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Application of Stochastic Queuing Principle for Analysis of Neural Pulse Sequence.

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あらまし

We introduced a detailed explanation for the mathematical method for solving the linear differential - difference equations with absorbing barrier that characterize the signal processing function of the neural cell. The equations are the same form of the birth and death processes. This method was firstly introduced by Saaty 1961. The probability Qn(t) of n impulses in a neural cell at an arbitrary time t was obtained by applying the generator expansion in combination with the Laplace integral transformation. Qn(t) showed definite peak as a function of the ratio β between the impulse arrival rate and the signal processing rate of the neuron. As the number of maximum available impulse number N, the dependency of Qn(t) on the β value was markedly changes. The present method,though basic will be available for evaluating the neural processing function.

和文キーワード Neural impulse, Signal Processing, Absorbing barrier, State transitional probability, Generator

神経パルス発生に対する待ち行列解析の応用

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Abstract

神経系を通過する神経パルスが神経細胞によってどのように処理されるかを定量的に評価する目的で生成消滅過程に関する線形微分差分方程式の解法を詳細に紹介した。基本的手法は Saaty 1961に準ずる。 任意の時刻における神経細胞内におけるパルス数がn個である確率Qn(t) は変数係数を含む形式でありこれに対して母関数展開ぼよびラプラス変換を施してQn(t)の過渡的変化を決定する式を得た。 Qn (t) はインパルスの到着確率と神経細胞内での処理確率の比βに依存し、ある特定の比で最大値をしめした。Qn (t) の平均値は最大許容インパルス数Nに依存して変化した。NがおおきくなるとQn (t) の平均値はβに依存して急激に増加したが Nが小さい場合β依存性は弱かった。 本研究は初歩的ではあるが発展させることで神経系の情報処理の基本特性を解析できると推定される。

英文 key words 神経インパルス、信号処理、吸収壁、状態確率、母関数、微分差分方程式、ラプラス変換

1. Introduction

Signal filtering through the excitable tissue is one of the most important properties of the physiological cellular functions. In the present study, we introduce a method for solving the absorbing barrier problem which may describe the impulse processing through the neural cell.

2. Mathematical method.

The main text is contributed to show the mathematical processes for solving the linear differential -difference equations describing the transitional changes of the system

$P'_{i,n}(t) = -(\lambda_i + \mu_i)P'_{i,n}(t) + \lambda_i P_{i+1,n}(t) + \mu_i P_{i-1,n}(t)$

$P'_{0,n}(t) = -\lambda_0 P_{0,n}(t) + \lambda_0 P_{1,n}(t)$ ------(2-3a, 2-3b)

$\lambda_n = n\lambda$ and $\mu_n = n\mu$ ---for case 1. and

$\lambda_n = \lambda$ and $\mu_n = \mu$ ---for case 2 ---(2.4)

Since $P_{n,N}(t)$ is the cumulative time distribution for the unnormalized conditional probabilities $Q_n(t)\,dt$, we have

$Q_n(t) = dP_{n,N}(t)/dt$ ---(2.5) Applying (2,5) to he (2,3) where putting i =n and n=N, using (2,4a), we arrive

$$Q_n'(t) = -n\cdot(\lambda+\mu)Q_n(t) + n\mu Q_{n-1}(t) + n\lambda Q_{n+1}(t) \quad n = 1,\cdots,N-1$$

$$Q_0(t) = Q_N(t) = 0 \qquad -(3,\ 1)$$

$$Q_n(0) = \delta_{n,N-1}\cdot\lambda(N-1) \qquad -(3,\ 3)$$

We set generating function for

$$Q(z,t) = \sum_{n=1}^{N-1} Q_n(t) z^{n-1} \qquad -(3,\ 4)$$

On (3, 1) multiply z^{n-1} and sum as $\sum_{n=1}^{N-1}(3,1)\cdot z^{n-1}$

$$\sum_{n=1}^{N-1} z^{n-1}\cdot Q_n'(t) = -(\lambda+\mu)\sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_n(t) + \mu\cdot\sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_{n-1}(t) + \lambda\cdot\sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_{n+1}(t)$$

since

$$Q = Q_1\cdot z^0 + Q_2\cdot z^1 + Q_3\cdot z^2 + \cdots + Q_{N-1}\cdot z^{N-2}$$

we have

$$\partial Q/\partial z = 1\cdot z^0\cdot Q_2 + 2\cdot z\cdot Q_3 + 3\cdot z^2\cdot Q_4 + \cdots + (N-2)\cdot z^{N-3}\cdot Q_{N-1}$$

$$= \sum_{n=1}^{N-2} z^{n-1}\cdot n\cdot Q_{n+1}$$

$$= \sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_{n+1} - z^{N-2}\cdot(N-1)\cdot Q_N$$

and

$$z^2\cdot\partial Q/\partial z = 1\cdot z^2\cdot Q_2 + 2\cdot z^3\cdot Q_3 + 3\cdot z^4\cdot Q_4 + \cdots + (N-2)\cdot z^{N-1}\cdot Q_{N-1}$$

$$z\cdot\partial Q/\partial z = 1\cdot z\cdot Q_2 + 2\cdot z^2\cdot Q_3 + 3\cdot z^3\cdot Q_4 + \cdots + (N-2)\cdot z^{N-2}\cdot Q_{N-1}$$

$$2\cdot z\cdot Q = 2\cdot z\cdot Q_1 + 2\cdot z^2\cdot Q_2 + 3\cdot z^3\cdot Q_3 + \cdots + 2\cdot z^{N-1}\cdot Q_{N-1}$$

$$Q_0 = 0$$

we have

$$\mu\cdot\sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_{n-1} = \mu\cdot\left[1\cdot z^0\cdot Q_0 + 2\cdot z^1\cdot Q_1 + 3\cdot z^2\cdot Q_2 + \cdots + (N-1)\cdot z^{N-2}\cdot Q_{N-2}\right]$$

$$= \mu\cdot\left[2\cdot z\cdot Q_1 + (2\cdot z^2 + 1\cdot z^2)Q_2 + (2\cdot z^3 + 2\cdot z^3)Q_3 + (2\cdot z^4 + 3\cdot z^4)Q_4 \cdots + (2\cdot z^{N-1} + (N-2)\cdot z^{N-1})Q_{N-1} - N\cdot z^{N-1}\cdot Q_{N-1}\right]$$

$$= \mu\cdot\left[2\cdot z\cdot Q + z^2\cdot\partial Q/\partial z - N\cdot z^{N-1}\cdot Q_{N-1}\right]$$

Further

$$(\lambda+\mu)\sum_{n=1}^{N-1} z^{n-1}\cdot n\cdot Q_n(t)$$

$$= (\lambda+\mu)\cdot\left[1\cdot z^0\cdot Q_1 + 2\cdot z^1\cdot Q_2 + 3\cdot z^2\cdot Q_3 + \cdots + (N-1)\cdot z^{N-2}\cdot Q_{N-1}\right]$$

$$= (\lambda+\mu)\cdot\left[1\cdot z\cdot Q_1 + (1\cdot z + 1\cdot z)Q_2 + (z^2 + 2\cdot z^2)Q_3 + (z^3 + 3\cdot z^3)\cdot Q_4 \cdots + (z^{N-2} + (N-2)\cdot z^{N-2})\cdot Q_{N-1}\right]$$

$$= (\lambda+\mu)\cdot\left[z\cdot\partial Q/\partial z + Q\right]$$

Associating these , (3-1) is transformed to

$$\frac{\partial Q}{\partial t} = \lambda\cdot\frac{\partial Q}{\partial z} + \mu\cdot\left[2\cdot z\cdot Q + z^2\cdot\frac{\partial Q}{\partial z} - N\cdot z^{N-1}\cdot Q_{N-1}\right] - (\lambda+\mu)\cdot\left[z\cdot\frac{\partial Q}{\partial z} + Q\right]$$

$$-\frac{\partial Q}{\partial t} + \left\{\lambda + \mu z^2 - (\lambda+\mu)\cdot z\right\}\frac{\partial Q}{\partial z} = (\lambda+\mu-2\mu z)Q + \mu N\cdot Q_{N-1}z^{N-1} \qquad -(3,\ 6)$$

-----Solving this partial differential equation ----

$$\frac{dt}{1} = \frac{dz}{(\mu z-\lambda)(z-1)}$$

$$= \frac{1}{(\mu-\lambda)}\cdot\left[\frac{-\mu}{(\mu z-\lambda)} + \frac{1}{(z-1)}\right]$$

since $\alpha = \lambda - \mu$

$$= \frac{\mu/\alpha}{(\mu z-\lambda)} - \frac{1/\alpha}{(z-1)}$$

$$\therefore t = 1/\alpha\cdot\left[\log(\mu z-\lambda) - \log(z-1)\right] \quad \text{then, we have}$$

$$-\alpha\cdot t = \log\left[\frac{(z-1)}{(\mu z-\lambda)}\right]$$

$$\frac{(z-1)}{\mu z-\lambda} = e^{-\alpha t}$$

$$\therefore z = (1-\lambda e^{-\alpha t})/(1-\mu e^{-\alpha t})$$

Another combination of the equations is

$$Q'(t) + (\lambda+\mu-2\mu z)\cdot Q = \mu\cdot N\cdot Q_{N-1}\cdot z^{N-1}$$

As a result the solution of (3、6) is

$$Q(z,t) = -N\mu\alpha^2\int_0^t Q_{N-1}(s)^{-\alpha(t-s)}\frac{\left\{\lambda\left(1-e^{-\alpha(t-s)}\right) - z\cdot\left(\mu-\lambda e^{-\alpha(t-s)}\right)\right\}^{N-1}}{\left\{\left(\lambda-\mu e^{-\alpha(t-s)}\right) - \mu\cdot z\left(1-e^{-\alpha(t-s)}\right)\right\}^{N+1}}\alpha$$

$$+ (N-1)\lambda\cdot\alpha^2\cdot e^{-\alpha t}\frac{\left\{\lambda\cdot\left(1-e^{-\alpha t}\right) - z\cdot\left(\mu-\lambda e^{-\alpha t}\right)\right\}^{N-2}}{\left\{\left(\lambda-\mu e^{-\alpha t}\right) - \mu\cdot z\cdot\left(1-e^{-\alpha t}\right)\right\}^{N}} \qquad (3、7)$$

$$= \sum_{n=1}^{N-1} Q_n(t)\cdot z^{n-1}$$

$$\left(= z^0\cdot Q_1 + z^1\cdot Q_2 + z^2\cdot Q_3 + \cdots + z^{N-2}\cdot Q_{N-1}(t)\right)$$

