

# Micro Hydro Dynamical Analysis of Creeping Motions of a Non Spherical Bio Molecular Particle toward the Biomembrane.

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あらまし

A mathematical and mechanical engineering method was introduced to describe the interaction process between the bio molecular particle as a bio signal information carrier and the target receptor. The basic strategy lies in micro hydro dynamics. We show a detailed process proposed by Fan and Wu, Wangi (1987) which was originally proposed by Sampson (1898) founded upon singular points analysis. The stoke stream function, flow velocity and pressure were expressed by linear sum of Legendre polynomials and Gegenbauer polynomials in combination with irrational functions regarding with the distance between the points. The coefficients in series expansion of the stream function and velocity were determined by the non slip conditions on the body.

Bio molecular particle, Axisymmetric, Prolate, Micro hydrodynamics, Stokes equation, Legendre

## 非球状生体粒子の分子動力学的解析

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### Abstract

生体膜近傍における生体分子の挙動を解析し、生体情報伝達物質と受容体との機械力学的結合を記述する目的で微小力学的解析を行った。生体分子の一般形状を考慮し回転楕円体が無限の平面にむかって運動する場合をY. Fan, Wu, Wangi (1987)らの理論を応用して分析する解析的微小流体力学的手法を紹介した。円柱座標型ストークス式とSampson (1898)の有限特異点法とを組み合わせ、流速、流れ関数はレジェンドレ関数とゲーゲンバウエル関数および無理関数の線形級数として、表現された。楕円体上での滑りなしを境界条件として、これをみたすための未定係数を代数方程式を解くことで決定した。楕円体周囲の摩擦力は対称軸からの距離によって変化した。本研究は任意の形状を有する生体分子の生体膜近傍へ接近する際の挙動解析の基礎を与えられらる。

生体分子, 回転楕円体, 微小流体力学, 円柱座標型ストークス式, レジェンドレ関数

## 1. Introduction.

A mathematical method proposed by Fan (1987) was introduced for the creeping motion of a prolate axisymmetric body translating perpendicularly towards an infinite plane along its major axis. The coordinates  $(R, z, \phi)$  is chosen with its origin at the body center (Fig). The governing

Fig 1

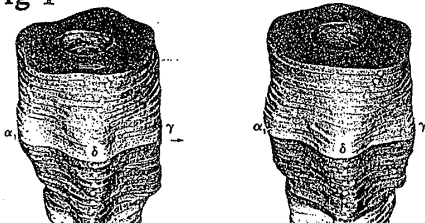


Fig 2. Bio molecules emphasizing non sphere.

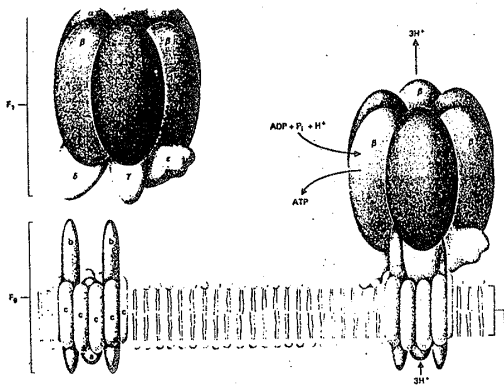


Fig 3. Coordinate systems and singular points.

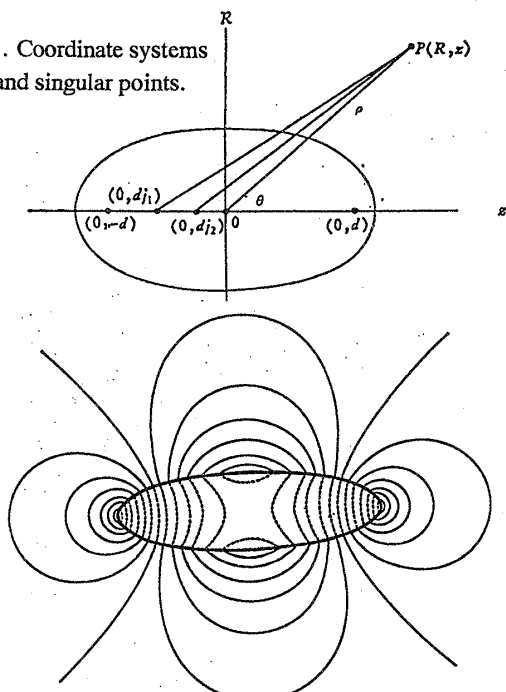


Fig 4. Contrors around the prolate.

equation are  $\nabla^2 v = \nabla p$ ,  $\nabla \cdot v = 0$  where  $v$  and  $p$  are the dimensionless velocity vector and pressure. The stream function was introduced in cylindrical coordinate as  $v_R = -1/R \partial \psi / \partial z$  and  $v_z = 1/R \partial \psi / \partial R$

$$\frac{\partial p}{\partial z} = \frac{1}{R} \frac{\partial}{\partial R} (D^2 \psi), \quad \frac{\partial p}{\partial R} = -\frac{1}{R} \frac{\partial}{\partial z} (D^2 \psi) \quad (2.5)$$

