Institute of Electronics, Information, and Communication Engineers

社団法人 電子情報通信学会
THE INSTITUTE OF ELECTRONICS,
INFORMATION AND COMMUNICATION ENGINEERS

信学技報
TECHNICAL REPORT OF IEICE.
CAS2001-2, VLD2001-19
DSP2001-21(2001-06)

Molecular Dynamical Method for Analyzing the Bio Molecular Transmission across the Bio Signal Network Circuit.

H, Hirayama, *Y. Okita., *T, Sugiura,
*M, Kimura and **T. Kazui
Department of Public Health Asahikawa medical college.
E mail , hirayama@asahikawa-med.ac.jp
*Shizuoka university. **Hamamatsu medical college.

あらまし We introduce a mathematical method for predicting micro hydrodynamical behavior of one bio molecules at very narrow space that has been proposed by Ganatos (1982). One solution was an infinite series in spherical co-ordinates which have planar symmetry about the plane y=0 which is perpendicular to the longitudinal axis of the cylinder coordinate. This solution vanishes as r approaches to infinite. Another solution was a double Fourier integral in rectangular co-ordinates which produce finite velocities in the flow filed. The basic form of them were composed of Legendre spherical function, modified Bessel functions. The coefficients of these solutions were set to satisfy the no-slip boundary conditions on both infinite confining walls simultaneously for an arbitrary disturbance representing a sphere of unspecified size position d velocity. The present method will be available for predicting bio molecular particle at narrow biological space.

和文キーワード Microhydro dynamics. Cylindrical coordinate. Spherical Coordinate. Lamb. Legendre function.

生体情報伝達回路における分子動力学的解析法

平山博史、*沖田善光、*杉浦敏文、*木村元彦、**数井輝久

旭川医科大学、旭川市緑が丘東2条1丁目
*静岡大学. **浜松医科大学

Abstract

細胞間などの狭い生体空間内を運動する生体分子の微小力学的挙動を推定する目的て Ganatos 1980が提唱した方法を紹介する。運動は並進、回転運動および1個の球周囲を通過する流れの3成分に分離して記述された。系は球粒子が通過する領域を円柱座標系で、球自身は球座表系で記述した。両系とも原点は球の中心とし、原点が必ずしも円柱の中心軸には一致しない一般的な場合を想定した。解析解はまず無限遠でゼロ、円管垂直軸周囲で平面対照な球座標系での無限級数でLamb の特別な形式である。 もう一つの解は流れの場に任意の箇所で有限な速度生み出す、直交座標系における解である。 解はレジェンドレ陪球関数、 変形ベッセル関数、双曲線関数の複雑な有限積分形式で与えられた。級数の係数は境界条件によって決定された。 本研究を発展させることにより、微小動力学的な生体空間における生体分子の挙動を推定することが可能である。

英文 key words 微小力学的挙動. 円柱座標系. 球座表系. Lamb. レジェンドレ陪関数、 変形ベッセル関数.

NII-Electronic Library Service

1. Introduction.

Following our previous paper (IEICE. Tech. rep NC 2000-1, p1-p7), we introduce micro hydrodynamical approach for creeping motion of a bio molecular particle that travels between the narrow space bounded by two planar cellular surfaces. The original one was proposed by Ganatos (J.F.M. vol 99. pp 755-783. 1980.) The present article gives detailed explanation for driving Vw component.

$$D_1(\alpha,\beta,z) = \left[A^*\left(1+\frac{\alpha^2}{\kappa}z\right) - A^{**}\frac{\alpha\beta}{\kappa}z - A^{***}\alpha z\right]e^{\kappa z} + \left[B^*\left(1-\frac{\alpha^2}{\kappa}z\right) + B^{**}\frac{\alpha\beta}{\kappa}z - B^{***}\alpha z\right]e^{-\kappa z}, \quad (2.9a)$$

$$D_2(\alpha,\beta,z) = \left[-A^*\frac{\alpha\beta}{\kappa}z + A^{**}\left(1+\frac{\beta^2}{\kappa}z\right) + A^{***}\beta z\right]e^{\kappa z} + \left[B^*\frac{\alpha\beta}{\kappa}z + B^{**}\left(1-\frac{\beta^2}{\kappa}z\right) + B^{***}\beta z\right]e^{-\kappa z}, \quad (2.9b)$$

$$D_3(\alpha,\beta,z) = [A^*\alpha z - A^{**}\beta z + A^{***}(1-\kappa z)]e^{\kappa z} + [B^*\alpha z - B^{**}\beta z + B^{***}(1+\kappa z)]e^{-\kappa z}, \quad (2.9c)$$

we substitute $(2{,}17,a,b,c)$

$$V_\omega = \hat{i}\int_0^\infty\int_0^{\pi/2}\kappa\cdot D_1(\kappa,\gamma,z)g_1(\kappa,\gamma)d\gamma\cdot d\kappa + \hat{j}\int_0^\infty\int_0^{\pi/2}\kappa\cdot D_2(\kappa,\gamma,z)g_2(\kappa,\gamma)d\gamma\cdot d\kappa + \hat{k}\int_0^\infty\int_0^{\pi/2}\kappa\cdot D_3(\kappa,\gamma,z)g_3(\kappa,\gamma)d\gamma\cdot d\kappa$$

For this, we substitute the terms in $(2{,}17,a,b,c)$ by

Converting $(\alpha,\beta)\rightarrow(\kappa,\gamma)$

****

II] Expansion of V_ω

$$g_1(\kappa,\gamma) = \cos(\kappa\chi\cos\gamma)\cdot\cos(\kappa y\sin\gamma)$$
$$g_2(\kappa,\gamma) = \sin(\kappa\chi\cos\gamma)\cdot\sin(\kappa y\sin\gamma)$$
$$g_3(\kappa,\gamma) = \sin(\kappa\chi\cos\gamma)\cdot\cos(\kappa y\sin\gamma)$$

$$\begin{aligned}
V_\omega = \hat{i}\Bigg[\Bigg[&\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_5(\eta)\cdot D_1(\kappa,\gamma,-b)\cdot g_1(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_5(\sigma)\cdot D_1(\kappa,\gamma,c)\cdot g_1(\kappa,\gamma)d\gamma d\kappa \\
&+\int_0^\infty\int_0^{\pi/2}\kappa G_6(\sigma,\eta)\cdot\left[\cos^2\gamma\cdot D_1(\kappa,\gamma,-b)-\cos\gamma\cdot\sin\gamma\cdot D_2(\kappa,\gamma,-b)\right]\cdot g_1(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa G_6(\eta,\sigma)\cdot\left[\cos^2\gamma\cdot D_1(\kappa,\gamma,c)-\cos\gamma\cdot\sin\gamma\cdot D_2(\kappa,\gamma,c)\right]\cdot g_1(\kappa,\gamma)d\gamma d\kappa \\
&+\int_0^\infty\int_0^{\pi/2}\kappa\, G_1(\sigma,\eta)\cdot\cos\gamma\cdot D_3(\kappa,\gamma,-b)\cdot g_1(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_1(\eta,\sigma)\cos\gamma\cdot D_3(\kappa,\gamma,c)g_1(\kappa,\gamma)d\gamma d\kappa\Bigg]\Bigg] \\
+\hat{j}\Bigg[\Bigg[&\int_0^\infty\int_0^{\pi/2}(-\kappa)\cdot G_6(\sigma,\eta)\cdot\left[\cos\gamma\cdot\sin\gamma\cdot D_1(\kappa,\gamma,-b)+\cos^2\gamma\cdot D_2(\kappa,\gamma,-b)\right] \\
&+\int_0^\infty\int_0^{\pi/2}\kappa G_6(\eta,\sigma)\cdot\left[\cos\gamma\cdot\sin\gamma\cdot D_1(\kappa,\gamma,c)+\cos^2\gamma\cdot D_2(\kappa,\gamma,c)\right]\cdot g_2(\kappa,\gamma)d\gamma d\kappa \\
&+\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_3(\sigma,\eta)\cdot D_2(\kappa,\gamma,-b)\cdot g_2(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\, G_3(\eta,\sigma)\cdot D_2(\kappa,\gamma,c)\cdot g_2(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\, G_1(\sigma,\eta)\cdot\sin\gamma\cdot D_3(\kappa,\gamma,-b)\cdot g_2(\kappa,\gamma)d\gamma d\kappa \\
&+\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_1(\eta,\sigma)\sin\gamma\cdot D_3(\kappa,\gamma,c)g_2(\kappa,\gamma)d\gamma d\kappa\Bigg]\Bigg] \\
+\hat{k}\Bigg[\Bigg[&\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_2(\sigma,\eta)\cdot\left[\cos\gamma\cdot D_1(\kappa,\gamma,-b)-\sin\gamma\cdot D_2(\kappa,\gamma,-b)\right]g_3(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\, G_2(\eta,\sigma)\cdot\left[\cos\gamma\cdot D_1(\kappa,\gamma,c)-\sin\gamma\cdot D_2(\kappa,\gamma,c)\right]\cdot g_3(\kappa,\gamma)d\gamma d\kappa \\
&+\int_0^\infty\int_0^{\pi/2}\kappa\, G_4(\sigma,\eta)\cdot D_3(\kappa,\gamma,-b)\cdot g_3(\kappa,\gamma)d\gamma d\kappa \\
&-\int_0^\infty\int_0^{\pi/2}\kappa\cdot G_4(\eta,\sigma)\cdot D_3(\kappa,\gamma,c)g_3(\kappa,\gamma)d\gamma d\kappa\Bigg]\Bigg]
\end{aligned}$$

where $D_{1,2,3}(\kappa,\gamma,-b)$ $D_{1,2,3}(\kappa,\gamma,c)$ are given by equation (2、16). As

$$\begin{aligned}
&\int_0^\infty\int_0^{\pi/2}\kappa\, G_n(\sigma,\eta)\cdot D_m(\kappa,\gamma,-b,c)\cdot g_l(\kappa,\gamma)d\gamma d\kappa \\
&=\int_0^\infty\int_0^{\pi/2}\kappa\, G_n(\sigma,\eta)\cdot\sum_{n=1}\left[A_n\cdot A_n^{\bullet}(\kappa,\gamma,-b,c)+B_n\cdot B_n^{\bullet}(\kappa,\gamma,-b,c)\right. \\
&\qquad\left.+C_n\cdot C_n^{\bullet}(\kappa,\gamma,-b,c)\right]\cdot g_l(\kappa,\gamma)d\gamma d\kappa \\
&=\sum_{n=1}A_n\int_0^\infty\int_0^{\pi/2}\kappa\, G_n(\sigma,\eta)\cdot A_n^{\bullet}(\kappa,\gamma,-b,c)\cdot g_l(\kappa,\gamma)d\gamma d\kappa \\
&+\sum_{n=1}B_n\int_0^\infty\int_0^{\pi/2}\kappa\, G_n(\sigma,\eta)\cdot B_n^{\bullet}(\kappa,\gamma,-b,c)\cdot g_l(\kappa,\gamma)d\gamma d\kappa \\
&+\sum_{n=1}C_n\int_0^\infty\int_0^{\pi/2}\kappa\, G_n(\sigma,\eta)\cdot C_n^{\bullet}(\kappa,\gamma,-b,c)\cdot g_l(\kappa,\gamma)d\gamma d\kappa
\end{aligned}$$

while $G_n(\sigma,\eta)$ is given by (2, 18)

$A_n^{\bullet}$、$B_n^{\bullet}$、$C_n^{\bullet}$ are given by $(A_1\sim A_9)$. Hence the integration about γ can be determined.

