1)^{1/2}$$ –5.40

Where the last equality derives from the fact that X2 is the closed –loop observability Gramian.

Remarks

1. The optimal controller uses only feed back from the states. Thus under the assumption that D11 =0, the FI and state-feed back (LQR) problems have the same optimal solution.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

1. Full Control Problem.

Lemma 5.6 For the full control case.

$$G_{FC}(s) = \left[\begin{array}{c|ccc} A & B_1 & I & 0 \\ \hline C_1 & 0 & 0 & I \\ C_2 & D_{21} & 0 & 0 \end{array}\right] \cdots 5.42$$

The following properties hold

1. The optimal H2 controller is
   $K_{FC} = [\,L_2 \;\; 0\,]^T$
2. The corresponding optimal value of the H2 cost is
   $\gamma_{opt} = \min_{u \in L_2}$
   $\|T_{zw}(s)\|_2^2 = \|C_1 G_f\|_2^2 = \mathrm{trace}\,(C_1 Y_2 C_1^T)^{1/2}$
3. Given $\gamma > \gamma_{opt}$, the set of all internally stabilizing controller such that
   $\|T_{zw}(s)\|_2^2 < \gamma$ is
   $\{K(s) : K = [\,L_2 \; Q(s)\,]^T \; Q(s) \in RH_2,$
   $\|Q\|_2^2 \leq \gamma^2 - \|C_1 G_f\|_2^2$

&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&

**Disturbance feed forward problem.**

Lemma 5.7 Consider the DF problem

$$G_{DF}(s) = \left[\begin{array}{cc|c} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right] \cdots 5.43$$

With assumption that $A - B_1 C_2$ is stable.

The, the following properties hold

1. $\gamma_{opt} = \min \|T_{zw}\|_2^2 = \|G_c B_1\|_2^2$
2. $$K_{opt} = \left[\begin{array}{c|c} A + B_2F_2 - B_1C_2 & B_1 \\ \hline F_2 & 0 \end{array}\right] \cdots 5.44$$
3. The set of all LTI controllers such that
   $\|T_{zw}\|_2^2 \leq \gamma$ is given by
   $K(s) = F_L(J_{DF}, Q_W)$, Q? RH2,
   $\|Q\|_2^2 \leq \gamma^2 - \|G_c B_1\|_2^2$
   Where
   $$J_{DF}(s) = \left[\begin{array}{c|cc} A + B_2F_2 - B_1C_2 & B_1 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right] \cdots 5.45$$

Proof. From Lemma 3.3 that under the additional hypothesis of stability of A-B1C2, the DF and FI problems are equivalent, in the sense that if $K_{FL} = [\,F \;\; Q\,]$ stabilizes GFI then $K_{DF} = F_L(J_{DF}, Q)$ stabilizes GDF and yields the same closed loop transfer function, $F_L(G_{FI}, K_{FI}) = F_L(G_{DF}, K_{DF})$. More over KDF parameterizes all the DF stabilizing controllers. The proof follows now by combining these facts with Lemma 5.5

As before, the solution to the OE case follows from the duality

&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&

**Output Estimation problem**

Lemma 5.8 For the OE case

$$G_{OE} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

A- B2C1 is stable - 5.46

The following properties hold

1. The optimal H2 controller is

$$K_{OE} = \left[\begin{array}{c|c} A + L_2C_2 - B_2 C_1 & L_2 \\ C_1 & 0 \end{array}\right]$$

2. $\gamma_{opt} = \min \|T_{zw}\|_2^2 = \|C_1 G_f\|_2^2$

2. The set of internally stabilizing controllers such that
   $\|T_{zw}\|_2^2 \leq \gamma$ is $K(s) = F_L(J_{OE}, Q)$, Q RH2,
   $\|Q\|_2^2 \leq \gamma^2 - \|C_1 G_f\|_2^2$
   where

$$J_{OE}(s) = \left[\begin{array}{c|cc} A + L_2C_2 - B_2C_1 & L_2 & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

Proof. of Theorem 5.2

Change the input variable $u(t) = v(t) + F_2\, x(t)$

This partitions the system into the two subsystems In Fig 5.2 Where G1(s) and Gtmp(t) have the realizations

$$G_1(s) = \left[\begin{array}{c|cc} A + B_2F_2 & B_1 & B_2 \\ \hline C_1 + D_{12}F_2 & 0 & D_{12} \end{array}\right]$$

$$G_{tmp}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F_2 & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right] \cdots 5.49$$

G1 is stable and Gtmp has an Output estimation form. K(s) internally stabilizes G(s) if and only if K(s) internally stabilizes Gtmp. From Fig 5.2

$$Z(s) = \left[\begin{array}{c|cc} A_F & B_1 & B_2 \\ \hline C_{1F} & 0 & D_{12} \end{array}\right] \begin{bmatrix} w(s) \\ v(s) \end{bmatrix}$$

$$= G_c(s)\, B_1\, w(s) + U(s)\, v(s) \cdots 5.50$$

or

$$T_{zw}(s) = G_c(s)\, B_1 + U(s)\, T_{vw}(s) \; –5.51$$

From Lemmas 5.2 and 5.3, we have

$$\min \|T_{zw}(s)\|_2^2 = \|G_c(s)\, B_1\|_2^2 + \min \|T_{vw}(s)\|_2^2 \cdots 5.52$$

But since Gtmp has an Out put estimation form, from Lemma 5.8 we have that the controller that minimizes this last transfer function is

$$K_{OE}(s) = \left[\begin{array}{c|c} A + L_2C_2 + B_2F_2 & L_2 \\ \hline -F_2 & 0 \end{array}\right] \cdots 5.53$$

( in 5.47 KOE, set $C_1 = -F_2$ )

Yielding $\min \|T_{vw}(s)\|_2^2 = \|F_2\, G_f(s)\|_2^2$.

It follows

$$\min \|T_{zw}(s)\|_2^2 = \|G_c(s) B_1\|_2^2 + \|F_2 G_f(s)\|_2^2 \cdots 5.54$$

Remark.

The optimal controller (5.33) exhibits the separation structure of the H2 problem. This can be made apparent by rewriting its state-space realization as

$$D\hat{x}(t)/dt = A\hat{x} + B_2 u + L_2(C_2\hat{x} - y)$$
$$U = F\hat{x}$$

Thus, the output of the controller is the optimal estimate of the LQR control action $u = F_2\, x$. Alternatively, the separation can be seen directly from the proof of the theorem. Because the state feed back used in obtaining G1 corresponds to the optimal LQR state feed back, The subsystem Gtmp leads to an optimal output estimation problem. The optimal cost is the optimal state feed back (LQR) cost plus the optimal filtering cost.