Since

$$\partial Q/\partial z = 1\cdot Q_2 + 2\cdot z\cdot Q_3 + 3\cdot z^2\cdot Q_4 + \cdots + (N-2)\cdot z^{N-3}\cdot Q_{N-1}$$

$$\partial^2 Q/\partial z^2 = 2\cdot Q_3 + 3\cdot 2\cdot z\cdot Q_4 + \cdots + (N-2)\cdot(N-3)\cdot z^{N-4}\cdot Q_{N-1}$$

$$\partial^3 Q/\partial z^3 = 3\cdot 2\cdot Q_4 + 4\cdot 3\cdot 2\cdot Q_5 + \cdots + (N-2)(N-3)(N-4)\cdot z^{N-5}\cdot Q_{N-1}$$

So we have

$$Q_2 = 1/1\cdot\partial Q/\partial z\,(z=0)$$

$$Q_3 = 1/2!\cdot\partial Q^2/\partial z^2\,(z=0)$$

$$Q_4 = 1/3!\cdot\partial Q^3/\partial z^3\,(z=0)$$

and generally

$$Q_n = \frac{1}{(n-1)!}\frac{\partial^{(n-1)}Q(z,t)}{\partial z^{(n-1)}}_{(z=0)}$$

Then, differentiate (3, 7) with z for $(n-1)$ times

$$1.\quad \frac{\partial^{n-1}Q_1^A}{\partial z^{n-1}} = \frac{\partial^{n-1}}{\partial z^{n-1}}\left[\frac{\left\{\lambda\cdot\left(1-e^{-\alpha(t-s)}\right) - z\cdot\left(\mu-\lambda e^{-\alpha(t-s)}\right)\right\}^{N-1}}{\left\{\left(\lambda-\mu e^{-\alpha(t-s)}\right) - \mu\cdot z\cdot\left(1-e^{-\alpha(t-s)}\right)\right\}^{N+1}}\right]$$

Putting ※$a = \lambda\left(1-e^{-\alpha(t-s)}\right)$ $\quad b = \mu - \lambda e^{-\alpha(t-s)}$

$c = \lambda - \mu e^{-\alpha(t-s)}$ $\quad d = \mu\left(1-e^{-\alpha(t-s)}\right)$

$$\frac{\partial^{n-1}Q_1^A}{\partial z^{n-1}} = \frac{\partial^{n-1}}{\partial z^{n-1}}\left\{\frac{(a-b\cdot z)^{N-1}}{(c-d\cdot z)^{N+1}}\right\} = \frac{\partial^{n-1}}{\partial z^{n-1}}\left[g^{N-1}(z)\cdot h^{N+1}(z)\right]$$

$$= \sum_{i=0}^{n-1} {}_{n-1}C_i \frac{\partial^{(n-1-i)} g_{(z)}^{N-1}}{\partial z^{(n-1-i)}}\cdot\frac{\partial^i h_{(z)}^{N+1}}{\partial z^i}$$

On the while

$$\frac{\partial^l x^k}{\partial x^l} = \frac{k!}{(k-l)!}x^{k-l}$$

About $\dfrac{\partial^l g_{(z)}^k}{\partial z^l} =$

$$l=1 \quad : \quad k\cdot g^{k-1}\cdot g' = k\cdot g^{k-1}\cdot(-b)$$
$$l=2 \quad : \quad k\cdot(k-1)\cdot g^{(k-2)}\cdot(-b)^2$$
$$l=3 \quad : \quad k\cdot(k-1)\cdot(k-2)\cdot g^{(k-3)}\cdot(-b)^3$$

So, generally

$$\frac{\partial^l g_{(z)}^k}{\partial x^l} = \frac{k!}{(k-l)!} g_{(z)}^{k-l}\cdot(-b)^l$$

Then, putting

$$l = n-1-i \quad , \quad k = N-1$$

we have

$$\frac{\partial^{(n-1-i)} g_{(z)}^{N-1}}{\partial z^{(n-1-i)}} = \frac{(N-1)!}{(N-1-n+1+i)!} g_{(z)}^{N-n+i}\cdot(-b)^{n-1-i}$$

$$= \frac{(N-1)!}{(N-n+i)!} g_{(z)}^{N-n+i}\cdot(-b)^{n-1-i}$$

similiarly

$$\frac{\partial^i h_{(z)}^{N+1}}{\partial z^i} = \frac{\partial^i (c-dz)^{-(N+1)}}{\partial z^i}$$

$$= \frac{(-N-1)!}{(-N-n-i)!}(c-dz)^{-N-1-i}\cdot(-d)^i$$

since

$$\frac{(-N-1)!}{(-N-1-i)!}$$
$$= \frac{(-N-1)(-N-2)\cdots(-N-1-i+1)(-N-1-i)(-N-1-i-1)\cdots}{(-N-1-i)!}$$
$$= (-N-1)(-N-2)\cdots(-N-1-i+1)$$
$$= (N+1)(N+2)\cdots(N+i)\cdot(-1)^i = \frac{(N+i)!(-1)^i}{N!}$$

Therefore

$$\frac{\partial^i h_{(z)}^{N+1}}{\partial z^i} = \frac{(N+i)!}{N!}(-1)^i(c-dz)^{-N-1-i}\cdot(-d)^i$$

Associating these

$$Q_n = \frac{1}{(n-1)!}\frac{\partial^{(n-1)}Q(z,t)}{\partial z^{(n-1)}}_{(z=0)}$$

$$= \frac{1}{(n-1)!}\left[\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{(N-1)!}{(N-n+i)!} g_{(z=0)}^{N-n+i}\cdot(-b)^{n-1-i} \cdot\frac{(N+i)!}{N!}(d)^i(c-dz)^{-N-1-i}{}_{(z=0)}\right]$$

$$= \frac{1}{(n-1)!}\left[\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{(N-1)!}{(N-n+i)!}\cdot\frac{(N+i)!}{N!}a^{N-n+i}\frac{(-b)^{n-1-i}\cdot d^i}{c^{N+1+i}}\right]$$

$$= \frac{(-1)^{n-1}}{(n-1)!\,N}\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{\Gamma(N+i+1)}{\Gamma(N-n+i+1)}\frac{a^{N-n}}{c^{N+1}}\cdot\left(-\frac{a\cdot d}{b\cdot c}\right)^i\cdot b^{n-1}$$

$$= \frac{(-1)^{n-1}}{(n-1)!\,N}\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{\Gamma(N+i+1)}{\Gamma(N-n+i+1)}\frac{\left[\lambda\cdot\left(1-e^{-\alpha(t-s)}\right)\right]^{-n+N}}{\left[\lambda-\mu e^{-\alpha(t-s)}\right]^{N+1}}\left[\mu-\lambda e^{-\alpha(t-s)}\right]^{-1+n}$$

$$\cdot\left[-\frac{\lambda\cdot\left(1-e^{-\alpha(t-s)}\right)\cdot\mu\left(1-e^{-\alpha(t-s)}\right)}{\left(\mu-\lambda e^{-\alpha(t-s)}\right)\left(\lambda-\mu e^{-\alpha(t-s)}\right)}\right]^i$$

here setting $\beta = \mu/\lambda$ then,

$$\lambda\cdot\mu = \lambda\cdot\beta\cdot\lambda = \beta\cdot\lambda^2$$

As a result,

$$Q_n^A = \frac{(-1)^{n-1}}{(n-1)!\,N}\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{\Gamma(N+i+1)}{\Gamma(N-n+i+1)}\cdot\left[\frac{-\beta\lambda^2\cdot\left(1-e^{-\alpha(t-s)}\right)^2}{\left(\mu-\lambda e^{-\alpha(t-s)}\right)\left(\lambda-\mu e^{-\alpha(t-s)}\right)}\right]^i$$

$$\cdot\frac{\left\{\lambda\left(1-e^{-\alpha(t-s)}\right)\right\}^{N-n}}{\left\{\lambda-\mu e^{-\alpha(t-s)}\right\}^{N+1}}\cdot\left\{\mu-\lambda e^{-\alpha(t-s)}\right\}^{n-1}$$

2. The second term of (3, 7) is

$$\frac{\left\{\lambda\cdot\left(1-e^{-\alpha t}\right)-z\cdot\left(\mu-\lambda e^{-\alpha t}\right)\right\}^{N-2}}{\left\{\left(\lambda-\mu e^{-\alpha t}\right)-\mu z\cdot\left(1-e^{-\alpha t}\right)\right\}^N}$$

Thus, setting $t-s\rightarrow t$, $N\rightarrow N-1$ in the first term

$$Q_n^B = \frac{(-1)^{n-1}}{(n-1)!(N-1)}\sum_{i=0}^{n-1} {}_{n-1}C_i \frac{\Gamma(N+i)}{\Gamma(N-n+i)}\cdot\left[\frac{-\beta\lambda^2\cdot\left(1-e^{-\alpha t}\right)^2}{\left(\mu-\lambda e^{-\alpha t}\right)\left(\lambda-\mu e^{-\alpha t}\right)}\right]^i$$

$$\cdot\frac{\left\{\lambda\cdot\left(1-e^{-\alpha t}\right)\right\}^{N-1-n}}{\left\{\lambda-\mu e^{-\alpha t}\right\}^N}\cdot\left\{\mu-\lambda e^{-\alpha t}\right\}^{n-1}$$

Therfore, on using the Leibnitz theorem

$$Q_n(t) = -\frac{(-1)^{n-1}\mu\alpha^2}{(n-1)!}\int_0^t\left[Q_{N-1}(s)e^{-\alpha(t-s)}\frac{\left\{\lambda\left(1-e^{-\alpha(t-s)}\right)\right\}^{N-n}}{\left\{\lambda-\mu e^{-\alpha(t-s)}\right\}^{N+1}}\left\{\mu-\lambda e^{-\alpha(t-s)}\right\}^{n-1}\right.$$

$$\left.\cdot\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{\Gamma(N+i+1)}{\Gamma(N-n+i+1)}\left\{\frac{-\beta\lambda^2\left(1-e^{-\alpha(t-s)}\right)^2}{\left(\mu-\lambda e^{-\alpha(t-s)}\right)\left(\lambda-\mu e^{-\alpha(t-s)}\right)}\right\}^i\right]ds$$

$$+\frac{(-1)^{n-1}\lambda\alpha^2}{(n-1)!}e^{-\alpha t}\cdot\frac{\left\{\lambda\cdot\left(1-e^{-\alpha t}\right)\right\}^{N-1-n}}{\left\{\lambda-\mu e^{-\alpha t}\right\}^N}\cdot\left\{\mu-\lambda e^{-\alpha t}\right\}^{n-1}$$

$$\cdot\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{\Gamma(N+i)}{\Gamma(N-n+i)}\left\{-\frac{\beta\lambda^2\left(1-e^{-\alpha t}\right)}{\left(\mu-\lambda e^{-\alpha t}\right)\left(\lambda-\mu e^{-\alpha t}\right)}\right\}^i$$