Since $F_m(\kappa,\gamma,-b,c,n,m)$ in A1 to A9 are determined by (2, 14)、 the nuclear function of the integrand is

$$\int_0^\infty\int_0^{\pi/2}F_l(\kappa,\gamma,-b,c,n,m)\cdot g_l(\kappa,\gamma)d\gamma d\kappa$$
$$=\int_0^\infty\int_0^{\pi/2}\sum S_{nmq}(\kappa,-b,c)\cdot K_{n-q-l}(\kappa,-b,c)\cdot g_l(\kappa,\gamma)d\gamma d\kappa$$

% from c−5, the series part can be expressed by $B_{n,m,j,l}(z_i=-b,c)$

$$=\int_0^\infty B_{n,m,j,l}(\kappa)\int_0^{\pi/2}g_l(\kappa,\gamma)d\gamma\cdot d\kappa$$

****

III] Integration of each term of V_ω

The terms are related to $\hat{k}$ and (2, 23)$V=\hat{i}u+\hat{j}v+\hat{k}w$

Hence, it leads to the form of (2, 24c)

$$F_4(\alpha,\beta,z_i,n,m) = \int_0^\infty\int_0^\infty\frac{s}{(s^2+t^2+z_i^2)^{\frac{1}{2}(n+1)}}P_n^m\left(\frac{z_i}{(s^2+t^2+z_i^2)^{\frac{1}{2}}}\right)\sin\alpha s\cos\beta t\,ds\,dt$$
$$=\frac{\pi}{2}\alpha|z_i|^3\sum_{q=0}^{[\frac{1}{2}n]}S_{nmq}(z_i)(\kappa|z_i|)^{n-q-\frac{3}{2}}K_{n-q-\frac{3}{2}}(\kappa|z_i|), \quad (2.14d)$$

$$S_{nmq}(z_i) = \frac{(2/\pi)^{\frac{1}{2}}}{(-2)^q q!(n-2q-m)!\,z_i^{n+m}} \quad (2.15)$$

and K_ν is the modified Bessel function of the second kind of order ν. Application of these results to (2.12) gives

$$\left.\begin{aligned}
D_1(\alpha,\beta,z_i) &= \sum_{n=1}^\infty[A_n\mathscr{A}_n^*(\alpha,\beta,z_i)+B_n\mathscr{B}_n^*(\alpha,\beta,z_i)+C_n\mathscr{C}_n^*(\alpha,\beta,z_i)], \\
D_2(\alpha,\beta,z_i) &= \sum_{n=1}^\infty[A_n\mathscr{A}_n^{**}(\alpha,\beta,z_i)+B_n\mathscr{B}_n^{**}(\alpha,\beta,z_i)+C_n\mathscr{C}_n^{**}(\alpha,\beta,z_i)], \\
D_3(\alpha,\beta,z_i) &= \sum_{n=1}^\infty[A_n\mathscr{A}_n^{***}(\alpha,\beta,z_i)+B_n\mathscr{B}_n^{***}(\alpha,\beta,z_i)+C_n\mathscr{C}_n^{***}(\alpha,\beta,z_i)],
\end{aligned}\right\}\quad i=1,2, \quad (2.16)$$

$$\begin{aligned}
D_1(\alpha,\beta,z) = &\,G_5(\eta)D_1(\alpha,\beta,-b)-G_5(\sigma)D_1(\alpha,\beta,c) \\
&+G_6(\sigma,\eta)\frac{\alpha}{\kappa^2}[\alpha D_1(\alpha,\beta,-b)-\beta D_2(\alpha,\beta,-b)] \\
&-G_6(\eta,\sigma)\frac{\alpha}{\kappa^2}[\alpha D_1(\alpha,\beta,c)-\beta D_2(\alpha,\beta,c)] \\
&+G_1(\sigma,\eta)\frac{\alpha}{\kappa}D_3(\alpha,\beta,-b) \quad -G_1(\eta,\sigma)\frac{\alpha}{\kappa}D_3(\alpha,\beta,c),
\end{aligned}\quad (2.17a)$$

$$\begin{aligned}
D_2(\alpha,\beta,z) = &-G_6(\sigma,\eta)\frac{\alpha\beta}{\kappa^2}\left[D_1(\alpha,\beta,-b)+\frac{\alpha}{\beta}D_2(\alpha,\beta,-b)\right] \\
&+G_6(\eta,\sigma)\frac{\alpha\beta}{\kappa^2}\left[D_1(\alpha,\beta,c)+\frac{\alpha}{\beta}D_2(\alpha,\beta,c)\right] \\
&+G_3(\sigma,\eta)D_2(\alpha,\beta,-b)-G_3(\eta,\sigma)D_2(\alpha,\beta,c) \\
&-G_1(\sigma,\eta)\frac{\beta}{\kappa}D_3(\alpha,\beta,-b) \quad +G_1(\eta,\sigma)\frac{\beta}{\kappa}D_3(\alpha,\beta,c),
\end{aligned}\quad (2.17b)$$

$$\begin{aligned}
D_3(\alpha,\beta,z) = &\,G_2(\sigma,\eta)\left[\frac{\alpha}{\kappa}D_1(\alpha,\beta,-b)-\frac{\beta}{\kappa}D_2(\alpha,\beta,-b)\right] \\
&-G_2(\eta,\sigma)\left[\frac{\alpha}{\kappa}D_1(\alpha,\beta,c)-\frac{\beta}{\kappa}D_2(\alpha,\beta,c)\right] \\
&+G_4(\sigma,\eta)D_3(\alpha,\beta,-b) \\
&-G_4(\eta,\sigma)D_3(\alpha,\beta,c),
\end{aligned}\quad (2.17c)$$

where

$$G_{1,2}(\mu,\nu) = 4\tau\mu\nu\left[\frac{\sinh\mu}{\mu}\pm\frac{\sinh\tau}{\tau}\frac{\sinh\nu}{\nu}\right]\Big/\delta_2, \quad (2.18a)$$

$$G_{3,4}(\mu,\nu) = 4\tau\left\{\nu\left[\cosh\mu-\frac{\sinh\tau}{\tau}\frac{\sinh\nu}{\nu}\right]\pm\mu\left[\frac{\sinh\mu}{\mu}-\frac{\sinh\tau}{\tau}\cosh\nu\right]\right\}\Big/\delta_2, \quad (2.18b)$$

$$G_5(\mu) = (-2\sinh\mu)/\delta_1, \quad (2.18c)$$

$$G_6(\mu,\nu) = 8\tau^2\left\{\mu\frac{\sinh\tau}{\tau}\left[\frac{\sinh\mu}{\mu}-\frac{\sinh\tau}{\tau}\cosh\nu\right] + \nu\left[\frac{\sinh\tau}{\tau}\cosh\mu-\frac{\sinh\nu}{\nu}\right]\right\}\Big/\delta_1\delta_2. \quad (2.18d)$$

NII-Electronic Library Service

Institute of Electronics, Information, and Communication Engineers

$$G_6(\eta,6)\Big[\int^{\infty}\int^{\pi/2} k\big[-\cos^2\gamma\cdot D_1(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot D_2(k\gamma c)\big]g_1(k\gamma c)d\gamma dk$$

$$+\int^{\infty}\int^{\pi/2} k\big[\cos\gamma\cdot\sin\gamma\cdot D_1(k\gamma c)+\cos^2\gamma\cdot D_2(k\gamma c)\big]g_2(k\gamma c)d\gamma dk\Big]\Big]$$

$$=G_6(\eta,6)\Big[\int^{\infty}\int^{\pi/2} k\cdot D_1(k\gamma c)\{-\cos^2\gamma\cdot g_1(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_2(k\gamma c)\}d\gamma dk$$

$$+\int^{\infty}\int^{\pi/2} k\cdot D_2(k\gamma c)\{\cos^2\gamma\cdot g_2(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)\}d\gamma dk\Big]$$

From equation (2. 16)

$$D_1(k\gamma c)=\sum_n\big[A_n\cdot A_n^*(k\gamma c)+B_n\cdot B_n^*(k\gamma c)+C_n\cdot C_n^*(k\gamma c)+\big]$$

$$D_2(k\gamma c)=\sum_n\big[A_n\cdot A_n^{**}(k\gamma c)+B_n\cdot B_n^{**}(k\gamma c)+C_n\cdot C_n^{**}(k\gamma c)+\big]$$

$$=G_6(\eta,6)[[$$

$$\sum_n A_n\Big[\int^{\infty}\int^{\pi/2} k\cdot A_n^*(k\gamma c)\{-\cos^2\gamma\cdot g_1(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_2(k\gamma c)\}d\gamma dk$$

$$+\int^{\infty}\int^{\pi/2} k\cdot A_n^{**}(k\gamma c)\{\cos^2\gamma\cdot g_2(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)\}d\gamma dk\Big]$$

$$+\sum_n B_n\Big[\int^{\infty}\int^{\pi/2} k\cdot B_n^*(k\gamma c)\{-\cos^2\gamma\cdot g_1(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_2(k\gamma c)\}d\gamma dk$$

$$+\int^{\infty}\int^{\pi/2} k\cdot B_n^{**}(k\gamma c)\{\cos^2\gamma\cdot g_2(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)\}d\gamma dk\Big]$$

$$+\sum_n C_n\Big[\int^{\infty}\int^{\pi/2} k\cdot C_n^*(k\gamma c)\{-\cos^2\gamma\cdot g_1(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_2(k\gamma c)\}d\gamma dk$$

$$+\int^{\infty}\int^{\pi/2} k\cdot C_n^{**}(k\gamma c)\{\cos^2\gamma\cdot g_2(k\gamma c)+\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)\}d\gamma dk\Big]$$