## III. The Expression for Plannar Reflection

of the Sampsonlet:

The expression for Sampsonlet at the point  $R=0, z=\xi$

$$\left. \begin{aligned} v_z &= \sum_{n=2}^{\infty} [C_n F_n^{(1)}(R, z-\xi) + D_n F_n^{(2)}(R, z-\xi)] \\ v_R &= \sum_{n=2}^{\infty} [C_n F_n^{(3)}(R, z-\xi) + D_n F_n^{(4)}(R, z-\xi)] \\ \psi &= \sum_{n=2}^{\infty} [C_n F_n^{(6)}(R, z-\xi) + D_n F_n^{(5)}(R, z-\xi)] \\ p &= p_{\infty} + \sum_{n=2}^{\infty} D_n \frac{4n-6}{n} F_n^{(1)}(R, z-\xi) \end{aligned} \right\} \quad (3.1)$$

$$F_n^{(1)}(R, z) = (R^2 + z^2)^{-\frac{n+1}{2}} P_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right)$$

$$F_n^{(2)}(R, z) = (R^2 + z^2)^{-\frac{n+1}{2}} \left[ P_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right) + 2J_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right) \right]$$

$$F_n^{(3)}(R, z) = (n+1)(R^2 + z^2)^{-\frac{n+1}{2}} \frac{1}{R} J_{n+1}\left(\frac{z}{\sqrt{R^2 + z^2}}\right)$$

$$F_n^{(4)}(R, z) = (n+1)(R^2 + z^2)^{-\frac{n+1}{2}} \frac{1}{R} J_{n+1}\left(\frac{z}{\sqrt{R^2 + z^2}}\right) - 2z(R^2 + z^2)^{-\frac{n+1}{2}} \frac{1}{R} J_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right)$$

$$F_n^{(5)}(R, z) = (R^2 + z^2)^{-\frac{n+1}{2}} J_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right)$$

$$F_n^{(6)}(R, z) = (R^2 + z^2)^{-\frac{n+1}{2}} J_n\left(\frac{z}{\sqrt{R^2 + z^2}}\right) \quad (3.2)$$

$P_n, J_n$  are the Legendre polynomials and Gegenbauer. The expressions for plannar reflection of Sampsonlet may be derived as follows. Decompose the stream function into two pairs

$$\psi = \psi_s + \psi_w \quad (3.3)$$

where  $\psi_s$  is the stream function of Sampsonlet given by the third one in (2.6).  $\psi_w$  represents the disturbance generated by the infinite wall and has following expressions in cylindrical coordinates.

$$\psi_w = \int_0^{\infty} R J_1(\omega R) [A^*(\omega) + z B^*(\omega)] \exp[\omega z] d\omega \quad (z \leq d) \quad (3.4)$$

According to (2.2) and (2.5)

$$v_z(R, z) =$$

$$\int_0^{\infty} \omega J_0(\omega R) [A^*(\omega) + z B^*(\omega)] \exp[\omega z] d\omega$$

$$v_R(R, z) = - \int_0^{\infty} \omega J_1(\omega R)$$

$$\left\{ \frac{1}{\omega} [\omega A^*(\omega) + (1 + \omega z) B^*(\omega)] \exp[\omega z] \right\} d\omega$$

$$p_w(R, z) = p_{-\infty} + 2 \int_0^{\infty} \omega J_0(\omega R) B^*(\omega) \exp[\omega z] d\omega$$

$$G^*(\omega, z) = [\omega A^*(\omega) + (1 + \omega z) B^*(\omega)] \exp[\omega z] / \omega$$

$$F^*(\omega, z) = [A^*(\omega) + zB^*(\omega)] \exp[\omega z] \quad (3.6)$$

$A^*(\omega)$  and  $B^*(\omega)$  can be expressed by

$$A^*(\omega) = (1 + \omega d) \exp[-\omega d] F^*(\omega, d) \quad (3.7)$$

$$- \omega d \exp[-\omega d] G^*(\omega, d)$$

$$B^*(\omega) = \omega [G^*(\omega, d) - F^*(\omega, d)] \exp[-\omega d]$$

Using (3.6) and (3.7),

$$F^*(\omega, z) = \{F^*(\omega, d) + \omega(z-d)$$

$$[G^*(\omega, d) - F^*(\omega, d)]\} \exp[\omega(z-d)]$$

$$G^*(\omega, z) = \{-\omega(z-d)F^*(\omega, d) \quad (3.8)$$

$$+ [1 + \omega(z-d)]G^*(\omega, d)\} \exp[\omega(z-d)]$$

Taking (3.6) and (3.3) into account, one has

$$v_z(R, z, \xi) = \sum_{n=2}^{\infty} [C_n F_n^{(1)}(R, z-\xi)$$

$$+ D_n F_n^{(2)}(R, z-\xi)] + \int_0^{\infty} \omega J_0(\omega R) F^*(\omega, z) d\omega$$

$$v_R(R, z, \xi) = \sum_{n=2}^{\infty} [C_n F_n^{(3)}(R, z-\xi)$$

$$+ D_n F_n^{(4)}(R, z-\xi)] - \int_0^{\infty} \omega J_1(\omega R) G^*(\omega, z) d\omega$$