----------- (3、8) $\binom{n-1}{i} = {}_{n-1}C_i$

where $\beta = \mu/\lambda$

The integration of (3.8) is a convolution of between $Q_{N-1}(s)$ and $f(t-s)$ as

$$L\left[\int_0^t F(t-s)\cdot g(s)ds\right] = L[F(t)]\cdot L[g(t)]$$

1. We transform the second term of (3, 8)

$$\frac{(-1)^{n-1}\lambda\alpha^2}{(n-1)!}e^{-\alpha t}\cdot\frac{\left\{\lambda\cdot\left(1-e^{-\alpha t}\right)\right\}^{N-n-1}}{\left\{\lambda-\mu e^{-\alpha t}\right\}^N}\cdot\left\{\mu-\lambda e^{-\alpha t}\right\}^{n-1}$$

$$\cdot\sum_{i=0}^{n-1} {}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}\left\{-\frac{\beta\lambda^2\left(1-e^{-\alpha t}\right)^2}{\left(\mu-\lambda e^{-\alpha t}\right)\left(\lambda-\mu e^{-\alpha t}\right)}\right\}^i$$

$$= \frac{\lambda\alpha^2\cdot e^{-\alpha t}}{(n-1)!}(-1)^{n-1}\cdot\cdot\sum_{i=0}^{n-1} {}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-\beta)^i$$

$$\cdot\lambda^{N-n-1+2i}\cdot\lambda^{-N-i}\cdot\mu^{n-1-i}$$

$$\cdot\left(1-e^{-\alpha t}\right)^{N-n-1}\cdot\left(1-\lambda/\mu\, e^{-\alpha t}\right)^{n-1}\cdot\left(1-\mu/\lambda\, e^{-\alpha t}\right)^{-N}$$

$$\cdot\left(1-e^{-\alpha t}\right)^{2i}\cdot\left(1-\lambda/\mu\, e^{-\alpha t}\right)^{-i}\cdot\left(1-\mu/\lambda\, e^{-\alpha t}\right)^{-i}$$

since $\mu/\lambda = \beta$

$$= \frac{\lambda\cdot\alpha^2\cdot e^{-\alpha t}}{(n-1)!}(-1)^{n-1}\cdot\sum_{i=0}^{n-1} {}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-\beta)^i\cdot\lambda^{-n-1+i}\cdot\mu^{n-1-i}$$

$$\cdot\left(1-e^{-\alpha t}\right)^{N-n-1+2i}\cdot\left(1-\beta e^{-\alpha t}\right)^{n-1-i}\cdot\left(1-\beta e^{-\alpha t}\right)^{-N-i}$$

since $\lambda^{-n-1+i}\cdot\mu^{n-1-i}=\left(\mu/\lambda\right)^{n-i}\cdot 1/(\mu\lambda) \quad =\beta^{n-i}/(\mu\lambda)$

$$=\frac{\lambda\cdot\alpha^2\cdot e^{-\alpha t}\cdot(-1)^{n-1}}{(n-1)!(\mu\lambda)}\cdot\sum_{i=0}^{n-1}{}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-1)^i\cdot\beta^i\cdot\beta^{n-i}$$

$$\cdot\left(1-e^{-\alpha t}\right)^{N-n-1+2i}\cdot\left(1-\beta\cdot e^{-\alpha t}\right)^{-N-i}\cdot\left(1-e^{-\alpha t}/\beta\right)^{n-1-i}$$

$$=\frac{\lambda\cdot\alpha^2\cdot e^{-\alpha t}\cdot(-1)^{n-1}}{(n-1)!(\mu\lambda)}\cdot\sum_{i=0}^{n-1}{}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-1)^i\cdot\beta^n$$

$$\cdot\left(1-e^{-\alpha t}\right)^{N-n-1+2i}\cdot\left(1-\beta\cdot e^{-\alpha t}\right)^{-N-i}\cdot\sum_{j=0}^{n-1-i}{}_{n-1-i}C_j\left(-e^{-\alpha t}/\beta\right)^j$$

$$=\frac{\alpha^2\cdot(-1)^{n-1}}{\mu\cdot(n-1)!}\cdot\sum_{i=0}^{n-1}{}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-1)^i\cdot\beta^n\cdot\left(1-e^{-\alpha t}\right)^{N-n-1+2i}$$

$$\cdot\left(1-\beta\cdot e^{-\alpha t}\right)^{-N-i}\cdot\sum_{j=0}^{n-1-i}{}_{n-1-i}C_j\left(-1/\beta\right)^j\cdot e^{-\alpha t-j\alpha t}$$

Therefore, the Laplace transform related term is

$$e^{-\alpha t(1+j)}\cdot\left(1-e^{-\alpha t}\right)^{N-n-1+2i}\cdot\left(1-\beta e^{-\alpha t}\right)^{-N-i}$$

From oherhettinger p 40, 5-22 in table

$$L\left[\left(1-e^{-t}\right)^{\nu-1}\cdot\left(1-z\cdot e^{-t}\right)^{-\mu}\right]=B(s,\nu)\,{}_2F_1(\mu,s:s+\nu:z)$$

setting

$$t\to\alpha\cdot t$$

$$\nu-1=N-n+2i-1 \quad : \quad \nu=N-n+2i$$

$$\mu=N+i \qquad z\to\beta$$

and since

$$L[f(\alpha t)]=1/\alpha\cdot F(s/\alpha)$$

Then

$$L\left[\left(1-e^{-\alpha t}\right)^{N-n+2i-1}\cdot\left(1-\beta\cdot e^{-\alpha t}\right)^{-N-i}\right]$$

$$=1/\alpha\cdot B\left(s/\alpha,N-n+2i\right){}_2F_1\left(N+i,s/\alpha:s/\alpha+N-n+2i:\beta\right)$$

More over, from the trabsitional theorem

$$L\left[e^{-\alpha t}\cdot f(t)\right]=F(s+\alpha)$$

we have

$$L\left[e^{-\alpha t(j+1)}\cdot\left(1-e^{-\alpha t}\right)^{N-n+2i-1}\cdot\left(1-\beta\cdot e^{-\alpha t}\right)^{-N-i}\right]$$

$$=1/\alpha\cdot B\left[(s+\alpha\cdot(j+1))/\alpha,N-n+2i\right]\cdot$$

$$\cdot{}_2F_1\left(N+i,(s+\alpha\cdot(j+1))/\alpha:(s+\alpha\cdot(j+1))/\alpha+N-n+2i:\beta\right)$$

$$=1/\alpha\cdot B\left(s/\alpha+j+1,N-n+2i\right)\cdot$$

$$\cdot{}_2F_1\left(N+i,s/\alpha+j+1:s/\alpha+j+1+N-n+2i:\beta\right)$$

where B is a beta function. Hence, the second term in the Laplace integral transform is

$$\frac{\alpha^2\cdot(-1)^{n-1}}{\mu\cdot(n-1)!}\sum_{i=0}^{n-1}{}_{n-1}C_i\frac{\Gamma(N+i)}{\Gamma(N-n+i)}(-1)^i\cdot\beta^n\cdot\sum_{j=0}^{n-1-i}{}_{n-1-i}C_j\left(-1/\beta\right)^j$$

$$\cdot 1/\alpha\cdot B\left(s/\alpha+j+1,N-n+2i\right){}_2F_1\left(N+i,s/\alpha+j+1:s/\alpha+j+1+N-n+2i:\beta\right)$$

The first term can be obtained by replacing $N\to N+1$ and then, multiply $\mu/\lambda\cdot Q_{N-1}(s)$

$$\frac{\alpha^2\cdot(-1)^{n-1}}{\mu\cdot(n-1)!}Q_{N-1}(s)\sum_{i=0}^{n-1}{}_{n-1}C_i\frac{\Gamma(N+1+i)}{\Gamma(N+1-n+i)}(-1)^i\cdot\beta^n\cdot\sum_{j=0}^{n-1-i}{}_{n-1-i}C_j\left(-1/\beta\right)^j$$

$$\cdot 1/\alpha\cdot B\left(s/\alpha+j+1,N+1-n+2i\right)\cdot{}_2F_1\left(N+1+i,s/\alpha+j+1:\right.$$

$$\left.:s/\alpha+j+1+N+1-n+2i:\beta\right)$$

5. Extinction

In the Eq (5, 1), the significant term is only the second term, since ($Q_{N-1}(s)=0$ for large N)

$$Q(z,t)=\mu\cdot(\lambda-\mu)^2e^{-(\lambda-\mu)t}\frac{1}{\left[\left(\lambda-\mu e^{-(\lambda-\mu)t}\right)-\mu\left(1-e^{-(\lambda-\mu)t}\right)z\right]^2}$$

$$=\sum_{n=1}^{N-1}Q_n(t)\cdot z^{n-1}$$

Then

$$Q_n(t)=\frac{1}{(n-1)!}\frac{\partial^{(n-1)}}{\partial z^{(n-1)}}Q(z,t)$$

$$=\frac{\mu(\lambda-\mu)^2\cdot e^{-(\lambda-\mu)t}}{(n-1)!}\frac{\partial^{(n-1)}}{\partial z^{(n-1)}}[c-dz]^{-2}$$

differentiation with z for $n-1$ times

$$\frac{(-2)!}{(-2-(n-1))!}\frac{(c-dz)^{-2-(n-1)}}{(z=0)}\cdot(-d)^{(n-1)}$$

$$※\frac{(-2)!}{(-1-n)!}=\frac{(-2)\cdot(-3)\cdots(-n)\cdot(-n-1)\cdot(-n-2)\cdots}{(-1-n)!}$$

$$=(-2)\cdot(-3)\cdots(-n)$$

$$=n\cdot(n-1)\cdots3\cdot2\cdot(-1)^{n-1}=n!(-1)^{n-1}$$

therefore

$$\frac{\partial^{(n-1)}}{\partial z^{(n-1)}}[c-dz]^{-2}=\frac{n!\cdot(-1)^{n-1}\cdot(-d)^{n-1}}{c^{n+1}}=n!\left(\frac{d}{c}\right)^n\frac{1}{c\cdot d}$$