****

③− a : Integration for the terms timed by $g_1(k\gamma c)$

$$\int^{\infty}\int^{\pi/2}K\big[A_n^*(-\cos^2\gamma)+A_n^{**}\cdot\cos\gamma\cdot\sin\gamma\big]g_1(k\gamma c)d\gamma dk$$

$$=\left(4/\pi^2\right)\int^{\infty}\int^{\pi/2}K\big[\{n\cdot(2n-1)\cdot F_2(\alpha,\beta,c,n,1)$$

$$+(n-2)/2\cdot[F_2(\alpha,\beta,c,n-1,2)-F_2(\beta,\alpha,c,n-1,2)]$$

$$-n(n+1)(n-2)\cdot F_1(\alpha,\beta,c,n-1,0)\}(-\cos^2\gamma)$$

$$+\{n\cdot(2n-1)F_3(\alpha,\beta,c,n,1)+(n-2)\cdot F_3(\alpha,\beta,c,n-1,2)\}\cos\gamma\cdot\sin\gamma\big]$$

$$\cdot g_1(k\gamma c)d\gamma dk$$

$$=\left(4/\pi^2\right)\int^{\infty}\int^{\pi/2}K\big[\{n\cdot(2n-1)\cdot F_2(\alpha,\beta,c,n,1)$$

$$+(n-2)/2\cdot[F_2(\alpha,\beta,c,n-1,2)-F_2(\beta,\alpha,c,n-1,2)]$$

$$-n(n+1)(n-2)\cdot F_1(\alpha,\beta,c,n-1,0)\}\big](-\cos^2\gamma)\cdot g_1(k\gamma c)d\gamma dk$$

$$+\left(4/\pi^2\right)\int^{\infty}\int^{\pi/2}K\big[\{n\cdot(2n-1)\cdot F_3(\alpha,\beta,c,n,1)$$

$$-(n-2)\cdot F_3(\alpha,\beta,c,n-1,2)\}\big]\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)d\gamma dk$$

$$F_2(\alpha,\beta,c,n,1)=\pi/2\cdot c^3\Bigg[\sum_{g=0}^{[n/2]}S_{n,1,q}(kc)^{n-g-3/2}\cdot K_{n-q-3/2}$$

$$-\sum_{g=0}^{[n/2]}S_{n,1,q}(kc)^{n-g-5/2}\cdot c^2\cdot\alpha\cdot K_{n-q-5/2}\Bigg]$$

①

$$=\pi/2\cdot c^2\Bigg[\sum_{g=0}^{[n/2]}S_{n,1,q}(kc)^{n-g-3/2+1}\Big/k\cdot K_{n-q-3/2}$$

$$-\sum_{g=0}^{[n/2]}S_{n,1,q}(kc)^{n-g-5/2+3}\Big/k^3\cdot k^2\cdot\cos^2\gamma\cdot\cdot K_{n-q-5/2}\Bigg]$$

※ from the definition of c-5

$$B_{n,m,j,l}=\sum_{q=0}^{[n/2]}S_{n,m,q}(kc)^{n-q+l-1/2}\cdot K_{n-q-j-1/2}$$

putting : $+1/2=l-1/2$ $-3/2=-j-1/2$ $\therefore l=0\ j=1$

putting : $1/2=l-1/2$ $-5/2=-j-1/2$ $\therefore l=1\ j=2$

$$=\pi/2\cdot c^2\left[B_{n,1,1,0}/k-B_{n,1,2,1}/k^3\cdot k^2\cos^2\gamma\right]=\pi/2\cdot c^2\left[B_{n,1,1,0}-\cos^2\gamma B_{n,1,2,1}\right]/k$$

Setting $c=z_i$ $=\pi/2\cdot z_i^2\left[B_{n,1,1,0}-\cos^2\gamma B_{n,1,2,1}\right]/k$

=================================

②$F_2(\alpha,\beta,c,n-1,2)$

$$=\pi/2\cdot c^3\Bigg[\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}\cdot(kc)^{n-1-q-3/2}K_{n-1-q-3/2}$$

$$-\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}\cdot(kc)^{n-1-q-5/2}\cdot\alpha^2\cdot c^2\cdot K_{n-1-q-5/2}\Bigg]$$

$$=\pi/2\cdot c^2\Bigg[\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}\cdot(kc)^{n-q-5/2+1}\Big/k\cdot K_{n-1-q-5/2}$$

$$-\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}\cdot(kc)^{n-q-7/2+3}\Big/k^3\cdot k^2\cos^2\gamma K_{n-q-7/2}\Bigg]$$

※ from the definition of c-5

$$B_{n-1,2,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q+l-1/2}K_{n-1-q-j-1/2}$$

putting $-3/2=-1+l-1/2$ $-5/2=-1-j-1/2$ $\therefore l=0\ j=1$

putting $-1/2=l-3/2$ $-7/2=-1-j-1/2$ $\therefore l=1\ j=2$

$$=\pi/2\cdot c^2\left[B_{n-1,2,1,0}/k-B_{n-1,2,2,1}/k\cos^2\gamma\right]$$

$$=\pi/2\cdot c^2\left[B_{n-1,2,1,0}-\cos^2\gamma\cdot B_{n-1,2,2,1}\right]/k$$

By the similar procedure, we have

$$F_2(\gamma,k,c,n-1,2)=\pi/2\cdot c^2\left[B_{n-1,2,1,0}-\sin^2\gamma B_{n-1,2,2,1}\right]/k$$

$$F_2(\alpha,\beta,c,n-1,2)-F_2(\beta,\alpha,c,n-1,2)=\pi/2\cdot c^2/k\left[\sin^2\gamma-\cos^2\gamma\right]\cdot B_{n-1,2,2,1}$$

here, inter changing c and z_i

$$=\pi/2\cdot z_i^2(1-2\cos^2\gamma)\cdot B_{n-1,2,2,1}/k$$

---

③$F_1(\alpha,\beta,c,n-1,0)$

$$=\pi/2\cdot c\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n-1-q-1/2}K_{n-1-q-1/2}$$

$$=\pi/2\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n+1-q-3/2}/k\,K_{n-q-3/2}$$

※ from the definition

$$B_{n-1,0,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n-1-q+l-1/2}\cdot K_{n-1-q-j-1/2}$$

$-1/2=-1+l-1/2$ $-3/2=-1-j-1/2$ $l=1$ $j=0$

$$=\pi/2\cdot B_{n-1,0,0,1}/k$$

As a result

The 1st integ $=\left(4/\pi^2\right)\int^{\infty}\int^{\pi/2}\Big[\Big[K\cdot\{n\cdot(2n-1)\cdot\pi/2\cdot z_i^2\left[B_{n,1,1,0}-\cos^2\gamma B_{n,1,2,1}\right]/k$

$$+(n-2)/2\left[\pi/2\cdot z_i^2/k\cdot(1-2\cos^2\gamma)\cdot B_{n-1,2,2,1}\right]$$

$$-n(n+1)(n-2)/2\cdot\pi/2\cdot B_{n-1,0,0,1}/k\}\Big]\cos^2\gamma g_1 d\gamma dk$$

$$=\left(4/\pi^2\right)\pi/2\int^{\infty}\int^{\pi/2}\Big[\Big[n\cdot(2n-1)\cdot z_i^2(B_{n,1,1,0}-\cos^2\gamma B_{n,1,2,1})$$

$$+(n-2)/2\cdot z_i^2\cdot(1-2\cos^2\gamma)\cdot B_{n-1,2,2,1}$$

$$-n(n+1)(n-2)/2\cdot B_{n-1,0,0,1}\Big]\cos^2\gamma g_1 d\gamma dk$$

① $F_3(\alpha,\beta,c,n,1)=\pi/2\cdot\alpha\cdot\beta\cdot c^5\cdot\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2}\cdot K_{n-q-5/2}$

$$=\pi/2\cdot k^2\cos\gamma\cdot\sin\gamma\cdot c^2\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2+3}/k^3K_{n-q-5/2}$$

$$=\pi/2\cdot c^2\cos\gamma\cdot\sin\gamma\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q+1/2}/k\,K_{n-q-5/2}$$

※ from the def.

$$B_{n,1,j,l}=\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q+l-1/2}K_{n-q-j-1/2}$$

$1/2=l-1/2$ $-5/2=-j-1/2$ $\therefore l=1\ j=2$

$$=\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\cdot B_{n,1,2,1}/k$$

NII-Electronic Library Service

② $F_3(\alpha,\beta,c,n-1,2)=\pi/2\cdot\alpha\cdot\beta\cdot c^5\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,q}(kc)^{n-1-q-5/2}\cdot K_{n-1-q-5/2}$

$=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n\ \ -q-7/2+3}/k^3\cdot K_{n\ \ -q-7/2}$

※ from the definition of c-5

$$B_{n-1,2,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q+l-1/2}K_{n-1-q-j-1/2}$$

$-1/2=-1+l-1/2 \qquad -7/2=-1-j-1/2 \qquad \therefore l=1\ j=2$

$=\pi/2\cos\gamma\cdot\sin\gamma\cdot c^2/k\cdot B_{n-1,2,2,1}$

Therefore, the second integral is

$=(-4/\pi^2)\int_0^\infty\int_0^{\pi/2}k\{n\cdot(2n-1)\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\cdot B_{n,1,2,1}\cdot 1/k$

$\quad+(n-2)\cdot\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2/k\cdot B_{n-1,1,2,1}\}\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)d\gamma dk$

※ inter changing c and z_i

$=(2/\pi)\int\int\{n\cdot(2n-1)B_{n,1,2,1}+(n-2)\cdot B_{n-1,1,2,1}\}z_i^2$

$\quad\cos^2\gamma\cdot\sin^2\gamma\cdot g_1(k\gamma c)d\gamma dh$

Adding this to the first integral

$=(2/\pi)\int_0^\infty\int_0^{\pi/2}[[n\cdot(2n-1)z_i^2[B_{n,1,1,0}-\cos^2\gamma\cdot B_{n,1,2,1}-\sin^2\gamma\cdot B_{n,12,1}]$

$\quad+(n-2)/2\cdot z_i^2\cdot B_{n-1,2,2,1}[1-2\cos^2\gamma-2\sin^2\gamma]$

$\quad-n(n+1)(n-2)/2\cdot B_{n-1,0,0,1}]]\cos^2\gamma g_1\cdot d\gamma dk$

$=(2/\pi)\int_0^\infty\int_0^{\pi/2}[[n\cdot(2n-1)z_i^2(B_{n,1,1,0}-B_{n,1,2,1})-(n-2)/2\cdot z_i^2\cdot B_{n-1,2,2,1}$

$\quad-n(n+1)(n-2)/2\cdot B_{n-1,0,0,1}]]\cos^2\gamma\cdot g_1 d\gamma dk$

Since the integration

※ $\int^{\pi/2}\cos^2\gamma\cdot g_1 d\gamma dk=\pi/2\,1/(k^2\rho^2)\cdot[k^2x^2J_0+(y^2-x^2)k^2J_1/(k\rho)]$