$$\psi(R, z, \xi) = \sum_{n=2}^{\infty} [C_n F_n^{(5)}(R, z-\xi)$$

$$+ D_n F_n^{(6)}(R, z-\xi)] + \int_0^{\infty} R J_1(\omega R) F^*(\omega, z) d\omega$$

$$p(R, z, \xi) = p_{-\infty} + \sum_{n=2}^{\infty} D_n \frac{4n-6}{n} F_{n-1}^{(1)}(R, z-\xi)$$

$$+ 2 \int_0^{\infty} \omega^2 J_0(\omega R) [G^*(\omega, d)$$

$$- F^*(\omega, d)] \exp[\omega(z-d)] d\omega \quad (3.9)$$

the nonslip condition on the plane i.e.  $v_z = 0, v_R = 0$  at

$z = d$ , it yields

$$\int_0^{\infty} \omega J_0(\omega R) F^*(\omega, d) d\omega$$

$$= - \sum_{n=2}^{\infty} [C_n F_n^{(1)}(R, d-\xi) + D_n F_n^{(2)}(R, d-\xi)]$$

$$\int_0^{\infty} \omega J_1(\omega R) G^*(\omega, d) d\omega \quad (3.10)$$

$$= \sum_{n=2}^{\infty} [C_n F_n^{(3)}(R, d-\xi) + D_n F_n^{(4)}(R, d-\xi)]$$

Inversing above Hankel transformation gives

$$F^*(\omega, d) = - \sum_{n=2}^{\infty} [C_n \Pi_n^{(1)}(\omega) + D_n \Pi_n^{(2)}(\omega)] \quad (3.11)$$

$$G^*(\omega, d) = - \sum_{n=2}^{\infty} [C_n \Pi_n^{(3)}(\omega) + D_n \Pi_n^{(4)}(\omega)]$$

$$\Pi_n^{(1)}(\omega) = \frac{\omega^{n-1}}{n!} \exp[-\omega(d-\xi)]$$

$$\Pi_n^{(2)}(\omega) = \frac{\omega^{n-3}}{n!} [(2n-3)\omega(d-\xi) - (n-1)(n-3)] \exp[-\omega(d-\xi)]$$

$$\Pi_n^{(3)}(\omega) = \frac{\omega^{n-1}}{n!} \exp[-\omega(d-\xi)] \quad (3.12)$$

$$\Pi_n^{(4)}(\omega) = \frac{\omega^{n-3}}{n!} [(2n-3)\omega(d-\xi) - n(n-2)] \exp[-\omega(d-\xi)]$$

Using (3.11), the properties of Legendre polynomials

$$\int_0^{\infty} \frac{x}{(a^2 + x^2)^{\mu/2}} P_{\mu-1}^{\nu}(x) dx$$

$$= \left[ \frac{a}{(a^2 + x^2)^{1/2}} \right] J_{\nu}(xy) dx = \frac{y^{\mu-2} \exp[-ay]}{\Gamma(\mu+\nu)}$$

$$(Re(\nu) > -1, Re(\mu) > 1/2)$$

Substituting (3.11) into (3.8) and (3.9) and

$$\int_0^{\infty} x^{n-1/2} \exp[-ax] (xy)^{1/2} J_0(xy) dx$$

$$= \frac{n! y^{1/2}}{(a^2 + y^2)^{\frac{n+1}{2}}} P_n \left[ \frac{a}{(a^2 + y^2)^{1/2}} \right] \quad (y > 0, Re(a) > 0)$$

$$U(R, z, \xi) = \sum_{n=2}^{\infty} [C_n S_n^{(o)}(R, z, \xi) + D_n S_n^{(p)}(R, z, \xi)] \quad (3.13)$$