By these conputation

$$Q_n(t)=\frac{\mu\cdot(\lambda-\mu)^2\cdot e^{-(\lambda-\mu)t}n!}{(n-1)!}\left[\frac{\mu\left(1-e^{-(\lambda-\mu)t}\right)}{\left(\lambda-\mu e^{-(\lambda-\mu)t}\right)}\right]^n$$

$$\cdot\left[\mu\left(1-e^{-(\lambda-\mu)t}\right)\cdot\left(\lambda-\mu e^{-(\lambda-\mu)t}\right)\right]^{-1}$$

$$=n\cdot\mu^n(\lambda-\mu)^2\cdot e^{-(\lambda-\mu)t}\cdot\left(1-e^{-(\lambda-\mu)t}\right)^{n-1}\cdot\left(\lambda-\mu e^{-(\lambda-\mu)t}\right)^{-(n+1)} \quad (5,\ 2)$$

6. The case $\lambda_n=\lambda$, $\mu_n=\mu$

$$\left.\begin{array}{l} Q_n'(t)=-(\lambda+\mu)Q_n(t)+\lambda\cdot Q_{n+1}+\mu\cdot Q_{n-1}\\ Q_0(t)=Q_N(t)=0\\ Q_n(0)=\lambda\cdot\delta_{n\cdot N-1}\end{array}\right\} \quad (6\text{-}1)$$

Setting generating function

$$G=\sum_{n=1}^{N-1}z^{n-1}\cdot Q_n(t)$$

Multiply z^n on (6-1) and sum as $\sum_{n=1}^{N-1}z^n\cdot(6,1)$

$$\sum_{n=1}^{N-1}z^n\cdot Q_n'(t)=-(\lambda+\mu)\cdot\sum_{n=1}^{N-1}z^n\cdot Q_n(t)+\lambda\cdot\sum_{n=1}^{N-1}z^n\cdot Q_{n+1}$$

$$+\mu\cdot\sum_{n=1}^{N-1}z^n\cdot Q_{n-1}$$

since

$$\sum_{n=1}^{N-1}z^n\cdot Q_n(t)=z\cdot\sum_{n=1}^{N-1}z^{n-1}\cdot Q_n(t)=z\cdot G$$

$$\sum_{n=1}^{N-1}z^n\cdot Q_{n+1}=z^1\cdot Q_2+z^2\cdot Q_3+\cdots+z^{N-1}\cdot Q_N$$

$$=-Q_1+Q_1+z\cdot Q_2+z^2\cdot Q_3+\cdots+z^{N-2}\cdot Q_{N-1}+z^{N-1}\cdot Q_N$$

$$=-Q_1+\sum_{n=1}^{N-1}z^{n-1}\cdot Q_n \qquad \because Q_N=0$$

$$=-Q_1+G$$

$$\sum_{n=1}^{N-1}z^n\cdot Q_{n-1}=z\cdot Q_0+z^2\cdot Q_1+\cdots+z^{N-1}\cdot Q_{N-2}$$

$$=z^2\left(Q_1+z\cdot Q_2+\cdots+z^{N-3}\cdot Q_{N-2}\right)$$

$$\because Q_0=0$$

$$=z^2\left(Q_1+z\cdot Q_2+\cdots+z^{N-3}\cdot Q_{N-2}+z^{N-2}\cdot Q_{N-1}-z^{N-2}\cdot Q_{N-1}\right)$$

$$=z^2\left(G-z^{N-2}\cdot Q_{N-1}\right)$$

Therefore

$$z\cdot\partial G/\partial t=-(\mu+\lambda)\cdot z\cdot G+\mu\cdot z^2\cdot G-\mu\cdot z^N\cdot Q_{N-1}+\lambda G-\lambda Q_1$$

$$=\left[\mu z^2-(\lambda+\mu)z+\lambda\right]G-\mu\cdot z^N\cdot Q_{N-1}-\lambda\cdot Q_1 \quad -(6,\ 2)$$

The Laplace transform of this function is

$$z[sG^*-G(0)]=[\mu z^2-(\lambda+\mu)z+\lambda]G^*-\mu\cdot z^N\cdot Q^*_{N-1}-\lambda\cdot Q^*_1$$

from (3、5)

$$G(0)=\lambda\cdot z^{N-2}$$

Therfore

$$G^*=\frac{[\mu\cdot z^N\cdot Q^*_{N-1}+\lambda\cdot Q^*_1-\lambda\cdot z^{N-1}]}{[\mu z^2-(\lambda+\mu+s)\cdot z+\lambda]} \qquad (6,\ 3)$$

The solution of characteristic equation is

$$\alpha_{1,2}=\frac{(\lambda+\mu+s)\pm\sqrt{(\lambda+\mu+s)^2-4\mu\lambda}}{2\mu} \qquad (6,\ 4)$$

To converge G^* as a series, the numerator of (6, 3) has to be zero for $z=\alpha_1\cdot and\alpha_2$. Since

$$\mu\cdot Q^*_{N-1}\cdot\alpha_1^N-\lambda\cdot\alpha_1^{N-1}+\lambda\cdot Q^*_1=0 \qquad :\alpha_2^N$$

$$\mu\cdot Q^*_{N-1}\cdot\alpha_2^N-\lambda\cdot\alpha_2^{N-1}+\lambda\cdot Q^*_1=0 \qquad :\alpha_1^N$$

Then, Q1* and Qn-1* as solutions of these equations

$$\mu(\alpha_1^N\cdot\alpha_2^N-\alpha_1^N\cdot\alpha_2^N)Q^*_{N-1}+\lambda(\alpha_2^N-\alpha_1^N)Q^*_1=\lambda\cdot(\alpha_1^{N-1}\alpha_2^N-\alpha_2^{N-1}\alpha_1^N)$$

$$\therefore Q^*_1=\alpha_1^{N-1}\cdot\alpha_2^{N-1}\frac{(\alpha_2-\alpha_1)}{(\alpha_2^N-\alpha_1^N)}$$

$$Q^*_{N-1}=\frac{\lambda(\alpha_1^{N-1}-\alpha_2^{N-1})}{\mu(\alpha_1^N-\alpha_2^N)}$$

Substitute then in (6, 3)

$$G^*=\frac{1}{[\mu z^2-(\lambda+\mu+s)\cdot z+\lambda]}\cdot\left\{\mu\cdot z^N\frac{\lambda(\alpha_1^{N-1}-\alpha_2^{N-1})}{\mu(\alpha_1^N-\alpha_2^N)}-\lambda\cdot z^{N-1}+\lambda\cdot\alpha_1^{N-1}\cdot\alpha_2^{N-1}\frac{(\alpha_2-\alpha_1)}{(\alpha_2^N-\alpha_1^N)}\right\}$$

$$=\frac{\lambda}{\mu\left[z^2-\frac{(\lambda+\mu+s)}{\mu}\cdot z+\lambda/\mu\right]}\left[\frac{z^N\cdot(\alpha_1^{N-1}-\alpha_2^{N-1})-z^{N-1}\cdot(\alpha_1^N-\alpha_2^N)}{(\alpha_1^N-\alpha_2^N)}+(\alpha_1\alpha_2)^{N-1}\cdot(\alpha_1-\alpha_2)\right]$$

$$=\frac{\lambda}{\mu(z-\alpha_1)(z-\alpha_2)(\alpha_1^N-\alpha_2^N)}\left[z^N\cdot(\alpha_1^{N-1}-\alpha_2^{N-1})-z^{N-1}(\alpha_1^N-\alpha_2^N)+(\alpha_1\alpha_2)^{N-1}\cdot(\alpha_1-\alpha_2)\right]$$

Since $\alpha_1\cdot\alpha_2=\lambda/\mu$

Then, the numerator is

$$=z^N\cdot\alpha_1^{N-1}-z^{N-1}\cdot\alpha_1^N-z^N\cdot\alpha_2^{N-1}+z^{N-1}\cdot\alpha_2^N+(\lambda/\mu)^{N-1}(\alpha_1-\alpha_2)$$

$$=z^{N-1}\cdot\alpha_1^{N-1}\cdot(z-\alpha_1)-z^{N-1}\cdot\alpha_2^{N-1}\cdot(z-\alpha_2)+(\lambda/\mu)^{N-1}(\alpha_1-\alpha_2)$$

Therefore

$$G^*=\frac{\lambda}{\mu(\alpha_1^N-\alpha_2^N)}\left[\frac{(z\cdot\alpha_1)^{N-1}}{(z-\alpha_2)}-\frac{(z\cdot\alpha_2)^{N-1}}{(z-\alpha_1)}+\left(\frac{\lambda}{\mu}\right)^{N-1}\frac{(\alpha_1-\alpha_2)}{(z-\alpha_1)(z-\alpha_2)}\right]$$

$$=\frac{\lambda}{\mu(\alpha_1^N-\alpha_2^N)}\left[\frac{(z\cdot\alpha_1)^{N-1}}{(z-\alpha_2)}-\frac{(z\cdot\alpha_2)^{N-1}}{(z-\alpha_1)}+\left(\frac{\lambda}{\mu}\right)^{N-1}\left[\frac{1}{(z-\alpha_1)}-\frac{1}{(z-\alpha_2)}\right]\right]$$

since $\alpha_1=\lambda/\mu\cdot 1/\alpha_2 \qquad \alpha_2=\lambda/\mu\cdot 1/\alpha_1$

$$=\frac{\lambda}{\mu(\alpha_1^N-\alpha_2^N)}\left[\frac{(z\cdot\lambda/\mu\cdot 1/\alpha_2)^{N-1}-(\lambda/\mu)^{N-1}}{(z-\alpha_2)}-\frac{(z\cdot\lambda/\mu\cdot 1/\alpha_1)^{N-1}-(\lambda/\mu)^{N-1}}{(z-\alpha_1)}\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\cdot\left[\frac{(z/\alpha_2)^{N-1}-1}{(z-\alpha_2)}-\frac{[(z/\alpha_1)^{N-1}-1]}{(z-\alpha_1)}\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\left[\frac{1}{\alpha_2}\sum_{k=0}^{N-2}(z/\alpha_2)^k-\frac{1}{\alpha_1}\sum_{k=0}^{N-2}(z/\alpha_1)^k\right]$$