$\quad=\pi/2\cdot 1/\rho^2[x^2\cdot J_0+(y^2-x^2)\cdot J_1/(k\rho)]$

the terms of coefficient of A_n that are timed by $g_1(k\gamma c)$

$=-\int_0^\infty[-n\cdot(2n-1)\cdot z_i^2\cdot(B_{n,1,1,0}-B_{n,1,2,1})$

$\quad+(n-2)/2\cdot z_i^2\cdot B_{n-1,2,2,1}+n(n+1)(n-2)/2\cdot B_{n-1,0,0,1}]B_1/\rho^2\,d\gamma$

$=+H_2(z_i)$

****

③－6：

1 Integration for $g_2(k,\gamma,c)$

$\int_0^\infty\int_0^{\pi/2}k\cdot[A_n^*\cos\gamma\sin\gamma\ \ g_2(k\gamma c)+A_n^{**}\cos^2\gamma g_2(k\gamma c)]d\gamma dk$

$=(-4/\pi)\int_0^\infty\int_0^{\pi/2}k[[\{n\cdot(2m-1)\cdot F_2(\alpha,\beta,c,n,1)$

$\quad+(n-2)/2\cdot[F_2(\alpha,\beta,c,n-1,2)-F_2(\beta,\alpha,c,n-1,2)]$

$\quad-n\cdot(n+1)(n-2)/2\cdot F_1(\alpha,\beta,c,n-1,0)\}\cdot\cos\gamma\cdot\sin\gamma$

$\quad+\{n\cdot(2n-1)\cdot F_2(\alpha,\beta,c,n,1)+(n-2)F_3(\alpha,\beta,c,n-1,2)\}]]$

$\quad\cos^2\gamma\, g_2(k\gamma c)d\gamma dk$

① $F_2(\alpha,\beta,c,n,1)$

$=\pi/2\cdot c^3\cdot\left[\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-3/2}\cdot K_{n-q-3/2}-\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2}\cdot\alpha^2\cdot c^2\cdot K_{n-q-5/2}\right]$

$=\pi/2\cdot\left[\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-3/2+3}/k^3\cdot K_{n-q-3/2}-\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2+5}/k^5\cdot k^2\cdot\cos^2\gamma\cdot K_{n-q-5/2}\right]$

※ from the definition of c-5

$$B_{n,1,j,l}=\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q+l-1/2}K_{n-q-j-1/2}$$

putting $3/2=l-1/2 \quad -3/2=-j-1/2 \quad \therefore l=2\ j=1$

putting $5/2=l-1/2 \quad -5/2=-j-1/2 \quad \therefore l=3\ j=2$

$=\pi/2\cdot[B_{n,1,1,2}-\cos^2\gamma B_{n,1,2,3}]/k^3$

==================================

② $F_2(\alpha,\beta,c,n-1,2)$

$=\pi/2\cdot c^3\cdot\left[\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q-3/2}K_{n-1-q-3/2}\right.$

$\quad\left.-\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q-5/2}\cdot\alpha^2\cdot c^2K_{n-1-q-5/2}\right]$

$=\pi/2\cdot\left[\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n\ \ -q-5/2+3}/k^3\cdot K_{n\ \ -q-3/2}\right.$

$\quad\left.-\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n\ \ -q-7/2+5}/k^5\cdot k^2\cdot\cos^2\gamma\cdot K_{n\ \ -q-7/2}\right]$

$$B_{n-1,2,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,1,q}(kc)^{n-1-q+l-1/2}K_{n-1-q-j-1/2}$$

putting $1/2=-1+l-1/2 \quad -5/2=-1-j-1/2 \quad \therefore l=2\ j=1$

putting $3/2=l-3/2 \qquad -7/2=-j-3/2 \qquad \therefore l=3\ j=1$

$=\pi/2\cdot[B_{n-1,2,1,2}/k^3-k^2\cos^2\gamma/k^5\cdot B_{n-1,2,2,3}]$

By the similar procedure, we have

$F_2(\beta,\alpha,c,n-1,2)=\pi/2\cdot 1/k^3[B_{n-1,2,1,2}-\sin^2\gamma B_{n-1,2,2,3}]$

G③ $F_1(\alpha,\beta,c,n-1,0)$

$=\pi/2\cdot c\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n-1-q-1/2}K_{n-1-q-1/2}$

$=\pi/2\cdot 1/c^2\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n\ \ -q-3/2+3}/k^3K_{n\ \ -q-3/2}$

$$B_{n-1,0,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,0,q}(kc)^{n-1-q+l-1/2}K_{n-1-q-j-1/2}$$

$3/2=-1+l-1/2 \quad -3/2=-1-j-1/2 \quad \therefore l=3\ j=0$

$=\pi/2\cdot 1/c^2\cdot B_{n-1,0,0,3}/k^3=\pi/2\cdot 1/z_i^2\cdot B_{n-1,0,0,3}/k^3$

+===

④ $F_3(\alpha,\beta,c,n\ \ ,1)$

$=\pi/2\cdot\alpha\cdot\beta\cdot c^5\cdot\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2}K_{n-q-5/2}$

$=\pi/2\cdot k^2\cdot\cos\gamma\sin\gamma\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q-5/2+5}/k^5K_{n-q-5/2}$

$$B_{n,1,j,l}=\sum_{q=0}^{[n/2]}S_{n,1,q}(kc)^{n-q+l-1/2}K_{n-q-j-1/2}$$

$5/2=l-1/2 \quad -5/2=-j-1/2 \quad \therefore l=3\ j=2$

$=\pi/2\cdot\cos\gamma\sin\gamma\cdot B_{n-1,1,2,3}/k^3$

==

⑤ $F_3(\alpha,\beta,c,n-1,2)$

$=\pi/2\cdot\alpha\cdot\beta\cdot c^5\cdot\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q-5/2}K_{n-1-q-5/2}$

$=\pi/2\cdot k^2\cdot\cos\gamma\sin\gamma\sum_{q=0}^{[n/2]}S_{n-1,2,q}(kc)^{n-q-7/2+5}/k^5K_{n-q-7/2}$

$$B_{n-1,2,j,l}=\sum_{q=0}^{[(n-1)/2]}S_{n-1,2,q}(kc)^{n-1-q+l-1/2}K_{n-1-q-j-1/2}$$

$3/2=-1+l-1/2 \quad -7/2=-1-j-1/2 \quad \therefore l=3\ j=2$

$=\pi/2\cdot\cos\gamma\sin\gamma/k^3\,B_{n,1,2,2,3}$

Associating these

$(-4/\pi^2)\int_0^\infty\int_0^{\pi/2}k\cdot[[\{n\cdot(2n-1)\cdot\pi/2\cdot[B_{n,1,1,2}-\cos^2\gamma\cdot B_{n1,2,3}]/k^3$

$\quad+(n-2)/2\cdot[\pi/2\cdot 1/k^3(1-2\cos^2\gamma)]B_{n-1,2,2,3}$

$\quad-n(n+1)(n-2)/2\cdot\pi/2\cdot 1/z_i^2\cdot B_{n-1,0,0,3}/k^3\}\cos\gamma\cdot\sin\gamma$

$\quad+\{n\cdot(2n-1)\cdot\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot B_{n,1,2,3}/k^3$

$\quad+(n-2)\cdot\pi/2\cdot\cos\gamma\cdot\sin\gamma/k^3\cdot B_{n-1,2,2,3}\}\cos^2\gamma]]g_2(k\gamma c)d\gamma dk$

NII-Electronic Library Service

Institute of Electronics, Information, and Communication Engineers

$$= \left(-4/\pi^4\right)2/\pi\int^{\infty}\int^{\pi/2} 1/k^2 \cdot \Big[\Big[\Big\{n\cdot(2n-1)\cdot B_{n,1,1,2}\cdot\cos\gamma\sin\gamma$$
$$+ n(2n-1)\left\{-\cos^2\gamma\cdot\cos\gamma\sin\gamma+\cos\gamma\sin\gamma\cdot\cos^2\gamma\right\}B_{n,1,2,3}$$
$$+(n-2)/2\left\{1-2\cos^2\gamma)\cdot\cos\gamma\sin\gamma+2\cos\gamma\sin\gamma\cdot\cos^2\gamma\right\}B_{n-1,2,2,3}$$
$$-n\cdot(n+1)(n-2)/2\cdot 1/z_i^2\cdot B_{n-1,0,0,3}\cdot\cos\gamma\sin\gamma\Big]\Big]g_2(k\gamma c)d\gamma dk$$

$$= -2/\pi\int^{\infty}\int^{\pi/2} 1/k^2\cdot\Big[\Big[n\cdot(2n-1)\cdot B_{n,1,1,2}\cdot\cos\gamma\sin\gamma$$
$$+(n-2)/2\cdot\cos\gamma\sin\gamma\cdot B_{n-1,2,2,3}$$
$$-n(n+1)(n-2)/2\cdot 1/z_i^2 B_{n-1,0,0,3}\cdot\cos\gamma\sin\gamma\Big]\Big]g_2(k\gamma c)d\gamma dk$$
$$= +2/\pi\int^{\infty}\int^{\pi/2} 1/k^2\cdot\Big[\Big[-n\cdot(2n-1)\cdot B_{n,1,1,2}\qquad -(n-2)/2\cdot B_{n-1,2,2,3}$$
$$+n(n+1)(n-2)/(2\cdot z_i^2)\cdot B_{n-1,0,0,3}\Big]\Big]\cos\gamma\sin\gamma\cdot g_2(k\gamma c)d\gamma dk$$

※ $\int_0^{\pi/2}\cos\gamma\sin\gamma g_2(k\gamma c)d\gamma dk = -\pi/2\cdot k^2 xy/(k^2\rho^2)\left[J_0(k\rho)-2\cdot J_1/(k\rho)\right]$

$$= \int^{\infty} xy/\left(k^2\rho^2\right)\left[J_0(k\rho)-2\cdot J_1(k\rho)/(k\rho)\right]$$
$$\left[-n\cdot(2n-1)\cdot B_{n,1,1,2}-(n-2)/2\cdot B_{n-1,2,2,3}+n(n+1)(n-2)\cdot B_{n-1,0,0,3}/(2z_i^2)\right]$$
$$= x\cdot y\cdot B2\cdot\left[-n\cdot(2n-1)\cdot B_{n,1,1,2}-(n-2)/2\cdot B_{n-1,2,2,3}\right.$$
$$\left.+n(n+1)(n-2)\cdot B_{n-1,0,0,3}/(2z_i^2)\right]$$
$$= H_{10}$$

****

③－b : Coefficients of $\sum_n B_n$

③－b・1 B_n^* is given by A_2 and B_n^{**} is given by A_5

$$\int^{\infty}\int^{\pi/2} k\cdot B_n^*(k\gamma c)(-\cos^2\gamma\cdot g_1(k\gamma c))d\gamma dr$$
$$+\int^{\infty}\int^{\pi/2} k\cdot B_n^{**}(k\gamma c)\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)d\gamma d$$
$$= \left(-4/\pi^2\right)\cdot\int^{\infty}\int^{\pi/2}\Big[\Big[k\cdot\left\{-1/2\cdot\left[F_2(k,\gamma,c,n+1,2)-F_2(\gamma,k,c,n+1,2)\right]\right.$$
$$+n(n+1)/2\cdot F_1(k,\gamma,c,n+1,0)\Big\}(-\cos^2\gamma\cdot g_1(k\gamma c))$$
$$+k\left(-F_3(k,\gamma,c,n+1,2)\cdot\cos\gamma\cdot\sin\gamma\cdot g_1(k\gamma c)\right)\Big]\Big]d\gamma dk$$