$$U(R, z, \xi) = (v_z, v_R, \psi, p - p_{\infty})$$

$$S_n^{(o)}(R, z, \xi) = (S_n^{(1)}, S_n^{(3)}, S_n^{(5)}, S_n^{(7)})$$

$$S_n^{(p)}(R, z, \xi) = (S_n^{(2)}, S_n^{(4)}, S_n^{(6)}, S_n^{(8)})$$

$$S_n^{(1)}(R, z, \xi)$$

$$= F_n^{(1)}(R, z-\xi) - F_n^{(1)}(R, 2d-z-\xi)$$

$$+ 2(z-d)(n+1) F_{n+1}^{(1)}(R, 2d-z-\xi)$$

$$S_n^{(2)}(R, z, \xi)$$

$$= F_n^{(2)}(R, z-\xi) - F_n^{(2)}(R, 2d-z-\xi) - 2(n-2)$$

$$(z-d) F_{n-1}^{(1)}(R, 2d-z-\xi) + 2(2n-3)(z-d)$$

$$(d-\xi) F_n^{(1)}(R, 2d-z-\xi)$$

$$S_n^{(3)}(R, z, \xi) = F_n^{(3)}(R, z-\xi) - F_n^{(3)}(R, 2d-z-\xi)$$

$$- 2(n+1)(z-d) F_{n+1}^{(3)}(R, 2d-z-\xi)$$

$$S_n^{(4)}(R, z, \xi) = F_n^{(4)}(R, z-\xi)$$

$$- F_n^{(4)}(R, 2d-z-\xi) - 2(2n-3)(z-d)$$

$$(d-\xi) F_n^{(3)}(R, 2d-z-\xi) + \frac{2(n-1)(n-3)}{n}$$

$$(z-d) F_{n-1}^{(3)}(R, 2d-z-\xi)$$

$$S_n^{(6)}(R, z, \xi) = F_n^{(6)}(R, z-\xi) - F_n^{(6)}(R, 2d-z-\xi) + 2(z-d) \\ (n+1) F_{n+1}^{(6)}(R, 2d-z-\xi)$$

$$S_n^{(6)}(R, z, \xi) = F_n^{(6)}(R, z-\xi) - F_n^{(6)}(R, 2d-z-\xi) - 2(n-2) \\ (z-d) F_{n-1}^{(6)}(R, 2d-z-\xi)$$

$$+ 2(2n-3)(z-d)(d-\xi) F_n^{(6)}(R, 2d-z-\xi) \\ S_n^{(7)}(R, z, \xi) = 4(n+1) F_{n+1}^{(1)}(R, 2d-z-\xi)$$

$$S_n^{(8)}(R, z, \xi) = \frac{4n-6}{n} F_{n-1}^{(1)}(R, z-\xi) \\ + 4(2n-3)(d-\xi) F_n^{(1)}(R, 2d-z-\xi) \\ - \frac{2(2n^2-6n+3)}{n} F_{n-1}^{(1)}(R, 2d-z-\xi)$$

$$G_{njh}^{(i)}(R, z) = \int_{d_{j1}}^{d_{j2}} \xi^{k-1} F_n^{(i)}(R, z-\xi) d\xi \quad (4.18)$$

The necessary formula for  $G_{njh}^{(i)}$  can be

$$G_{njh}^{(i)}(R, z) = d_{j2}^{(k-1)} J_n^{(i)}(R, z-d_{j2})$$

$$- d_{j1}^{(k-1)} J_n^{(i)}(R, z-d_{j1}) - \frac{k-1}{n} G_{n-1,j,k-1}^{(i)}(R, z) \\ (i=1, 3; k=1, 2, 4; n \geq k; j=1, \dots, M)$$

$$G_{njh}^{(2)}(R, z) = d_{j2}^{(k-1)} J_n^{(2)}(R, z-d_{j2}) \\ - d_{j1}^{(k-1)} J_n^{(2)}(R, z-d_{j1}) - \frac{k-1}{n} G_{n-1,j,k-1}^{(2)}(R, z)$$

$$+ \frac{2(k-1)}{n(n-1)} \left[ G_{n-2,j,k}^{(1)}(R, z) \right. \\ \left. - z G_{n-2,j,k-1}^{(1)}(R, z) - \frac{1}{n-2} G_{n-3,j,k-1}^{(1)}(R, z) \right] \\ (k=1, 2, 3, 4; n \geq k+2; j=1, \dots, M)$$

$$G_{njh}^{(4)}(R, z) = d_{j2}^{(k-1)} J_n^{(4)}(R, z-d_{j2}) \\ - d_{j1}^{(k-1)} J_n^{(4)}(R, z-d_{j1}) - \frac{k-1}{n} G_{n-1,j,k-1}^{(3)}(R, z)$$

$$+ \frac{2(k-1)}{n(n-1)} [G_{n-2,j,k}^{(3)}(R, z) - z G_{n-2,j,k-1}^{(3)}(R, z)] \\ (k=1, 2, 3, 4; n \geq k+2; j=1, \dots, M)$$

$$J_n^{(i)}(R, z-\xi) = \int F_n^{(i)}(R, z-\xi) d\xi$$

$$= \frac{1}{n} F_{n-1}^{(i)}(R, z-\xi) \quad (i=1, 3, n \geq 1)$$

$$J_n^{(2)}(R, z-\xi) = \int F_n^{(2)}(R, z-\xi) d\xi$$

$$= \frac{1}{n} F_{n-1}^{(i)}(R, z-\xi)$$

$$+ \frac{2}{n(n-1)} (z-\xi) F_{n-2}^{(i-1)}(R, z-\xi)$$

$$+ \frac{2}{n(n-1)(n-2)} F_{n-3}^{(i-1)}(R, z-\xi) \quad (n \geq 3)$$

$$J_n^{(4)}(R, z-\xi) = \frac{1}{n} F_{n-1}^{(4)}(R, z-\xi)$$

$$+ \frac{2}{n(n-1)} (z-\xi) F_{n-2}^{(3)}(R, z-\xi) +$$

$$+ \frac{2}{n(n-1)(n-2)} F_{n-3}^{(i-1)}(R, z-\xi) \quad (n \geq 2)$$

$$G_{0j1}^{(1)} = - \left. \operatorname{sh}^{-1} \frac{z-\xi}{R} \right|_{d_{j1}}^{d_{j2}}$$