since $1/\alpha_2=\alpha_1\cdot(\lambda/\mu)^{-1} \qquad 1/\alpha_1=\alpha_2\cdot(\lambda/\mu)^{-1}$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\left[\alpha_1\cdot\left(\frac{\lambda}{\mu}\right)^{-1}\sum_{k=0}^{N-2}z^k\cdot\alpha_1^k(\lambda/\mu)^{-k}-\alpha_2\cdot\left(\frac{\lambda}{\mu}\right)^{-1}\sum_{k=0}^{N-2}z^k\cdot\alpha_2^k(\lambda/\mu)^{-k}\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\left[\sum_{k=0}^{N-2}z^k\cdot\alpha_1^{k+1}\cdot(\lambda/\mu)^{-(k+1)}-\sum_{k=0}^{N-2}z^k\cdot\alpha_2^{k+1}\cdot(\lambda/\mu)^{-(k+1)}\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\left[\sum_{n=1}^{N-1}z^{n-1}\cdot\alpha_1^n\cdot(\lambda/\mu)^{-n}-\sum_{n=1}^{N-1}z^{n-1}\cdot\alpha_2^n\cdot(\lambda/\mu)^{-n}\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left[\frac{\lambda}{\mu}\right]^{N-1}\left[\sum_{n=1}^{N-1}\left(\frac{\lambda}{\mu}\right)^{-n}z^{n-1}\cdot(\alpha_1^n-\alpha_2^n)\right]$$

$$=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\sum_{n=1}^{N-1}\left(\frac{\lambda}{\mu}\right)^{N-1-n}(\alpha_1^n-\alpha_2^n)\cdot z^{n-1}$$

As a result

$$Q_n^*(s)=\frac{\lambda}{\mu}\cdot\frac{1}{(\alpha_1^N-\alpha_2^N)}\left(\frac{\lambda}{\mu}\right)^{N-1-n}(\alpha_1^n-\alpha_2^n)$$

$$=\left(\frac{\lambda}{\mu}\right)^{N-n}\cdot\left[\frac{\alpha_1^n}{(\alpha_1^N-\alpha_2^N)}-\frac{\alpha_2^n}{(\alpha_1^N-\alpha_2^N)}\right]$$

$$=\left(\frac{\lambda}{\mu}\right)^{N-n}\cdot\left[\frac{\alpha_1^n}{\alpha_1^N(1-(\alpha_2/\alpha_1)^N)}-\frac{\alpha_2^n}{\alpha_1^N(1-(\alpha_2/\alpha_1)^N)}\right]$$

$$=\left(\frac{\lambda}{\mu}\right)^{N-n}\cdot\left[\sum_{k=0}^{\infty}(\alpha_1)^{n-N}\cdot(\alpha_2/\alpha_1)^{N\cdot k}-\sum_{k=0}^{\infty}\alpha_2^n\alpha_1^{-N}(\alpha_2/\alpha_1)^{N\cdot k}\right]$$

since $\alpha_1\cdot\alpha_2=\lambda/\mu$ and $\alpha_2/\alpha_1=\lambda/\mu\cdot 1/\alpha_1^2$

$$(\alpha_1)^{n-N}\cdot(\alpha_2/\alpha_1)^{N\cdot k}=(\lambda/\mu)^{N\cdot k}\cdot 1/\alpha_1^{(-n+(2k+1)N)}$$

$$\alpha_2^n\cdot\alpha_1^{-N}\cdot(\alpha_2/\alpha_1)^{N\cdot k}=(\lambda/\mu)^{n+N\cdot k}\cdot 1/\alpha_1^{n+N+2N\cdot k}$$

$$=\left(\frac{\lambda}{\mu}\right)^{N-n}\sum_{k=0}^{\infty}\left\{\frac{(\lambda/\mu)^{N\cdot k}}{\alpha_1^{-n+(2k+1)N}}-\frac{(\lambda/\mu)^{n+N\cdot k}}{\alpha_1^{n+(1+2k)N}}\right\} \qquad -\ (6,\ 7)$$

****

In the next we operate Laplace invert transform on (6, 7)

$$L^{-1}[\alpha_1^{-(2k+1)N+n}]$$

$$=L^{-1}\left[\frac{\left[\lambda+\mu+s+((\lambda+\mu+s)^2-4\lambda\mu)^{1/2}\right]^{-(2k+1)N+n}}{(2\mu)^{-(2k+1)N+n}}\right]$$

$$=L^{-1}\left[(s+(s^2-4\lambda\mu)^{1/2})^{-(2k+1)N+n}\right]e^{-(\lambda+\mu)t}\cdot(2\mu)^{(2k+1)N-n}$$

$$=L^{-1}\left[\left\{\frac{(s+(s^2-4\lambda\mu)^{1/2})(s-(s^2-4\lambda\mu)^{1/2})}{(s-(s^2-4\lambda\mu)^{1/2})}\right\}^{-(2k+1)N+n}\cdot e^{-(\lambda+\mu)t}\cdot(2\mu)^{(2k+1)N-n}\right]$$

$$=L^{-1}\left[(s^2-s^2+4\lambda\mu)^{-(2k+1)N+n}\cdot\left[s-(s^2-4\lambda\mu)^{1/2}\right]^{(2k+1)N-n}\cdot e^{-(\lambda+\mu)t}\cdot(2\mu)^{(2k+1)N-n}\right]$$

$$=e^{-(\lambda+\mu)t}\cdot(4\lambda\mu)^{-(2k+1)N+n}\cdot(2\mu)^{(2k+1)N-n}\cdot L^{-1}\left[(s-(s^2-4\lambda\mu)^{1/2})^{(2k+1)N-n}\right]$$

O b e r h e t t i n g e r p 242. 4-41 of table

$$L^{-1}\left[(s-(s^2-4\lambda\mu)^{1/2})^{\nu}\right]=\nu/t\cdot(2\sqrt{\lambda\mu})^{\nu}\cdot I_\nu(2\sqrt{\lambda\mu}\cdot t)$$

Setting $\nu=(2k+1)\cdot N-n$

For the recurrent formula of modified Bessel function, setting $z = at$

$$\frac{2\cdot\nu}{at} I_\nu(at) = I_{\nu-1}(at) - I_{\nu+1}(at)$$

Hence, by setting $a = 2\sqrt{\lambda\mu}$

$$\frac{\nu}{t}\left(2\sqrt{\lambda\mu}\right)^{\nu} I_\nu\left(2\sqrt{\lambda\mu}\cdot t\right) = \left(2\sqrt{\lambda\mu}\right)^{\nu}\frac{\left(2\sqrt{\lambda\mu}\right)}{2}\cdot\left[I_{\nu-1}\left(2\sqrt{\lambda\mu t}\right) - I_{\nu+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$

$$= \frac{\left(2\sqrt{\lambda\mu}\right)^{\nu+1}}{2}\cdot\left[I_{\nu-1}\left(2\sqrt{\lambda\mu t}\right) - I_{\nu+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$

$$= \frac{\left(2\sqrt{\lambda\mu}\right)^{(2k+1)N-n+1}}{2}\cdot\left[I_{(2k+1)N-n--1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$

Therefore

$$L^{-1}\left[\alpha_1^{-(2k+1)N+n}\right]$$
$$= e^{-(\lambda+\mu)t}\cdot 2^{-2(2k+1)N+2n}\cdot 2^{(2k+1)N-n}\cdot 2^{(2k+1)N-n}$$
$$\cdot(\lambda\mu)^{-(2k+1)N+n}\cdot\mu^{(2k+1)N-n}\cdot(\lambda\mu)^{(2k+1)N/2-(n-1)/2}$$
$$\cdot\left[I_{(2k+1)N-n--1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$
$$= e^{-(\lambda+\mu)t}\cdot\left(\lambda^{1/2}\right)^{-(2k+1)N+n}\cdot\left(\mu^{1/2}\right)^{(2k+1)N-n}\cdot(\lambda\mu)^{1/2}$$
$$\cdot\left[I_{(2k+1)N-n--1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$
$$= e^{-(\lambda+\mu)t}\cdot(\mu/\lambda)^{(k+1/2)N-n/2}\cdot(\lambda\mu)^{1/2}$$
$$\cdot\left[I_{(2k+1)N-n-1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$

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By the similar procesure, we have

$$L^{-1}\left[\alpha_1^{-(2k+1)N-n}\right]$$
$$= e^{-(\lambda+\mu)t}\cdot(\mu/\lambda)^{(k+1/2)N+n/2}\cdot(\lambda\mu)^{1/2}$$
$$\cdot\left[I_{(2k+1)N+n-1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N+n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$

As a result

$$Q_n^*(s) = \left(\frac{\lambda}{\mu}\right)^{N-n}\left\{\sum_{k=0}^{\infty}(\lambda/\mu)^{k\cdot N}(\mu/\lambda)^{(k+1/2)N-n/2}\cdot\begin{bmatrix} I_{(2k+1)N-n-1}\left(2\sqrt{\lambda\mu t}\right) \\ -I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right)\end{bmatrix}\right.$$
$$-\sum_{k=0}^{\infty}(\lambda/\mu)^{k\cdot N+n}\cdot(\mu/\lambda)^{(k+1/2)N+n/2}\cdot\left[I_{(2k+1)N+n-1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N+n+1}\left(2\sqrt{\lambda\mu t}\right)\right]$$
$$\left.\cdot e^{-(\lambda+\mu)t}\cdot(\lambda\mu)^{1/2}\right]\Bigg\}$$
$$= \left(\frac{\lambda}{\mu}\right)^{(N-n)/2} e^{-(\lambda+\mu)t}\cdot(\lambda\mu)^{1/2}\left\{\sum_{k=0}^{\infty}\left[I_{(2k+1)N-n-1}\left(2\sqrt{\lambda\mu t}\right)\right.\right.$$
$$\left.\left.-I_{(2k+1)N-n+1}\left(2\sqrt{\lambda\mu t}\right) - I_{(2k+1)N+n-1}\left(2\sqrt{\lambda\mu t}\right) + I_{(2k+1)N+n+1}\left(2\sqrt{\lambda\mu t}\right)\right]\right\}$$