1\] $F_1(k,\gamma,c,n+1,0)=\pi/2\cdot c\cdot\sum_{q=0}^{[(n+1)/2]} S_{n+1,0,q}(c)\cdot(kc)^{n+1-q-1/2}K_{n+1-q-1/2}(kc)$

$$= \pi/2\cdot\sum_{q=0}^{[(n+1)/2]} S_{n+1,0,q}(c)\cdot(kc)^{n\ -q+3/2}K_{n\ -q-1/2}(kc)/k$$

※

$$\sum_{q=0}^{[(n+1)/2]} S_{n+1,0}q(c)(kc)^{n+1-q+l-1/2}K_{n+1-q-j-1/2}(kc) = B_{n+1,0,j,l}$$
$$\therefore 1/2+l=3/2\qquad 1/2=1-j-1/2\qquad\therefore l=1\ \ j=0$$

$\therefore F_1(k,\gamma,c,n+1,0)=\pi/2\cdot B_{n+1,0,0,1}/k$

2\] $F_2(k,\gamma,c,n+1,2)=\pi/2\cdot c^3\cdot\left[\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}\cdot(kc)\cdot(kc)^{n+1-q-3/2}K_{n+1-q-3/2}(kc)\right.$

$$\left.-\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}\cdot(kc)^{n+1-q-5/2}k^2\cos^2\gamma c^2 K_{n+1-q-5/2}(kc)\right]$$

$$= \pi/2\cdot c^2\left[\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(kc)^{n+1-q-3/2+1}/k\cdot K_{n+1-q-3/2}\right.$$
$$\left.-\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(c)k^2\cos^2\gamma c^3(kc)^{n+1-q-5/2}/k\cdot K_{n+1-q-5/2}(kc)\right]$$
$$= \pi/2\cdot c^2\left[\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(kc)^{n+1-q+1/2}\cdot K_{n\ -q-1/2}/k\right.$$
$$\left.-\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(c)(kc)^{n\ -q-3/2+3}/k^3\cdot K_{n\ -q-3/2}\cdot k^2\ \cos^2\gamma\right]$$

※

$$\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(c)(kc)^{n+1-q+l-1/2}\cdot K_{n+1-q-j-1/2} = B_{n+1,2,k,l}$$

putting $1/2=l+1/2\qquad -1/2=1/2-j\quad\therefore l=0\ \ j=1$
putting $3/2=l+1/2\qquad -3/2=1/2-j\quad\therefore l=1\ \ j=2$

$$= \pi/2\cdot c^2\left[B_{n+1,2,1,0}-\cos^2\gamma\cdot B_{n+1,2,2,1}\right]k$$

By the similar procedure, we have

$$F_2(\gamma,k,c,n+1,2)=\pi/2\cdot c^2\left[B_{n+1,2,1,0}-\sin^2\gamma\cdot B_{n+1,2,2,1}\right]k$$

3\] $F_3(k,\gamma,c,n+1,2)=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(kc)^{n+1-q-5/2}\cdot c^5\cdot K_{n+1-q-5/2}(kc)$

$$= \pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(kc)^{n\ -q-3/2+3}/k^3\cdot K_{n\ -q-3/2}$$

※

$$\sum_{q=0}^{[(n+1)/2]} S_{n+1,2,q}(kc)^{n-q+l+1/2}\cdot K_{n\ -q-j+1/2} = B_{n+1,2,j,l}$$
$$3/2=l+1/2\qquad -3/2=-j+1/2\qquad\therefore l=1\ \ j=2$$

$$= \pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2/k\cdot B_{n+1,2,2,1}$$

Hence

③・b・1

$$= (-4/\pi^2)\int^{\infty}\int^{\pi/2}\Big[\Big[k\cdot(-1/2)\cdot\pi/2\cdot c^2/k\cdot\left[B_{n+1,2,1,0}-\cos^2\gamma\cdot B_{n+1,2,2,1}\right.$$
$$\left.-B_{n+1,2,1,0}+\sin^2\gamma\cdot B_{n+1,2,2,1}\right]$$
$$+k\cdot n\cdot(n+1)/2\cdot\pi/2\cdot B_{n+1,0,0,1}/k\Big\}(-\cos^2\gamma\cdot g_1(k\gamma c))$$
$$-k\cdot\pi/2\cdot\cos\gamma\sin\gamma\cdot c^2/k\cdot B_{n+1,2,2,1}\cdot\cos\gamma\sin\gamma\cdot g_1(k\gamma c)\Big]\Big]$$
$$= (-4/\pi^2)\pi/2\int^{\infty}\int^{\pi/2}$$
$$\Big[\Big[\left\{\left(1-2\cos^2\gamma\right)B_{n+1,2,2,1}^{(-1/2)}\cdot c^2(-\cos^2\gamma)-\cos^2\gamma\sin^2\gamma\cdot c^2\cdot B_{n+1,2,2,1}\right\}g_1(k\gamma c)$$
$$+n(n+1)/2B_{n+1,0,0,1}\Big](-\cos^2\gamma\cdot g_1(k\gamma c))\Big]\Big]$$
$$= (-4/\pi^2)\pi/2\int^{\infty}\int^{\pi/2}$$
$$\Big[\Big[(\cos^2\gamma-1/2+\sin^2\gamma)\cdot c^2\cdot B_{n+1,2,2,1}(-\cos^2\gamma\cdot g_1(k\gamma c)$$
$$+n(n+1)/2B_{n+10,0,1})(-\cos^2\gamma g_1(k\gamma c))\Big]\Big]$$
$$= (-4/\pi^2)\pi/2\cdot 1/2\int^{\infty}\int^{\pi/2}(-1)$$
$$\left[c^2B_{n+1,2,2,1}+(n)(n+1)\cdot B_{n+10,0,1}\right]\cos^2\gamma\cdot g_1(k\gamma c)$$
$$= (-4/\pi^2)\pi/2\cdot 1/2\cdot H_5$$

****

③・6・2 The components of $g_2(k\gamma c)$ are

$$\int^{\infty}\int^{\pi/2} k\left[B_n^*(k\gamma c)\cdot\cos\gamma\cdot\sin\gamma g_2(k\gamma c)+B_n^{**}\cos^2\gamma g_2(k\gamma c)\right]d\gamma dk$$
$$= \int^{\infty}\int^{\pi/2}\left(-4/\pi^2\right)k\Big[\Big[\left\{-1/2\left[F_2(k,\gamma,c,n+1,2)-F_2(,k,c,n+1,2)\right]\right.$$
$$+n(n+1)/2\cdot F_1(k,\gamma,c,n+1,0)\Big\}\cos\gamma\cdot\sin\gamma$$
$$-F_3(k,\gamma,c,n+1,2)\cos^2\gamma\Big]\Big]g_2(k,\gamma,c)dkd\gamma$$

1\] $F_1(k,\gamma,c,n+1,0)=\pi/2\cdot c\cdot\sum_{q=0}^{[(n+1)/2]} S_{n+1,0,q}(c)(kc)^{n-q+1/2}\cdot K_{n-q+1/2}(kc)$

$$= \pi/2\cdot 1/c^2\cdot\sum_{q=0}^{[(n+1)/2]} S_{n+1,0,q}(c)(kc)^{n-q+1/2+3}/k^3\cdot K_{n-q+1/2}(kc)$$

※

$$Bn+1,0,j,l = \sum_{q=0}^{[(n+1)/2]} S_{n+1,0,q}(c)(kc)^{n-q+l+1/2}\cdot K_{n-q-j+1/2}$$
$$7/2=l+1/2\qquad 1/2=-j+1/2\qquad\therefore l=3\ \ j=0$$

$$= \pi/2\cdot 1/c^2\cdot B_{n+1,0,0,3}/k^3$$

NII-Electronic Library Service

2] $F_2(k\gamma,c,n+1,2)=\pi/2\cdot c^3\left[\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n+1-q-3/2}K_{n+1-q-3/2}(kc)\right.$

$\left.-\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n+1-q-5/2}\cdot k^2\cdot\cos^2\gamma\cdot c^2K_{n+1-q-5/2}(kc)\right]$

$=\pi/2\cdot\left[\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n+1-q-3/2+3}/k^3K_{n-q-1/2}(kc)\right.$

$\left.-\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n-q-3/2+5}/k^5\cdot k^2\cdot\cos^2\gamma\cdot\quad K_{n-q-3/2}(kc)\right]$

※

$$B_{n+1,2,j,l}=\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n-q+l+1/2}\cdot K_{n-q-j+1/2}$$

putting $1+3/2=l+1/2\quad -1/2=-j+1/2\quad \therefore l=2\ j=1$

putting $7/2=l+1/2\quad -3/2=-j+1/2\quad \therefore l=3\ j=2$

$=\pi/2\cdot\left[B_{n+1,2,1,2}-\cos^2\gamma B_{n+1,2,2,3}\right]/k^3$

Hence,

$-1/2[F_2(k,\gamma,c,n+1,2)-F_2(\gamma,k,c,n+1,2)]$

$=-1/2\left[B_{n+1,2,1,2}-\cos^2\gamma\cdot B_{n+1,2;2,3}-B_{n+1,2,1,2}+\sin^2\gamma\cdot B_{n+1,2,2,3}\right]/k^3\cdot\pi/2$

$=-1/2(1-2\cos^2\gamma)/k^3\cdot B_{n+1,2,2,3}\cdot\pi/2$

****

3] $F_3(k,\gamma,c,n+1,2)=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^5\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)\cdot(kc)^{n+1-q-5/2}K_{n+1-q-5/2}(kc)$

$=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)\cdot(kc)^{n-q-3/2+5}/k^5K_{n-q-3/2}(kc)$

※

$$B_{n+1,2,j,l}=\sum_{q=0}^{[(n+1)/2]}S_{n+1,2,q}(c)(kc)^{n-q+l+1/2}\cdot K_{n-q-j+1/2}$$

$7/2=l+1/2\quad -3/2=-j+1/2\quad \therefore l=3\ j=2$

$=\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot B_{n+1,2,2,3}/k^3$

As a result,

$\int^\infty\int^{\pi/2}(-4/\pi^2)\cdot k\cdot\left[\left[(\cos^2\gamma-1/2)/k^3B_{n+1,2,2,3}\cdot\pi/2+n(n+1)/2\cdot\pi/2\cdot1/c^2\right.\right.$