$$G_{0j2}^{(1)} = \left. \sqrt{R^2 + (z-\xi)^2} \right|_{d_{j1}}^{d_{j2}} + z G_{0j1}^{(1)}$$

$$G_{2j1}^{(2)} = - \left[ \operatorname{sh}^{-1} \frac{z-\xi}{R} - \frac{z-\xi}{2\sqrt{R^2 + (z-\xi)^2}} \right]_{d_{j1}}^{d_{j2}}$$

$$G_{2j2}^{(2)} = \left[ \sqrt{R^2 + (z-\xi)^2} + \frac{R^2}{2\sqrt{R^2 + (z-\xi)^2}} \right]_{d_{j1}}^{d_{j2}} \\ + z G_{2j1}^{(2)}$$

$$G_{2j3}^{(2)} = \left[ R^2 \operatorname{sh}^{-1} \frac{z-\xi}{R} - \frac{z-\xi}{2} \sqrt{R^2 + (z-\xi)^2} \right]_{d_{j1}}^{d_{j2}}$$

$$= \frac{R^2(z-\xi)}{2\sqrt{R^2 + (z-\xi)^2}} \Big|_{d_{j1}}^{d_{j2}} + 2z G_{2j2}^{(2)} - z^2 G_{2j1}^{(2)}$$

$$G_{0j1}^{(3)} = \frac{1}{R} \left. \sqrt{R^2 + (z-\xi)^2} \right|_{d_{j1}}^{d_{j2}}$$

$$G_{0j2}^{(3)} = \frac{1}{2R} \left[ R^2 \operatorname{sh}^{-1} \frac{z-\xi}{R} - (z-\xi) \sqrt{R^2 + (z-\xi)^2} \right]_{d_{j1}}^{d_{j2}}$$

$$+ z G_{0j1}^{(3)}$$

$$G_{2j1}^{(4)} = \left[ \frac{1}{\sqrt{R^2 + (z-\xi)^2}} - \frac{1}{2\sqrt{R^2 + (z-\xi)^2}} \right]_{d_{j1}}^{d_{j2}}$$

$$G_{2j2}^{(4)} = \frac{R}{2} \left[ \operatorname{sh}^{-1} \frac{z-\xi}{R} - \frac{z-\xi}{\sqrt{R^2 + (z-\xi)^2}} \right]_{d_{j1}}^{d_{j2}} + z G_{2j1}^{(4)}$$

$$G_{2j3}^{(4)} = - \frac{R}{2} \left[ \sqrt{R^2 + (z-\xi)^2} + \frac{R^2}{\sqrt{R^2 + (z-\xi)^2}} \right]_{d_{j1}}^{d_{j2}}$$

$$+ 2z G_{2j2}^{(4)} - z^2 G_{2j1}^{(4)}$$

### 3. Results.

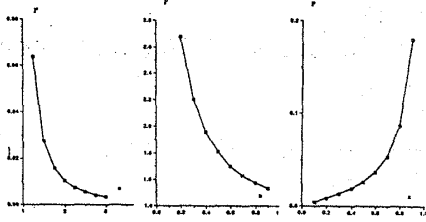


Fig 5. Pressure around the body

The pressure changed significantly depending on the geometric parameters  $a$ ,  $b$  and axial distance  $z$ .

### 4. Reference

1. Y. Fan and Wu, Wangi. Applied. Mathematics and Mechanics. vol8, pp 17-29. 1987.

**APPENDIX:** Induction of  $G$  through  $F$  in (3-2)

$$G_{0j1}^{(3)} = \int \xi^{3-1} F_0^{(3)}(z - \xi, R) d\xi$$

$$= \int \frac{1}{R} C_1 \left( \frac{(z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} \right) d\xi$$

From, Gradshteyn p83 2.264.2

$$\int \frac{x}{\sqrt{cx^2 + bx + a}} dx = \frac{\sqrt{cx^2 + bx + a}}{c} - \frac{b}{2c} I_F$$

$$= \frac{1}{R} \sqrt{(z - \xi)^2 + R^2}$$

$$G_{0j2}^{(3)} = \int \xi^{2-1} F_0^{(3)}(z - \xi, R) d\xi$$

$$= \frac{1}{R} \int \frac{\xi(z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} d\xi$$

Since,  $\xi = \xi - z + z$

$$= -\frac{1}{R} \left[ \int \frac{(\xi - z)^2 + z(\xi - z)}{\sqrt{(z - \xi)^2 + R^2}} d\xi \right]$$

$$= -\frac{1}{R} \left[ \int \frac{(\xi - z)^2}{\sqrt{(z - \xi)^2 + R^2}} d\xi + z \int \frac{(\xi - z)}{\sqrt{(z - \xi)^2 + R^2}} d\xi \right]$$

th, 1st, term, can, be, from Gradshteyn p83 2.264.3

$$\int \frac{x^2}{\sqrt{x^2 + R^2}} dx = \frac{x}{2} (x^2 + R^2) - \frac{R^2}{2} I_F$$

while

$$\text{The 2nd, term,} = \frac{z}{R} \int \frac{(\xi - z)}{\sqrt{(z - \xi)^2 + R^2}} d\xi = z \cdot G_{0j1}^{(3)}$$