****

7. Solution for $P_n(t)$

$$P_0'(t) = \mu\cdot P_1(t) \quad \text{------} \quad \text{(7-1-a)}$$
$$P_1'(t) = -(\lambda+\mu)\cdot P_1(t) + 2\mu\cdot P_2(t) \qquad (7-1-b)$$
$$P_n'(t) = -(\lambda+\mu)n\cdot P_n(t) + \lambda\cdot(n-1)\cdot P_{n-1} + \mu\cdot(n+1)\cdot P_{n+1}(t)$$
$$(7\text{-}1\text{-}c)\quad 1\le n\le N-2$$
$$P_{N-1}'(t) = -(\lambda+\mu)(N-1)\cdot P_{N-1}(t) + \lambda\cdot(N-2)\cdot P_{N-2}(t) \quad \text{(7-1-d)}$$
$$P_N'(t) = -\lambda\cdot(N-1)\cdot P_{N-1}(t) \qquad (7-1-e)$$
$$P_n(0) = \delta_{i,n}$$

Setting generating function

$$P(z,t) = \sum_{n=0}^{N} P_n(t)\cdot z^n$$

$$(7-1-a)\cdot z^0 + (7-1-b)\cdot z^1 + \sum_{n=2}^{N-2}(7-1-c)\cdot z^n + (7-1-d)\cdot z^{N-1} + (e)\cdot z^N$$

By this operation

$$P_0'(t) = \mu\cdot P_1(t)$$
$$z\cdot P_1'(t) = -(\lambda+\mu)\cdot z\cdot P_1(t) + 2\mu\cdot z\cdot P_2(t)$$
$$\sum_{n=2}^{N-2} z^n\cdot P_n'(t) = -(\lambda+\mu)\cdot\sum_{n=2}^{N-2} n\cdot z^n\cdot P_n(t) + \lambda\cdot\sum_{n=2}^{N-2}(n-1)P_{n-1}(t)\cdot z^n$$
$$+\mu\sum_{n=2}^{N-2}(n+1)\cdot P_{n+1}(t)z^n$$
$$z^{N-1}\cdot P_{N-1}'(t) = -(\lambda+\mu)(N-1)\cdot z^{N-1}\cdot P_{N-1}(t) + \lambda\cdot(N-2)\cdot z^{N-1}\cdot P_{N-2}(t)$$
$$z^N\cdot P_N'(t) = -\lambda\cdot(N-1)\cdot z^N\cdot P_{N-1}(t)$$

$$\text{left side} = P_0'(t) + z\cdot P_1'(t) + \sum_{n=2}^{N-2} z^n\cdot P_n'(t) + z^{N-1}\cdot P_{N-1}' + z^N\cdot P_N'$$
$$= \sum_{n=0}^{N} z^n\cdot P_n'(t)$$

The right side

1. For the first set of the terms

$$\mu\cdot\left[P_1(t) + 2\cdot z\cdot P_2(t) + \sum_{n=2}^{N-2}(n+1)\cdot P_{n+1}(t)\cdot z^n\right]$$
$$= \mu\sum_{n=1}^{N-1} n\cdot z^{n-1}\cdot P_n$$

since, $P = \sum_{n=0}^{N} P_n\cdot z^n = P_0 + z\cdot P_1 + z^2\cdot P_2 + \cdots + z^N\cdot P_N$

$$\partial P/\partial z = P_1 + 2\cdot z\cdot P_2 + \cdots + N\cdot z^{N-1}\cdot P_N$$
$$= \sum_{n=1}^{N} n\cdot z^{n-1}\cdot P_n = \sum_{n=1}^{N-1} n\cdot z^{n-1}\cdot P_n - N\cdot z^{N-1}\cdot P_N$$
$$= \mu\left[\partial P/\partial z - N\cdot P_N(t)\cdot z^{N-1}\right]$$

2. For the second set of the terms

$$-(\lambda+\mu)\cdot\left[z\cdot P_1(t) + \sum_{n=2}^{N-2} n\cdot z^n\cdot P_n(t) + (N-1)z^{N-1}P_{N-1}(t)\right]$$
$$= -(\lambda+\mu)\cdot z\cdot\left[P_1(t) + \sum_{n=2}^{N-2} n\cdot z^{n-1}\cdot P_n(t) + (N-1)\cdot z^{N-2}\cdot P_{N-1}(t)\right]$$
$$= -(\lambda+\mu)\cdot z\cdot\sum_{n=1}^{N-1} n\cdot z^{n-1}\cdot P_n(t)$$
$$= -(\lambda+\mu)\cdot z\cdot\left[\partial P/\partial z - N\cdot P_N\cdot z^{N-1}\right]$$

3. For the third set of the terms

$$\lambda\cdot\left[\sum_{n=2}^{N-2}(n-1)\cdot P_{n-1}(t)\cdot z^n + (N-2)\cdot z^{N-1}P_{N-2}(t) - (N-1)z^N P_N\right]$$
$$= \lambda z^2\left[\sum_{n=1}^{N-3} n\cdot z^{n-1}\cdot P_n + (N-2)\cdot z^{N-3}\cdot P_{N-2}(t) - (N-1)z^{N-2}P_{N-1}\right]$$
$$= \lambda z^2\left[\sum_{n=1}^{N-} n\cdot z^{n-1}\cdot P_n(t)\right] = \left[\partial P/\partial z - N\cdot P_N(t)\cdot z^{N-1}\right]\lambda z^2$$

Associate 1 to 3, we have following (7-5)

$$\frac{\partial P}{\partial t} = \left[\lambda z^2 - (\lambda+\mu)z + \mu\right]\frac{\partial P}{\partial z} - N\cdot P_N(t)\cdot z^{N-1}\left[\lambda z^2 - z(\lambda+\mu) + \mu\right]$$

The general solution of (7、5) is

$$P(z,t) = -N\cdot(\lambda-\mu)^2\cdot(1-z)(\lambda-\mu z)\int_0 P_N(s)^{(\lambda-\mu)(t-s)}ds$$
$$\cdot\frac{\left\{\mu\cdot\left(1-e^{(\lambda-\mu)(t-s)}\right) - z\cdot\left(\lambda-\mu e^{(\lambda-\mu)(t-s)}\right)\right\}^{N-1}}{\left\{\mu - \lambda e^{(\lambda-\mu)(t-s)} - \lambda\cdot z\cdot\left(1-e^{(\lambda-\mu)(t-s)}\right)\right\}^{N+1}}ds$$
$$+\left\{\frac{\mu\cdot\left(1-e^{(\lambda-\mu)t}\right) - z\cdot\left(\lambda-\mu e^{(\lambda-\mu)t}\right)}{\mu - \lambda e^{(\lambda-\mu)t} - \lambda\cdot z\cdot\left(1-e^{(\lambda-\mu)t}\right)}\right\}^{i} \qquad (7,\ 6)$$

=====Remark****************************.

A reflecting barrier at the orgin and an absorbing barrier at N with coefficients given as in the case (2).

In this case

$$P_N^*(s) = \frac{\left[\alpha^{i+1} - (\alpha_2-1)/(\alpha_1-1)\cdot\alpha_1^{i+1}\right]}{\alpha_2^N\cdot(\lambda\alpha_2-\mu)(\alpha_2-1) - \alpha_1^N\cdot(\lambda\alpha_1-\mu)(\alpha_2-1)}$$

$$\alpha_j = \left[(\lambda+\mu+s)\pm\left((\lambda+\mu+s)^2 - 4\lambda\mu\right)^{1/2}\right]/(2\lambda)$$

$\alpha_1\alpha_2 = \mu/\lambda$

But $\quad Q_i(t) = \dfrac{dP_N(t)}{dt}$

$$Q_i^*(s) = s\cdot P_N^*(s)$$

$$= \frac{s\cdot\alpha_2^{i+1}\cdot(\alpha_1-1) - s\cdot(\alpha_2-1)\cdot\alpha_1^{i+1}}{(\alpha_1-1)\left[\alpha_2^N\cdot(\lambda\cdot\alpha_2-\mu)(\alpha_2-1)-\alpha_1^N\cdot(\lambda\alpha_1-\mu)(\alpha_2-1)\right]}$$

$$= \frac{\lambda\cdot(\alpha_1^{i+1}-\alpha_2^{i+1})-\mu\cdot(\alpha_1^i-\alpha_2^i)}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$$

****

7. Solution for $P_n(t)$

$$Q_i^*(s) = s\cdot P_N^*(s) = \frac{\lambda\cdot(\alpha_1^{i+1}-\alpha_2^{i+1})-\mu\cdot(\alpha_1^i-\alpha_2^i)}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$$

1. $Q_i'^* = \dfrac{\lambda\cdot\alpha_1^{i+1}}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$

since $\alpha_1\cdot\alpha_2=\mu/\lambda$, putting $\alpha_2=\mu/\lambda\cdot 1/\alpha_1$

and reduce only the function of α_1

$$= \frac{1}{(\alpha_1^{N-i}-\alpha_2^{N+1}\cdot\alpha_1^{-i-1})-\mu/\lambda(\alpha_1^{N-i-1}-\alpha_2^N\cdot\alpha_1^{-i-1})}$$

$$= \frac{1}{(\alpha_1^{N-i}-\alpha_2^{N+1}/\alpha_1^{i+1})}\frac{1}{\left[1-\mu/\lambda(\alpha_1^{N-i-1}-\alpha_2^N\cdot\alpha_1^{-i-1})/(\alpha_1^{N-i}-\alpha_2^{N+1}\cdot\alpha_1^{-i-1})\right]}$$

$$= \frac{1}{(\alpha_1^{N-i}-\alpha_2^{N+1}/\alpha_1^{i+1})}\sum_{k=0}^{\infty}\left(\frac{\mu}{\lambda}\right)^k\left[\frac{(\alpha_1^{N-i-1}-\alpha_2^N\cdot\alpha_1^{-i-1})}{(\alpha_1^{N-i}-\alpha_2^{N+1}\cdot\alpha_1^{-i-1})}\right]^k$$

$$= \sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\frac{(\alpha_1^{N-i-1}-\alpha_2^N/\alpha_1^{i+1})^k}{(\alpha_1^{N-i}-\alpha_2^{N+1}/\alpha_1^{i+1})^{k+1}}$$

Numerator $=\alpha_1^{N-i-1}-(\mu/\lambda)^N/(\alpha_1)^{N+i+1}$

$=\alpha_1^{-i-1}\left[\alpha_1^N-(\mu/\lambda)^N/\alpha_1^N\right]$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\left\{\frac{\left[\alpha_1^{-i-1}(\alpha_1^N-(\mu/\lambda)^N/\alpha_1^N)\right]^k}{\left[\alpha_1^{-i-1}(\alpha_1^{N+1}-(\mu/\lambda)^{N+1}/\alpha_1^{N+1})\right]^{k+1}}\right\}$$