$\cdot\cos\gamma\cdot\sin\gamma\quad\cdot B_{n+1,0,0,3}/k$

$\left.\left.+(-\pi/2\cdot\cos\gamma\cdot\sin\gamma B_{n+1,2,2,3}/k^3)\cos^2\gamma\right]\right]g_2(k\gamma c)d\gamma dk$

$=(-4/\pi^2)\cdot(\pi/2)\int^\infty\int^{\pi/2}(1/k^2)\cdot g_2(k\gamma c)d\gamma dk$

$\left[\left[\{\cos^2\gamma-1/2)\cdot\cos\gamma\cdot\sin\gamma-\cos\gamma\cdot\sin\gamma\right\}B_{n+1,2,2,3}\right.$

$\left.+n(n+1)/2\cdot1/c^2\cdot B_{n+1,0,0,3}\cdot\cos\gamma\cdot\sin\gamma\right]\Big]$

※

$\cos^2\gamma-1/2-\cos^2\gamma=-1/2$

$=4/\pi^2\cdot\pi/2\cdot\int^\infty\int^{\pi/2}(1/k^2)\cdot g_2(k\gamma c)\cdot\cos\gamma\cdot\sin\gamma d\gamma dk$

$\left[\left[1/2\cdot B_{n+1,2,2,3}-n(n+1)\cdot1/2\cdot1/c^2\cdot B_{n+1,0,0,3}\right]\right]$

$=x\cdot y/2\cdot B_2\left[B_{n+1,2,2,3}-n(n+1)/c^2B_{n+1,0,0,3}\right]$

$=H_{13}$

---

③－c : Coefficients of $\sum_n C_n$

③－c－1 Integral for $(g_1(k\gamma c))$

$\int^\infty\int^{\pi/2}k\left[\left[C_n^*(k\gamma c)\cdot(-\cos^2\gamma)+C_n^{**}(k\gamma c)\cdot\cos\gamma\cdot\sin\gamma\right]\right]g_1(k\gamma c)d\gamma dk$

from (A_3,A_6)

$=\int^\infty\int^{\pi/2}(-4/\pi^2)\cdot k\left[\left[1/2\{F_2(k,\gamma,c,n,2)-F_2(\gamma,k,c,n,2)\}\right.\right.$

$\left.+n(n+1)/2F_2(k,\gamma,c,n,0)\right](-\cos^2\gamma)$

$\left.+F_3(k,\gamma,c,n,2)\cdot\cos\gamma\cdot\sin\gamma\right]g_1(k\gamma c)d\gamma dk$

1] $F1(k,\gamma,c,n,0)=\pi/2\cdot c\cdot\sum_{q=0}^{[n/2]}S_{n,0q}(c)(kc)^{n-q-1/2}K_{n-q-1/2}(kc)$

$=\pi/2\cdot\sum_{q=0}^{[n/2]}S_{n,0,q}(c)(kc)^{n-q-1/2+1}/k\cdot K_{n-q-1/2}(kc)$

$$B_{n,0,j,l}(c)=\sum_{q=0}^{[n/2]}S_{n,0,q}(c)(kc)^{n-q+l-1/2}K_{n-q--j1/2}$$

$1/2=l-1/2\quad -1/2=-j-1/2\quad \therefore l=1\ j=0$

$=\pi/2\cdot B_{n,0,0,1}/k$

2] $F_2(k,\gamma,c,n,2)=\pi/2\cdot c^3\left[\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q-3/2}\cdot K_{n-q-3/2}(kc)\right.$

$\left.-\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q-5/2}\cdot k^2\cdot\cos^2\gamma\cdot c^2K_{n-q-5/2}(kc)\right]$

$=\pi/2\cdot c^2\left[\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q-3/2+1}/k\cdot K_{n-q-3/2}(kc)\right.$

$\left.-\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q-5/2+3}/k^3\cdot k^2\cdot\cos^2\gamma\cdot K_{n-q-5/2}(kc)\right]$

※

$$B_{n,2,j,l}(c)=\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q+l-1/2}\cdot K_{n-q-j-1/2}(kc)$$

putting $-1/2=l-1/2\quad -3/2=-j-1/2\quad \therefore l=0\ \ j=1$

putting $1/2=l-1/2\quad -5/2=-j-1/2\quad \therefore l=1\quad j=2$

$=\pi/2\cdot c^2\left[B_{n,2,1,0}-\cos^2\gamma\cdot B_{n,2,2,1}\right]/k$

Hence

$1/2[F_2(k,\gamma,c,n,2)-F_2(\gamma,k,c,n,2]$

$=\pi/2\cdot c^2/k\left[(-\cos^2\gamma+\sin^2\gamma)\right]B_{n,2,2,1}\cdot1/2$

$=\pi/2\cdot c^2/k\cdot(1/2-\cos^2\gamma)\cdot B_{n,2,2,1}$

3] $F_3(k,\gamma,n,2)=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^5\sum_{q=0}^{[n/2]}S_{n,2,q}(kc)^{n-q-5/2}K_{n-q-5/2}$

$=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\sum_{q=0}^{[n/2]}S_{n,2,q}(kc)^{n-q-5/2+3}/k^3K_{n-q-5/2}$

$$B_{n,2,j,l}=\sum_{q=0}^{[n/2]}S_{n,2,q}(c)(kc)^{n-q+l-1/2}\cdot K_{n-q-j-1/2}(kc)$$

$1/2=l-1/2\quad -5/2=-j-1/2\quad \therefore j=2\quad l=1$

$=\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot c^2\cdot B_{n,2,2,1}/k$

Associating these

$(-\pi/4)\cdot\pi/2\cdot\int^\infty\int_K^{\pi/2}\left[\left[c^2/k\cdot(1/2-\cos^2\gamma)\cdot B_{n,2,2,1}+n(n+1)/2\cdot B_{n,0,0,1}/k\right](-\cos^2\gamma)\right.$

$\left.+\cos\gamma\cdot\sin\gamma\cdot c^2\cdot B_{n,2,2,1}/k\cdot\cos\gamma\cdot\sin\gamma\right]g_1(k\gamma c)d\gamma dk$

※

$-1/2+\cos^2\gamma+\sin^2\gamma=1/2$

$=(-\pi/4)\cdot\pi/2\cdot\int^\infty\int^{\pi/2}\left[\left[c^2\cdot1/2\cdot B_{n,2,2,1}-n(n+1)/2\cdot B_{n,0,0,1}\right]\right]\cos^2\gamma\cdot g_1(k\gamma c)d\gamma dk$

$=-\int^\infty1/2\left[c^2\cdot B_{n,2,2,1}-n(n+1)B_{n,0,0,1}\right]B_1/\rho^2$

$=-H_8(l)$

③・c－2 : Integration for $g_2(k\gamma c)$

$\int^\infty\int^{\pi/2}k\left[\left[C_n^*(k\gamma c)\cdot\cos\gamma\cdot\sin\gamma+C_n^{**}(k\gamma c)\cdot\cos^2\gamma\right]\right]g_2(k\gamma c)d\gamma dk$

(A_3,A_6)

$=\int^\infty\int^{\pi/2}k\cdot(-4/\pi^2)\left[\left[1/2\cdot\{F_2(k\gamma,c,n,2)-F_2(\gamma,k,c,n,2)\}\right.\right.$

$\left.+n(n+1)/2\cdot F_1(k,\gamma,c,n,0)\right]\cdot\cos\gamma\cdot\sin\gamma$

$\left.+F_3(k,\gamma,c,n,2)\cdot\cos^2\gamma\right]g_2(k\gamma c)d\gamma dk$

1] $F1(k,\gamma,c,n,0)=\pi/2\cdot c\cdot\sum_{q=0}^{[n/2]}S_{n,0,q}(kc)^{n-q-1/2}K_{n-q-1/2}(kc)$

$=\pi/2\cdot1/c^2\cdot\sum_{q=0}^{[n/2]}S_{n,0,q}(kc)^{n-q-1/2+3}/k^3\cdot K_{n-q-1/2}(kc)$

NII-Electronic Library Service

Institute of Electronics, Information, and Communication Engineers

※

$$B_{n,0,j,l}(z)=\sum_{q=0}^{[n/2]} S_{n,0,q}(c)(kc)^{n-q+l-1/2}K_{n-q-j-1/2}$$

$$5/2=l-1/2 \qquad -1/2=-j-1/2 \qquad \therefore l=3 \;\; j=0$$

$$=\pi/2\cdot 1/c^2\cdot B_{n,0,0,3}/k^3$$

2] $$F_2(k,\gamma,c,n,2)=\pi/2\cdot c^3\left[\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-3/2}K_{n-q-3/2}(kc)\right.$$

$$\left.-\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-5/2}\cdot k^2\cdot\cos^2\gamma\cdot c^2\cdot K_{n-q-5/2}(kc)\right]$$

$$=\pi/2\left[\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-3/2+3}/k^3\,K_{n-q-3/2}(kc)\right.$$

$$\left.-\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-5/2+5}/k^5\cdot k^2\cdot\cos^2\gamma\cdot K_{n-q-5/2}(kc)\right]$$

$$B_{n,2,j,l}(c)=\sum_{q=0}^{[n/2]} S_{n2,q}(c)(kc)^{n-q+l-1/2}K_{n-q-j-1/2}$$

putting $3/2=l-1/2 \qquad -3/2=-j-1/2 \qquad \therefore l=2 \;\; j=1$

putting $5/2=l-1/2 \qquad -5/2=-j-1/2 \qquad \therefore l=3 \;\; j=2$

$$=\pi/2\left[B_{n,2,1,2}-\cos^2\gamma\cdot B_{n,2,2,3}\right]/k^3$$

Hence

$$1/2\left[F_2(k,\gamma,c,n,2)-F_2(\gamma,k,c,n,2)\right]$$

$$=\pi/2\cdot 1/2\cdot 1/k^3\left[-\cos^2\gamma+\sin^2\gamma\right]B_{n,2,2,3}$$

3] $$F_3(k,\gamma,c,n,2)=\pi/2\cdot k^2\cdot\cos\gamma\cdot\sin\gamma\cdot c^5\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-5/2}\cdot K_{n-q-5/2}$$

$$=\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot k^2\sum_{q=0}^{[n/2]} S_{n,2,q}(c)(kc)^{n-q-5/2+5}/k^5\cdot K_{n-q-5/2}$$

$$5/2=l-1/2 \qquad -5/2=-j-1/2 \qquad \therefore l=3 \;\; j=2$$

$$=\pi/2\cdot\cos\gamma\cdot\sin\gamma\cdot 1/k^3\cdot B_{n,2,2,3}$$

Associating these, we have

$$\pi/2(-4/\pi^2)\int_0^\infty\int_0^{\pi/2} k\Big[\Big[1/2\cdot 1/k^3(-\cos^2\gamma+\sin^2\gamma)\cdot B_{n,2,2,3}+n(n+1)/2\cdot 1/c^2\cdot 1/k^3 B_{n,0,0,3}\Big]\cos\gamma\sin\gamma+\cos\gamma\sin\gamma\cdot 1/k^3\cdot B_{n,2,2,3}\cdot\cos^2\gamma\Big]g_2(k\gamma c)d\gamma dk$$

$$1/2(-\cos^2\gamma+\sin^2\gamma)\cos\gamma\cdot\sin\gamma+\cos\gamma\cdot\sin\gamma\cdot\cos^2\gamma$$

$$=1/2(-\cos^2\gamma+\sin^2\gamma+2\cos^2\gamma)\cos\gamma\cdot\sin\gamma$$

$$=1/2(\cos^2\gamma+\sin^2\gamma)\cos\gamma\cdot\sin\gamma$$

Hence

$$=\pi/2\cdot(-4/\pi^2)\int_0^\infty\int_0^{\pi/2}\left[1/2\cdot B_{n,2,2,3}+n(n+1)/2\cdot 1/c^2\,B_{n,0,0,3}\right]\cos\gamma\cdot\sin\gamma\cdot g_2(k\gamma c)d\gamma dk$$

$$=\pi/2\cdot(-4/\pi^2)\cdot H_{16}(c)$$

Appendix 1.