Hence,

$$G_{0j2}^{(3)} = \frac{-1}{R} \left[ \frac{x \cdot (x^2 + R^2)}{2} - \frac{R^2}{2} I_F \right] + z \cdot G_{0j1}^{(3)}$$

$$G_{0j3}^{(3)} = \int \xi^{3-1} F_0^{(3)}(z - \xi, R) d\xi$$

$$= \int \xi^2 \frac{1}{R} C_1 \left( \frac{(z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} \right) d\xi = -\frac{1}{R} \int \frac{\xi^2 \cdot (\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi$$

$$G_{0j3}^{(3)} = -\frac{1}{R} \left[ \int \frac{(\xi - z)^3}{\sqrt{(\xi - z)^2 + R^2}} d\xi + 2z \int \frac{\xi(\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi - z^2 \int \frac{(\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi \right]$$

$$\text{2nd} = -2 \cdot z \cdot (-G_{0j2}^{(3)}) = 2 \cdot z \cdot G_{0j2}^{(3)}$$

$$\text{3rd} = z^2 \cdot (-G_{0j1}^{(3)})$$

The 1st term, from, Grashytyn, p 86, 2,272,7

$$\int \frac{x^3}{(x^2 + a)^{n+1/2}} dx, \dots, \text{setting } u = \sqrt{x^2 + a^2}$$

$$= \frac{u^3}{3} - R^2 u$$

Hence, we have

$$G_{0j3}^{(3)} = -\frac{1}{R} \left[ \frac{((\xi - z)^2 + R^2)^{3/2}}{3} - R^2 ((\xi - z)^2 + R^2)^{1/2} \right]$$

$$+ 2 \cdot z \cdot G_{0j2}^{(3)} - z^2 \cdot G_{0j1}^{(3)}$$

$$G_{0j4}^{(3)} = \int \xi^{4-1} F_0^{(3)}(z - \xi, R) d\xi$$

$$= \int \xi^3 \frac{1}{R} C_1 \left( \frac{(z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} \right) d\xi = -\frac{1}{R} \int \frac{\xi^3 \cdot (\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi$$

Since,

$$\xi^3 = (\xi - z)^3 + 3z\xi^2 - 3z^2\xi + z^3$$

$$= -\frac{1}{R} \left[ \int \frac{(\xi - z)^4}{\sqrt{(\xi - z)^2 + R^2}} d\xi + 3z \int \frac{\xi^2 \cdot (\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi - 3z^2 \int \frac{\xi \cdot (\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi + z^3 \int \frac{(\xi - z)}{\sqrt{(\xi - z)^2 + R^2}} d\xi \right]$$

$$\text{2nd, term} = 3z \cdot G_{0j3}^{(3)}$$

$$\text{3rd, term} = -3z^2 \cdot G_{0j2}^{(3)}$$

$$\text{4th, term} = z^3 \cdot G_{0j1}^{(3)}$$

The 1st, from, the, table, of, Gradshteyn p87 2.273.3

$$\int \frac{x^4}{\sqrt{x^2 + a}} dx = \frac{x}{4} \sqrt{x^2 + R^2} - \frac{3}{8} R^2 \cdot x \cdot \sqrt{x^2 + a} + \frac{3}{8} R^4 \cdot I_F$$

Hence,

$$G_{0j4}^{(3)} = -\frac{1}{R} \left[ \frac{((\xi - z)^3}{4} \sqrt{(\xi - z)^2 + R^2} - \frac{3}{8} R^2 \cdot (\xi - z) \cdot \sqrt{(\xi - z)^2 + R^2} + \frac{3}{8} R^4 \cdot I_F \right]$$

$$+ 3z \cdot G_{0j3}^{(3)} - 3z^2 \cdot G_{0j2}^{(3)} + z^3 \cdot G_{0j1}^{(3)}$$

$$G_{0j5}^{(3)} = \int \xi^{5-1} F_0^{(3)}(z - \xi, R) d\xi$$

$$= -\frac{1}{R} \int \frac{\xi^4 \cdot (\xi - z)}{\sqrt{(z - \xi)^2 + R^2}} d\xi$$

, expand

$$\begin{aligned}\xi^4 &= (\xi-z)^4 + 4\xi^3z - 6\xi^2z^2 + 4\xi z^3 - z^4 \\ &= -\frac{1}{R} \left[ \int \frac{(\xi-z)^4}{\sqrt{(z-\xi)^2 + R^2}} d\xi + 4z \int \frac{\xi^3(\xi-z)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right. \\ &\quad \left. - 6z^2 \int \frac{\xi^2(\xi-z)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right. \\ &\quad \left. + 4z^3 \int \frac{\xi(\xi-z)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - z^4 \int \frac{(\xi-z)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right]\end{aligned}$$

$$\text{2nd, term} = 4 \cdot z \cdot G_{0j4}^{(3)} \quad \text{3rd, term} = -6 \cdot z^2 \cdot G_{0j3}^{(3)}$$

$$\text{4th, term} = 4 \cdot z^3 \cdot G_{0j2}^{(3)} \quad \text{5th, term} = -z^4 \cdot G_{0j1}^{(3)}$$

$$\text{1st, term, from, Gradshteyn p87 2.273.8 } u = \sqrt{x^2 + a}$$

$$\begin{aligned}\int \frac{x^5}{\sqrt{x^2 + R^2}} dx &= \\ &= \frac{(x^2 + R^2)^{3/2}}{5} - \frac{2}{3} R^2 \cdot (x^2 + R^2)^{1/2} + R^4 \cdot (x^2 + R^2)^{-1/2}\end{aligned}$$