Since N is sufficiently larger than i

$$\approx\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\left\{\alpha_1^{-(i+1)(k-k-1)}\cdot\alpha_1^{Nk-(N+1)(k+1)}\right\}$$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\alpha_1^{i-N-k}$$

Since $Q_i'(t)=L^{-1}\left[Q_i^{'}(s)\right]$. Then we seek the invert Laplace transform of α_1^{i-N-k}

$$L^{-1}\left[\alpha_1^{i-N-k}\right]$$

$$= e^{-(\lambda+\mu)t}\cdot L^{-1}\left[\frac{\left(s+\sqrt{s^2-4\lambda\mu}\right)^{i-N-k}}{(2\lambda)^{i-N-k}}\right]$$

$$=\frac{e^{-(\lambda+\mu)t}}{(2\lambda)^{i-N-k}}\cdot L^{-1}\left[\left\{\frac{\left(s+(s^2-4\lambda\mu)^{1/2}\right)\left(s-(s^2-4\lambda\mu)^{1/2}\right)}{\left(s-(s^2-4\lambda\mu)^{1/2}\right)}\right\}^{i-N-k}\right]$$

$$=\frac{e^{-(\lambda+\mu)t}}{(2\lambda)^{i-N-k}}\cdot L^{-1}\left[(4\lambda\mu)^{i-N-k}\cdot\left[s-(s^2-4\lambda\mu)^{1/2}\right]^{N+k-i}\right]$$

$$=\frac{e^{-(\lambda+\mu)t}}{(2\lambda)^{i-N-k}}(2\lambda)^{i-N-k}\cdot\mu^{i-N-k}\cdot 2^{i-N-k}\cdot L^{-1}\left[\left[s-(s^2-4\lambda\mu)^{1/2}\right]^{N+k-i}\right]$$

since $L^{-1}\left[\left[s-(s^2-4\lambda\mu)^{1/2}\right]^{N+k-i}\right]$

$$=\frac{(N+k-i)}{t}I_{N+k-i}\left(2\sqrt{\lambda\mu t}\right)\left(2\sqrt{\lambda\mu}\right)^{N+k-i}$$

$$=\frac{e^{-(\lambda+\mu)t}\cdot(N+k-i)}{t}I_{N+k-i}\left(2\sqrt{\lambda\mu t}\right)(\lambda/\mu)^{(N+k-i)/2}$$

2. The second term

$$Q_i^{*2}(s)=\frac{\lambda\cdot\alpha_2^{i+1}}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$$

$$=\frac{1}{(\alpha_1^{N+1}\cdot\alpha_2^{-i-1}-\alpha_2^{N+1-i-1})-\mu/\lambda(\alpha_1^N\cdot\alpha_2^{-i-1}-\alpha_2^{N-i-1})}$$

$$=\frac{1}{(\alpha_1^{N+1}\cdot\alpha_2^{-i-1}-\alpha_2^{N-i})}\frac{1}{\left[1-\mu/\lambda\cdot(\alpha_1^N\cdot\alpha_2^{-i-1}-\alpha_2^{N-i-1})/(\alpha_1^{N+1}\cdot\alpha_2^{-i-1}-\alpha_1^{N-i})\right]}$$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\frac{(\alpha_1^N\cdot\alpha_2^{-i-1}-\alpha_2^{N-i-1})^k}{(\alpha_1^{N+1}\cdot\alpha_2^{-i-1}-\alpha_2^{N-i})^{k+1}}$$

since $\alpha_2=\mu/\lambda\cdot\alpha_1^{-1}$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\frac{\left[\alpha_1^N(\mu/\lambda)^{-i-1}\cdot\alpha_1^{i+1}-(\mu/\lambda)^{N-i-1}\cdot\alpha_1^{i+1-N}\right]^k}{\left[\alpha_1^{N+1}(\mu/\lambda)^{-i-1}\cdot\alpha_1^{i+1}-(\mu/\lambda)^{N-i}\cdot\alpha_1^{i-N}\right]^{k+1}}$$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k(\mu/\lambda)^{(-i-1)(k-(k+1))}\frac{\left[\alpha_1^{N+i+1}-(\mu/\lambda)^N\cdot\alpha_1^{i+1-N}\right]^k}{\left[\alpha_1^{N+i+2}-(\mu/\lambda)^{N+1}\cdot\alpha_1^{i-N}\right]^{k+1}}$$

since $N\gg i$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^{k+i+1}\alpha_1^{(N+i+1)k-(N+i+2)(k+1)}$$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^{k+i+1}\alpha_1^{-(N+k+i+2)}$$

Because of

$$L^{-1}\left[\alpha_1^{-(N+k+i+2)}\right]=\frac{e^{-(\lambda+\mu)t}\cdot(N+k+i+2)}{t}I_{N+k+i+2}\left(2\sqrt{\lambda\mu t}\right)(\lambda/\mu)^{(N+k+i+2)/2}$$

Then, we have the invertion for the second term

$$Q_i^2(t)=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^{k+i+1}\left(\frac{\lambda}{\mu}\right)^{(N+k+i+2)/2}\cdot\frac{(N+k+i+2)}{t}I_{N+k+i+2}\left(2\sqrt{\lambda\mu t}\right)e^{-(\lambda+\mu)t}$$

3. The third term is

$$Q_i^{*3}(s)=\frac{\mu\cdot\alpha_1^i}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$$

$$=\frac{\mu}{\lambda}\cdot\left\{\frac{\lambda\cdot\alpha_1^i}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu(\alpha_1^N-\alpha_2^N)}\right\}$$

Then,

$$Q_i^3(t)=\frac{\mu}{\lambda}\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^k\cdot\frac{(N+k-(i-1))}{t}I_{N+k-i+1}\left(2\sqrt{\lambda\mu t}\right)(\lambda/\mu)^{(N+k-i+1)/2}$$

$$=\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^{k+1}\left(\frac{\lambda}{\mu}\right)^{(N+k-i+1)/2}\cdot\frac{(N+k-i+1)}{t}I_{N+k-i+1}\left(2\sqrt{\lambda\mu t}\right)$$

4. The fourth term is

$$Q_i^{*4}(s)=\frac{\mu\cdot\alpha_2^i}{\lambda\cdot(\alpha_1^{N+1}-\alpha_2^{N+1})-\mu\cdot(\alpha_1^N-\alpha_2^N)}$$

Then,

$$Q_i^4(t)=\left(\frac{\mu}{\lambda}\right)\sum_{k=0}\left(\frac{\mu}{\lambda}\right)^{k+i-1+1}\cdot(\lambda/\mu)^{(N+k+i+1)/2}\cdot\frac{(N+k+i-1+2)}{t}\cdot I_{N+k+i-1+2}\left(2\sqrt{\lambda\mu t}\right)$$

Thus, for I << N, we have finally

$$Q_i(t)=e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}\left(\frac{\mu}{\lambda}\right)^k\left(\frac{\lambda}{\mu}\right)^{(N+k-i)/2}\cdot\frac{N+k-i}{t}I_{N+k-i}\left(2\sqrt{\lambda\mu t}\right)$$

$$-\left(\frac{\mu}{\lambda}\right)^{k+i+1}\left(\frac{\lambda}{\mu}\right)^{(N+k+i+2)/2}\cdot\frac{N+k+i+2}{t}I_{N+k+i+2}\left(2\sqrt{\lambda\mu t}\right)$$

$$-\left(\frac{\mu}{\lambda}\right)^{k+1}\left(\frac{\lambda}{\mu}\right)^{(N+k-i+1)/2}\cdot\frac{N+k-i+1}{t}I_{N+k-i+1}\left(2\sqrt{\lambda\mu t}\right)$$

$$+\left(\frac{\mu}{\lambda}\right)^{k+i+1}\left(\frac{\lambda}{\mu}\right)^{(N+k+i+1)/2}\cdot\frac{N+k+i+1}{t}I_{N+k+i+1}\left(2\sqrt{\lambda\mu t}\right)$$

The mean is

$$\int_0^\infty tQ_i(t)dt=\lim_{p\to0}\int_0^\infty te^{-pt}Q_i(t)dt = \frac{1}{\lambda-\mu}\left\{(N-i)-\frac{\left(\frac{\mu}{\lambda}\right)^{i+1}}{1-\frac{\mu}{\lambda}}\right\}$$

4. Explicit Solution to Case (1) for Large N

There is a simple method of determining $Q_n(t)$ for large values of N, the interesting case for most applications, without resort to Laplace transforms. We now obtain this expression for large N.

In (3.8) we note that $Q_{N-1}(s)$ is a conditional probability distribution, with $Q_{N-1}(0)=(N-1)\lambda$. In fact by assuming a solution of the form $G+H\beta^n$ after integrating the set of differential-difference equations (3.1) on $[0,\infty]$, it is easily shown that

$$\int_0^\infty Q_n(t)\,dt=\frac{1-\beta^n}{1-\beta^N}.$$

Consequently for $\mu<\lambda$ (the case of interest) and with $n=N-1$,

$$\lim_{N\to\infty}\int_0^\infty Q_{N-1}(t)\,dt=1.$$

Also $\lim_{N\to\infty} Q_{N-1}(0)\to\infty$, and hence $Q_{N-1}(t)$ is almost a right-sided Dirac delta function.

But for such a function $\delta(x)$ we have

$$\int_0^t f(x)\,\delta(x)\,dx=f(0).$$

Therefore for large N and for t away from the origin we have

$$\int_0^t Q_{N-1}(s)\,f(t-s)\,ds\sim f(t),$$

i.e. equals the value of $f(t-s)$ at $s=0$, since for large s, $Q_{N-1}(s)$ is zero except near the origin where it has a spike of order N.