The An, Bn and Cn functions in (2.16)

$$A_n^*(\alpha,\beta,z_i)=-\frac{4}{\pi^2}\Big\{n(2n-1)F_2(\alpha,\beta,z_i,n,1)+\frac{1}{2}(n-2)\times\left[F_2(\alpha,\beta,z_i,n-1,2)-F_2(\beta,\alpha,z_i,n-1,2)\right]-\frac{1}{2}n(n+1)(n-2)F_1(\alpha,\beta,z_i,n-1,0)\Big\} \quad \text{(A-1)}$$

$$B_n^*(\alpha,\beta,z_i)=-\frac{4}{\pi^2}\Big\{-\frac{1}{2}\left[F_2(\alpha,\beta,z_i n+1,2)-F_2(\beta,\alpha,z_i,n+1,2)\right]+\frac{1}{2}n(n+1)F_1(\alpha,\beta,z_i,n+1,0)\Big\} \quad \text{(A2)}$$

$$C_n^*(\alpha,\beta,z_i)=-\frac{4}{\pi^2}\Big\{\frac{1}{2}\left[F_2(\alpha,\beta,z_i n,2)-F_2(\beta,\alpha,z_i,n,2)\right]+\frac{1}{2}n(n+1)F_1(\alpha,\beta,z_i,n,0)\Big\} \quad \text{(A-3)}$$

$$A_n^{**}(\alpha,\beta,z_i)=-\frac{4}{\pi^2}\left[n(2n-1)F_3(\alpha,\beta,z_i n,1)+(n-2)F_3(\alpha,\beta,z_i,n-1,2)\right] \quad \text{(A-4)}$$

$$B_n^{**}(\alpha,\beta,z_i)=\frac{4}{\pi^2}F_3(\alpha,\beta,z_i,n+1,2), \quad \text{(A-5)}$$

$$C_n^{**}(\alpha,\beta,z_i)=-\frac{4}{\pi^2}F_3(\alpha,\beta,z_i,n,2); \quad \text{(A-6)}$$

$$A_n^{***}(\alpha,\beta,z_i)=-\frac{4}{\pi^2}\left[n(2n-1)z_iF_4(\alpha,\beta,z_i n,1)-(n+1)(n-2)F_4(\alpha,\beta,z_i,n-1,1)\right]$$

$$B_n^{***}(\alpha,\beta,z_i)=\frac{4}{\pi^2}nF_4(\alpha,\beta,z_i,n+1,1), \quad \text{(A-8)}$$

$$C_n^{***}(\alpha,\beta,z_i)=\frac{4}{\pi^2}F_4(\alpha,\beta,z_i,n,1). \quad \text{(A-9)}$$

The inner set of integrals required by (2,22). $u=\sqrt{a*a+b*b}$,

J0 and J1 are Bessel function of the first and the second kind.

$$\int_0^{\frac{1}{2}\pi}\cos(a\cos\gamma)\cos(b\sin\gamma)d\gamma=\frac{\pi}{2}J_0(u), \quad \text{(B 1)}$$

$$\int_0^{\frac{1}{2}\pi}\cos^2\gamma\cos(a\cos\gamma)\cos(b\sin\gamma)d\gamma=\frac{\pi}{2u^2}\left[a^2J_0(u)+\frac{b^2-a^2}{u}J_1(u)\right], \quad \text{(B 2)}$$

$$\int_0^{\frac{1}{2}\pi}\cos\gamma\sin\gamma\sin(a\cos\gamma)\sin(b\sin\gamma)d\gamma=-\frac{\pi}{2}\frac{ab}{u^2}\left[J_0(u)-\frac{2}{u}J_1(u)\right], \quad \text{(B 3)}$$

$$\int_0^{\frac{1}{2}\pi}\cos\gamma\sin(a\cos\gamma)\cos(b\sin\gamma)d\gamma=\frac{\pi}{2}\frac{a}{u}J_1(u). \quad \text{(B 4)}$$

The primed An', Bn' and Cn' contained in (2-24)

$$A'_n=\int_0^\infty\left\{G_5(\eta)H_1(-b)-G_5(\sigma)H_1(c)+G_6(\sigma,\eta)H_2(-b)-G_6(\eta,\sigma)H_2(c)+G_1(\sigma,\eta)H_3(-b)-G_1(\eta,\sigma)H_3(c)\right\}d\kappa \quad \text{(C 1a)}$$

$$B'_n=\int_0^\infty\left\{G_5(\eta)H_4(-b)-G_5(\sigma)H_4(c)+G_6(\sigma,\eta)H_5(-b)-G_6(\eta,\sigma)H_5(c)+G_1(\sigma,\eta)H_6(-b)-G_1(\eta,\sigma)H_6(c)\right\}d\kappa \quad \text{(C 1b)}$$

$$C'_n=\int_0^\infty\left\{G_5(\eta)H_7(-b)-G_5(\sigma)H_7(c)+G_6(\sigma,\eta)H_8(-b)-G_6(\eta,\sigma)H_8(c)+G_1(\sigma,\eta)H_9(-b)-G_1(\eta,\sigma)H_9(c)\right\}d\kappa \quad \text{(C1 c)}$$

****

$$A''_n=\int_0^\infty\left\{G_6(\sigma,\eta)H_{10}(-b)-G_6(\eta,\sigma)H_{10}(c)+G_1(\sigma,\eta)H_{12}(-b)-G_1(\eta,\sigma)H_{12}(c)+G_3(\sigma,\eta)H_{11}(-b)-G_3(\eta,\sigma)H_{11}(c)\right\}d\kappa \quad \text{(C 1d)}$$

$$B''_n=\int_0^\infty\left\{G_6(\sigma,\eta)H_{13}(-b)-G_6(\eta,\sigma)H_{13}(c)+G_1(\sigma,\eta)H_{15}(-b)-G_1(\eta,\sigma)H_{15}(c)+G_3(\sigma,\eta)H_{14}(-b)-G_3(\eta,\sigma)H_{14}(c)\right\}d\kappa \quad \text{(C 1e)}$$

$$C''_n=\int_0^\infty\left\{G_6(\sigma,\eta)H_{16}(-b)-G_6(\eta,\sigma)H_{16}(c)+G_1(\sigma,\eta)H_{18}(-b)-G_1(\eta,\sigma)H_{18}(c)+G_3(\sigma,\eta)H_{17}(-b)-G_3(\eta,\sigma)H_{17}(c)\right\}d\kappa \quad \text{(C 1f)}$$

$$A'''_n=\int_0^\infty\left\{G_2(\sigma,\eta)H_{19}(-b)-G_2(\eta,\sigma)H_{19}(c)+G_4(\sigma,\eta)H_{20}(-b)-G_4(\eta,\sigma)H_{20}(c)\right\}d\kappa, \quad \text{(C 1g)}$$

$$B'''_n=\int_0^\infty\left\{G_2(\sigma,\eta)H_{21}(-b)-G_2(\eta,\sigma)H_{21}(c)+G_4(\sigma,\eta)H_{22}(-b)-G_4(\eta,\sigma)H_{22}(c)\right\}d\kappa, \quad \text{(C 1h)}$$

$$C'''_n=\int_0^\infty\left\{G_2(\sigma,\eta)H_{23}(-b)-G_2(\eta,\sigma)H_{23}(c)+G_4(\sigma,\eta)H_{24}(-b)-G_4(\eta,\sigma)H_{24}(c)\right\}d\kappa, \quad \text{(C 1i)}$$

Where

$$H_1(z_i)=-n(2n-1)z_i^2J_0(\kappa\rho)B_{n,1,1,0}(z_i)+n(2n-1)\frac{z_i^2}{\rho^2}B_1B_{n,1,2,1}(z_i)-\frac{1}{2}(n-2)(y^2-x^2)B_2B_{n-1,2,2,3}(z_i)+\frac{1}{2}n(n+1)(n-2)J_0(\kappa\rho)B_{n-1,0,0,1}(z_i) \quad \text{(C 2a)}$$

$$H_2(z_i)=\Big\{-n(2n-1)z_i^2)\left[B_{n,1,1,0}(z_i)-B_{n,1,2,1}(z_i)\right]+\frac{1}{2}(n-2)z_i^2B_{n-1,2,2,1}(z_i)+\frac{1}{2}n(n+1)(n-2)B_{n-1,0,0,1}(z_i)\Big\}\frac{B_1}{\rho^2} \quad \text{(C 2b)}$$