Hence,

$$\begin{aligned}G_{0j5}^{(3)} &= -\frac{1}{R} \left[ \frac{((\xi-z)^2 + R^2)^{3/2}}{5} - \frac{2}{3} R^2 \cdot ((\xi-z)^2 + R^2)^{1/2} \right. \\ &\quad \left. + R^4 \cdot ((\xi-z)^2 + R^2)^{-1/2} \right] \\ &\quad + 4 \cdot z \cdot G_{0j4}^{(3)} - 6 \cdot z^2 \cdot G_{0j3}^{(3)} + 4 \cdot z^3 \cdot G_{0j2}^{(3)} - z^4 \cdot G_{0j1}^{(3)}\end{aligned}$$

$$\begin{aligned}G_{2j1}^{(4)} &= \int \xi^{2-1} F_2^{(4)}(z-\xi, R) d\xi \\ &= \int \left[ 3 \cdot ((z-\xi)^2 + R^2)^0 \frac{1}{R} C_3 \left( \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} \right) \right. \\ &\quad \left. - 2 \cdot (z-\xi) \cdot ((z-\xi)^2 + R^2)^{-1/2} \frac{1}{R} C_2 \left( \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} \right) \right] d\xi\end{aligned}$$

From, the, relation, of, Legendre, and, Gegenbauer,

$$\begin{aligned}C_3(q) &= \frac{(P_1(q) - P_3(q))}{5} \quad \therefore q = \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} \\ &= \frac{1}{5} \left[ q - \frac{(5q^3 - 3q)}{2} \right] = \frac{1}{5} \left( q - \frac{5}{2} q^3 + \frac{3}{2} q \right) = \frac{q}{2} - \frac{q^3}{2} \\ &= \frac{1}{R} \left[ 3 \int \left( \frac{q}{2} - \frac{q^3}{2} \right) d\xi - 2 \int \frac{(1-q^2)}{2} \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right] \\ &= \frac{1}{R} \left[ \frac{3}{2} \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - \frac{3}{2} \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right. \\ &\quad \left. - \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi + \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right] \\ &= \frac{1}{R} \left[ \frac{1}{2} \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - \frac{1}{2} \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right]\end{aligned}$$

$$\int \frac{x}{\sqrt{x^2 + R^2}} dx = \sqrt{x^2 + R^2}$$

2nd, term, can, be, modified, Gradshteyn p86 2.272.7

$$\begin{aligned}u &= (x^2 + a)^{1/2} \\ \int \frac{x^3}{(x^2 + R^2)^{3/2}} dx &= (x^2 + R^2)^{-1/2} + R^2 (x^2 + R^2)^{-3/2}\end{aligned}$$

Thus,

$$G_{2j1}^{(4)} = \frac{-1}{2R} \left[ \sqrt{(\xi-z)^2 + R^2} - \sqrt{(\xi-z)^2 + R^2} - R^2 ((\xi-z)^2 + R^2)^{-1/2} \right]$$

$$= \frac{R}{2} \frac{1}{\sqrt{(\xi-z)^2 + R^2}}$$

$$G_{2j2}^{(4)} = \int \xi^{2-1} F_2^{(4)}(z-\xi, R) d\xi$$

$$= \frac{1}{2R} \left[ \int \frac{\xi \cdot (z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - \int \frac{\xi \cdot (z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right]$$

$$\xi = (z-\xi) - z$$

$$= \frac{1}{2R} \left[ \int \frac{(z-\xi)^2}{\sqrt{(z-\xi)^2 + R^2}} d\xi - z \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right. \\ \left. - \int \frac{(z-\xi)^4}{((z-\xi)^2 + R^2)^{3/2}} d\xi + z \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right]$$

$$\begin{aligned}\text{2nd} + \text{4th} &= -z \frac{1}{2R} \left[ \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - z \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right] \\ &= -z \cdot G_{2j1}^{(4)}\end{aligned}$$

$$\text{1st} = \int \frac{x^2}{\sqrt{x^2 + R^2}} dx = \frac{x \cdot \sqrt{x^2 + R^2}}{2} - \frac{R^2}{2} I_F$$

$$\text{3rd} = \int \frac{x^4}{(x^2 + R^2)^{3/2}} dx = \frac{x \cdot \sqrt{x^2 + R^2}}{2} + \frac{R^2 x}{\sqrt{x^2 + R^2}} - \frac{3R^2}{2} I_F$$