Applying these ideas to (3.8), one has

$$\left.\begin{aligned}\frac{Q_n(t)}{\lambda}\sim&-(-1)^{n-1}\frac{\beta\alpha^2}{(n-1)!}e^{-\alpha t}\frac{A^{N-n}}{C^{N+1}}B^{n-1}\\&\times\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(N+i)!}{(N-n+i)!}\left(-\frac{\beta A^2}{BC}\right)^i\\&+(-1)^{n-1}\frac{\alpha^2}{(n-1)!}e^{-\alpha t}\frac{A^{N-n-1}}{C^N}B^{n-1}\\&\times\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(N+i-1)!}{(N-n+i-1)!}\left(-\frac{\beta A^2}{BC}\right)^i,\end{aligned}\right\}\tag{4.1}$$

where $A=\lambda(1-e^{-\alpha t})$, $B=\mu-\lambda e^{-\alpha t}$ and $C=\lambda-\mu e^{-\alpha t}$.

$$\begin{aligned}\frac{\cancel{Q_n(t)}}{\lambda}\sim&\frac{(1-\beta)^2}{(n-1)!}e^{-\alpha t}\frac{(1-e^{-\alpha t})^{N-n-1}}{(1-\beta e^{-\alpha t})^{N+n}}\sum_{k=0}^{n-1}(1-\beta)^{2k}\beta^{n-k-1}e^{-k\alpha t}\\&\times\sum_{i=k}^{n-1}\left\{(1-\beta)-\frac{n}{N+1}(1-\beta e^{-\alpha t})\right\}\binom{n-1}{i}\binom{i}{k}n!\,(-1)^{n+i-1}\\&\times\left\{\binom{i}{0}\binom{N}{n}+\binom{i}{1}\binom{N}{n-1}+\dots+\binom{i}{i}\binom{N}{n-i}\right\}\\&\times\sum_{j=0}^{n-k-1}\sum_{m=0}^{n-k-1}\binom{n-k-1}{j}\binom{n-k-1}{m}(-1)^{j+m}e^{-(j+m)\alpha t}\beta^{-j+m}.\end{aligned}\quad(4\ldots$$

Also $N^{p_k}e^{-q_k\alpha i}=O(1)$, as $\alpha t\to\log N$ $(N\gg1, p_k<q_k)$. For example,

$$\frac{Q_1(t)}{\lambda}\sim(1-\beta)^3Ne^{-\alpha t}\frac{(1-e^{-\alpha t})^{N-2}}{(1-\beta e^{-\alpha t})^{N+1}}.$$

Therefore

$$\begin{aligned}\log\frac{Q_1(t)}{\lambda(1-\beta)^3}&=\log N-\alpha t+(N-2)\log(1-e^{-\alpha t})-(N+1)\log(1-\beta e^{-\alpha t})\\&\sim\log N-\alpha t-(1-\beta)Ne^{-\alpha t},\end{aligned}$$

and it is clear that the main contribution to $Q_1(t)$ is obtained for values of αt in the neighbourhood of $\log N$.

Hence, a term $a_kN^{p_k}e^{-q_k\alpha t}$ is either $O(1)$ if $\alpha t\sim\log N$ and $p_k=q_k$, or is negligible for $p_k<q_k$ and αt small, or is arbitrarily large for $\alpha t\gg\log N$, $N\gg1$, and $p_k>q_k$. However, since $Q_n(t)$ is continuous and bounded for all n and N with

$$\int_0^\infty Q_n(t)\,dt=\frac{1-\beta^n}{1-\beta^N},$$

it follows that all terms in which $p_k>q_k$ must cancel. When these ideas are applied to computing $Q_n(t)$ $(n=2,3,4)$, for example, we have

$$\frac{Q_2(t)}{\lambda}\sim\frac{(1-e^{-\alpha t})^{N-3}}{(1-\beta e^{-\alpha t})^{N+2}}\{(1-\beta)^5N^2e^{-2\alpha t}+2\beta(1-\beta)^3Ne^{-\alpha t}\},$$

$$\frac{Q_3(t)}{\lambda}\sim\frac{(1-e^{-\alpha t})^{N-4}}{(1-\beta e^{-\alpha t})^{N+3}}\left\{\frac{(1-\beta)^7N^3e^{-3\alpha t}}{2!}+3\beta(1-\beta)^5N^2e^{-2\alpha t}+3\beta^2(1-\beta)^3Ne^{-\alpha t}\right\},$$

$$\frac{Q_4(t)}{\lambda}\sim\frac{(1-e^{-\alpha t})^{N-4}}{(1-\beta e^{-\alpha t})^{N+3}}\left\{\frac{(1-\beta)^9N^4e^{-4\alpha t}}{3!}+2\beta(1-\beta)^7N^3e^{-3\alpha t}+6\beta^2(1-\beta)^5N^2e^{-2\alpha t}+4\beta^3(\ldots\beta)^3Ne^{-\alpha t}\right\}.$$

for large N may now be written as

$$\begin{aligned}\frac{Q_n(t)}{\lambda}&\sim\frac{(1-\beta)^3}{(n-1)!}\frac{(1-e^{-\alpha t})^{N-n-1}}{(1-\beta e^{-\alpha t})^{N+n}}\sum_{k=0}^{n-1}\frac{n!\,(n-1)!}{(n-k-1)!\,k!}\beta^{n-k-1}(1-\beta)^{2k}e^{-(k+1)\alpha t}\frac{N^{k+1}}{(k+1)!}\\&=(1-\beta)^3\frac{(1-e^{-\alpha t})^{N-n-1}}{(1-\beta e^{-\alpha t})^{N+n}}\sum_{k=0}^{n-1}\binom{n}{k+1}\beta^{n-k-1}(1-\beta)^{2k}\frac{(Ne^{-\alpha t})^{k+1}}{k!}.\end{aligned}\tag{4.5}$$

(a) *The Mean*

We can work to obtain the unnormalized mean μ'_n defined by

$$\frac{\mu'_n}{\lambda}=\int_0^\infty t\frac{Q_n(t)}{\lambda}dt=\sum_{k=0}^{n-1}\binom{n}{k+1}(1-\beta)^{k+2}\beta^{n-(k+1)}\frac{1}{\alpha^2}$$

$$\times\int^\infty\{y+\log N+\log(1-\beta)-\log(k+1)\}T_{k+1}(y)\,dy.\qquad(4.13)\qquad T_m(y)=\frac{m^m}{(m-1)!}e^{-my-me^{-y}},$$

Writing $\alpha^2=\lambda^2(1-\beta)^2$, and using (4.8) and (4.9), this expression becomes

$$\frac{\mu'_n}{\lambda}=\sum_{k=0}^{n-1}\binom{n}{k+1}(1-\beta)^{k+1}\left[\frac{\beta^{n-(k+1)}}{(1-\beta)\lambda^2}\left\{\log N+\log(1-\beta)+\gamma-\sum_{j=1}^{k-1}\frac{1}{j}\right\}\right].$$

Hence the normalized mean (dividing by $1-\beta^n$) is given by

$$\mu_n=\frac{1}{\lambda(1-\beta)}\left[\{\log N+\log(1-\beta)+\gamma\}-\frac{1}{\lambda(1-\beta^n)}\sum_{j=1}^{n-1}\frac{(1-\beta^j)(1-\beta^{n-j})}{j}\right].\tag{4.15}$$

(b) *The Variance*

We continue to work inside the summation sign in (4.7). The normalized σ_n^2 is obtained from

$$\frac{\sigma_n^2}{\lambda}=\int_0^\infty(t-\mu_n)^2\frac{Q_n(t)}{\lambda(1-\beta^n)}dt=\int_0^\infty t^2\frac{Q_n(t)}{\lambda(1-\beta^n)}dt-\frac{\mu_n^2}{\lambda},$$

which on using (4.9), (4.10) and (4.11) simplifies to:

$$\begin{aligned}\lambda^2\sigma_n^2=&\frac{1}{1-\beta^n}\left[\frac{1}{(1-\beta)^2}\left\{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{k=1}^{i}\binom{n}{k}(1-\beta)^k\beta^{n-k}\right.\right.\\&\left.\left.+(1-\beta^n)\sum_{i=n+1}^{\infty}\frac{1}{i^2}+\sum_{k=1}^{n}\binom{n}{k}(1-\beta)^k\beta^{n-k}\left(\sum_{j=1}^{k-1}\frac{1}{j}\right)^2\right\}\right]\\&-\frac{1}{(1-\beta)^2}\frac{1}{(1-\beta^n)^2}\sum_{j=1}^{n-1}\frac{(1-\beta^j)(1-\beta^{n-j})^2}{j}.\end{aligned}\tag{4.16}$$

It is significant to note here that the variance does not depend on N.

$$\sum_{k=0}^{n-1}\binom{n}{k+1}(1-\beta)^{k+2}\beta^{n-(k+1)}\frac{(k+1)^{k+1}}{k!}e^{-(k+1)y-(k+1)e^{-y}}.\tag{4.7}$$

Qn and beta at N=10

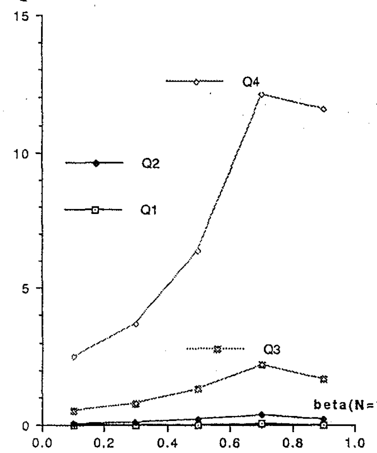

Fig 1

Mean vs beta at N =10

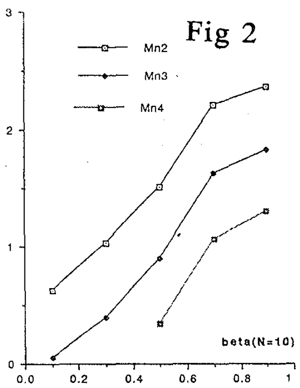

Fig 2

Mean vs Beta at N=100

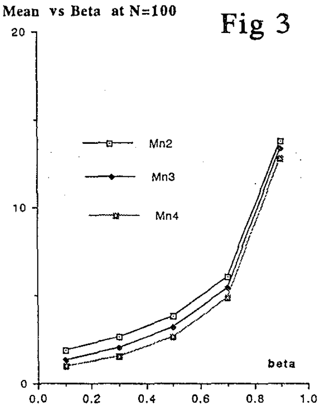

Fig 3

3. Results

Fig 1 shows the Qn (n=1,2,3,4) values as functions of beta (= μ/λ) for N =10, t =1.0 and α =0.1. As the beta value increased, Qn value showed definite peaks. Fig 2 and Fig 3 shows the beta value at N=100 on the mean of Qn. The dependency of beta was completely different between N=10 and N=100. The present approach ,when extended will be available for evaluating the signal filtering function of the neural cellular membrane.

4. Reference

1. Saaty.T.L. J. Roy. Statistics. Soc. SerB.23. pp 319-334.1961.