NII-Electronic Library Service

Institute of Electronics, Information, and Communication Engineers

$$H_3(z_i) = \kappa z_i^2\left[-n(2n-1)z_i B_{n,1,1,0}(z_i) + (n+1)(n-2)B_{n-1,1,1,0}(z_i)\right]\frac{B_1}{\rho^2} \quad \text{(C 2c)}$$

$$H_4(z_i) = \frac{1}{2}\left[(y^2 - x^2)B_2 B_{n+1,2,2,3}(z_i) - n(n+1)J_0(\kappa\rho)B_{n+1,0,0,1}(z_i)\right] \quad \text{(C 2d)}$$

$$H_5(z_i) = -\frac{1}{2}\left[z_i^2 B_{n+1,2,2,1}(z_i) + n(n+1)B_{n+1,0,0,1}(z_i)\right]\frac{B_1}{\rho^2} \quad \text{(C 2e)}$$

$$H_6(z_i) = n\frac{\kappa z_i^2}{\rho^2}B_1 B_{n+1,1,1,0}(z_i) \quad \text{(C 2f)}$$

$$H_7(z_i) = -\frac{1}{2}\left[(y^2 - x^2)B_2 B_{n,2,2,3}(z_i) + n(n+1)J_0(\kappa\rho)B_{n,0,0,1}(z_i)\right] \quad \text{(C 2g)}$$

$$H_8(z_i) = \frac{1}{2}\left[z_i^2 B_{n,2,2,1}(z_i) - n(n+1)B_{n,0,0,1}(z_i)\right]\frac{B_1}{\rho^2} \quad \text{(C 2h)}$$

$$H_9(z_i) = \kappa z_i^2 B_{n,1,1,0}(z_i)\frac{B_1}{\rho^2} \quad \text{(C 2i)}$$

$$H_{10}(z_i) = xyB_2\left[-n(2n-1)B_{n,1,1,2}(z_i) - \frac{1}{2}(n-2)B_{n-1,2,2,3}(z_i) + \frac{1}{2z_i^2}n(n+1)(n-2)B_{n-1,0,0,3}(z_i)\right] \quad \text{(C 2j)}$$

$$H_{11}(z_i) = xyB_2\left[n(2n-1)B_{n,1,2,3}(z_i) + (n-2)B_{n-1,2,2,3}(z_i)\right] \quad \text{(C 2k)}$$

$$H_{12}(z_i) = \kappa xyB_2\left[-n(2n-1)z_i B_{n,1,1,2}(z_i) + (n+1)(n-2)B_{n-1,1,1,2}(z_i)\right] \quad \text{(C 2l)}$$

$$H_{13}(z_i) = \frac{1}{2}xyB_2\left[B_{n+1,2,2,3}(z_i) - n(n+1)\frac{1}{z_i^2}B_{n+1,0,0,3}(z_i)\right] \quad \text{(C 2m)}$$

$$H_{14}(z_i) = -xyB_2 B_{n+1,2,2,3}(z_i) \quad \text{(C 2n)}$$

$$H_{15}(z_i) = n\kappa xyB_2 B_{n+1,1,1,2}(z_i) \quad \text{(C 2o)}$$

$$H_{16}(z_i) = -\frac{1}{2}xyB_2\left[B_{n,2,2,3}(z_i) + n(n+1)\frac{1}{z_i^2}B_{n,0,0,3}(z_i)\right] \quad \text{(C 2p)}$$

$$H_{17}(z_i) = xyB_2 B_{n,2,2,3}(z_i) \quad \text{(C 2q)}$$

$$H_{18}(z_i) = \kappa xyB_2 B_{n,1,1,2}(z_i) \quad \text{(C 2r)}$$

$$H_{19}(z_i) = -x\frac{J_1(\kappa\rho)}{\kappa^2\rho}\left\{n(2n-1)\left[B_{n,1,1,2}(z_i) - B_{n,1,2,3}(z_i)\right] - \frac{1}{2}(n-2)B_{n-1,2,2,3}(z_i) - \frac{1}{2}n(n+1)(n-2)\frac{1}{z_i^2}B_{n-1,0,0,3}(z_i)\right\} \quad \text{(C 2s)}$$

$$H_{20}(z_i) = -x\frac{J_1(\kappa\rho)}{\kappa\rho}\left[n(2n-1)z_i B_{n,1,1,2}(z_i) - (n+1)(n-2)B_{n-1,1,1,2}(z_i)\right] \quad \text{(C 2t)}$$

$$H_{21}(z_i) = -\frac{1}{2}x\frac{J_1(\kappa\rho)}{\kappa^2\rho}\left[B_{n+1,2,2,3}(z_i) + n(n+1)\frac{1}{z_i^2}B_{n+1,0,0,3}(z_i)\right] \quad \text{(C 2u)}$$

$$H_{22}(z_i) = nx\frac{J_1(\kappa\rho)}{\kappa\rho}B_{n+1,1,1,2}(z_i) \quad \text{(C 2v)}$$

$$H_{23}(z_i) = \frac{1}{2}x\frac{J_1(\kappa\rho)}{\kappa^2\rho}\left[B_{n,2,2,3}(z_i) - n(n+1)\frac{1}{z_i^2}B_{n,0,0,3}(z_i)\right] \quad \text{(C 2w)}$$

$$H_{24}(z_i) = x\frac{J_1(\kappa\rho)}{\kappa\rho}B_{n,1,1,2}(z_i) \quad \text{(C 2x)}$$

where

$$B_1 = x^2 J_0(\kappa\rho) + (y^2 - x^2)\frac{J_1(\kappa\rho)}{\kappa\rho} \quad \text{(C 3a)}$$

$$B_2 = \frac{1}{\kappa^2\rho^2}\left[J_0(\kappa\rho) - 2\frac{J_1(\kappa\rho)}{\kappa\rho}\right] \quad \text{(C 3b)}$$

$$\rho = (x^2 + y^2)^{\frac{1}{2}} \quad \text{(C 4)}$$

$$B_{n,m,j,l}(zl) = \sum_{q=0}^{\left[\frac{1}{2}n\right]} S_{nmq}(z_i)(\kappa|z_i|)^{n-q+l-\frac{1}{2}} K_{n-q-j-\frac{1}{2}}(\kappa|z_i|) \quad \text{(C 5)}$$

The equation (2,11) reveal that the unknown D1, D2 and D3 functions evaluated at the two walls are simply Fourier transforms of these disturbances. These equations are inverted to give

$$\left.\begin{aligned}
D_1(\alpha,\beta,z_i) &= -\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\left\{\sum_{n=1}^{\infty}\left[A_n A_n'(s,t,z_i) + B_n B_n'(s,t,z_i) + C_n C_n'(s,t,z_i)\right]\right\}\cos\alpha s\cos\beta t\; ds\, dt,\\
D_2(\alpha,\beta,z_i) &= -\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\left\{\sum_{n=1}^{\infty}\left[A_n A_n''(s,t,z_i) + B_n B_n''(s,t,z_i) + C_n C_n''(s,t,z_i)\right]\right\}\sin\alpha s\sin\beta t\; ds\, dt,\\
D_3(\alpha,\beta,z_i) &= -\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\left\{\sum_{n=1}^{\infty}\left[A_n A_n'''(s,t,z_i) + B_n B_n'''(s,t,z_i) + C_n C_n'''(s,t,z_i)\right]\right\}\sin\alpha s\cos\beta t\; ds\, dt,
\end{aligned}\right\} i = 1,2 \quad (2.12)$$

Evaluation of the double integrals required in (2,12) is based on the associated Legendre function by its polynomial representation (2,13).

$$P_n^m(\zeta) = \frac{(1-\zeta^2)^{\frac{1}{2}m}}{2^n}\sum_{q=0}^{[\frac{1}{2}n]}\frac{(-1)^q(2n-2q)!\,\zeta^{n-2q-m}}{q!(n-q)!(n-2q-m)!}. \quad (2.13)$$

Once this substitution of (2.6) has been made,the second integration is performed

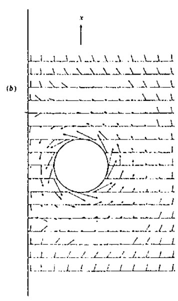
$$\begin{aligned}
F_1(\alpha,\beta,z_i,n,m) &= \int_0^\infty\int_0^\infty\frac{1}{(s^2+t^2+z_i^2)^{\frac{1}{2}(n+1)}}\frac{P_n^m\left(z_i/(s^2+t^2+z_i^2)^{\frac{1}{2}}\right)}{(s^2+t^2)^{\frac{1}{2}m}}\cos\alpha s\cos\beta t\,ds\,dt\\
&= \frac{\pi}{2}|z_i|\sum_{q=0}^{\left[\frac{1}{2}n\right]}S_{nmq}(z_i)(k|z_i|)^{n-q-\frac{1}{2}}K_{n-q-\frac{1}{2}}(k|z_i|), \quad (2.14a)
\end{aligned}$$

$$\begin{aligned}
F_2(\alpha,\beta,z_i,n,m) &= \int_0^\infty\int_0^\infty\frac{s^2}{(s^2+t^2+z_i^2)^{\frac{1}{2}(n+1)}}\frac{P_n^m\left(z_i/(s^2+t^2+z_i^2)^{\frac{1}{2}}\right)}{(s^2+t^2)^{\frac{1}{2}m}}\cos\alpha s\cos\beta t\,ds\,dt\\
&= \frac{\pi}{2}|z_i|^3\sum_{q=0}^{\left[\frac{1}{2}n\right]}S_{nmq}(z_i)(k|z_i|)^{n-q-\frac{3}{2}}\left[K_{n-q-\frac{3}{2}}(k|z_i|) - S_{nmq}(z_i)\alpha^2 z_i^2(k|z_i|)^{n-q-\frac{5}{2}}K_{n-q-\frac{5}{2}}(k|z_i|)\right]. \quad (2.14b)
\end{aligned}$$

$$\begin{aligned}
F_3(\alpha,\beta,z_i,n,m) &= \int_0^\infty\int_0^\infty\frac{st}{(s^2+t^2+z_i^2)^{\frac{1}{2}(n+1)}}\frac{P_n^m\left(z_i/(s^2+t^2+z_i^2)^{\frac{1}{2}}\right)}{(s^2+t^2)^{\frac{1}{2}m}}\sin\alpha s\sin\beta t\,ds\,dt\\
&= \frac{\pi}{2}\alpha\beta|z_i|^5\sum_{q=0}^{\left[\frac{1}{2}n\right]}S_{nmq}(z_i)(k|z_i|)^{n-q-\frac{5}{2}}K_{n-q-\frac{5}{2}}(k|z_i|), \quad (2.14c)
\end{aligned}$$

$$\begin{aligned}
F_4(\alpha,\beta,z_i,n,m) &= \int_0^\infty\int_0^\infty\frac{s}{(s^2+t^2+z_i^2)^{\frac{1}{2}(n+1)}}P_n^m\left(\frac{z_i}{(s^2+t^2+z_i^2)^{\frac{1}{2}}}\right)\sin\alpha s\cos\beta t\,ds\,dt\\
&= \frac{\pi}{2}\alpha|z_i|^3\sum_{q=0}^{\left[\frac{1}{2}n\right]}S_{nmq}(z_i)(k|z_i|)^{n-q-\frac{3}{2}}K_{n-q-\frac{3}{2}}(k|z_i|), \quad (2.14d)
\end{aligned}$$

$$S_{nmq}(z_i) = \frac{(2/\pi)^{\frac{1}{2}}}{(-2)^q q!(n-2q-m)!z_i^{n+m}} \quad (2.15) \qquad z_i = -b = c$$



Result.

Fig 1 shows the velocity field induced by the rotation of a sphere about an axis parallel to two plane parallel wall computed by P. Ganatos (1980).

Reference

1). P. Ganatos. J. Fluid. Mech, vol 99. pp 755-783. 1980. 2). P. Ganatos. J. Fluid. Mech. vol84. pp 79. 1978.

NII-Electronic Library Service