Associating, these,

$$G_{2j2}^{(4)} = \frac{1}{2R} \left[ -\frac{R^2(\xi-z)}{\sqrt{(z-\xi)^2 + R^2}} + R^2 I_F \right] - z \cdot G_{2j1}^{(4)}$$

$$G_{2j3}^{(4)} = \int \xi^2 F_2^{(4)}(z-\xi, R) d\xi$$

$$= \frac{1}{2R} \left[ \int \frac{\xi^2 \cdot (z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi - \int \frac{\xi^2 \cdot (z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right]$$

$$\text{Since, } \xi^2 = (z-\xi)^2 + 2z\xi - z^2$$

$$\begin{aligned}&= \frac{1}{2R} \left[ \int \frac{(z-\xi)^3}{\sqrt{(z-\xi)^2 + R^2}} d\xi + 2z \int \frac{\xi \cdot (z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right. \\ &\quad \left. - z^2 \int \frac{(z-\xi)}{\sqrt{(z-\xi)^2 + R^2}} d\xi \right. \\ &\quad \left. - \int \frac{(z-\xi)^5}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right. \\ &\quad \left. - 2z \int \frac{\xi \cdot (z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi + z^2 \int \frac{(z-\xi)^3}{((z-\xi)^2 + R^2)^{3/2}} d\xi \right]\end{aligned}$$

$$\begin{aligned} \text{2nd} + 5\text{th} &= 2z \cdot \frac{1}{2R} \left[ \int \frac{\xi \cdot (z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} d\xi + \int \frac{\xi \cdot (z - \xi)^3}{((z - \xi)^2 + R^2)^{3/2}} d\xi \right] \\ &= 2 \cdot z \cdot G_{2j2}^{(4)} \end{aligned}$$

$$\begin{aligned} \text{3rd} + 4\text{th} &= -z^2 \cdot \frac{1}{2R} \left[ \int \frac{(z - \xi)}{\sqrt{(z - \xi)^2 + R^2}} d\xi + \int \frac{(z - \xi)^3}{((z - \xi)^2 + R^2)^{3/2}} d\xi \right] \\ &= -z^2 \cdot G_{2j1}^{(4)} \end{aligned}$$

The, 1st, term, *Gradshteyn* p86 2.272.7  $u = \sqrt{x^2 + a}$

$$\begin{aligned} \int \frac{x^3}{(x^2 + R^2)^{3/2}} dx &= \frac{[(\xi - z)^2 + R^2]^{3/2}}{3} - R^2 [(\xi - z)^2 + R^2]^{1/2} \\ \int \frac{x^5}{(x^2 + R^2)^{3/2}} dx &= \\ &= \frac{(x^2 + R^2)^{3/2}}{3} - 2R^2 (x^2 + R^2)^{1/2} - R^4 (x^2 + R^2)^{-1/2} \end{aligned}$$

associating, these

$$\begin{aligned} G_{2j3}^{(4)} &= \frac{-1}{2R} \left[ R^2 \cdot (x^2 + R^2)^{1/2} + R^4 \cdot (x^2 + R^2)^{-1/2} \right] \\ &+ 2 \cdot z \cdot G_{2j2}^{(4)} - z^2 \cdot G_{2j1}^{(4)} \end{aligned}$$

$$\begin{aligned} G_{2j4}^{(4)} &= \int \xi^3 \cdot F_2^{(4)}(z - \xi, R) d\xi \\ &= \int \xi^3 \left[ \left( R^2 + (z - \xi)^2 \right)^{-1/2} \cdot \frac{1}{R} \cdot J_2 \left( \frac{(z - \xi)}{((z - \xi)^2 + R^2)^{1/2}} \right) \right. \\ &\quad \left. - 2 \cdot (z - \xi) \left[ R^2 + (z - \xi)^2 \right]^{-1/2} C_2 \left[ \frac{(z - \xi)}{((z - \xi)^2 + R^2)^{1/2}} \right] \right] d\xi \end{aligned}$$

$$\text{Since, } C_n^{-1/2} = (P_{n-2} - P_n) / (2n - 1)$$

$$J_3(q) = C_3^{-1/2}(q) = \frac{q}{2} - \frac{q^3}{2} \quad C_2(q) = (1 - q^2) / 2$$

$$\begin{aligned} &= \int \xi^3 \left[ \frac{3}{2R} [q - q^3] - \frac{2 \cdot (z - \xi)}{((z - \xi)^2 + R^2)^{1/2}} \cdot \frac{1}{R} \cdot \frac{(1 - q^2)}{2} \right] d\xi \\ &= \frac{1}{2R} \int \left[ 3\xi^3 \left( \frac{(z - \xi)}{((z - \xi)^2 + R^2)^{1/2}} - \frac{(z - \xi)^3}{((z - \xi)^2 + R^2)^{3/2}} \right) \right. \\ &\quad \left. - \frac{1}{2} \cdot \frac{1}{R} \int 2\xi^3 \frac{(z - \xi)}{((z - \xi)^2 + R^2)^{1/2}} d\xi \right. \\ &\quad \left. + \frac{1}{2R} \int 2\xi^3 \frac{(z - \xi)(z - \xi)^2}{(R^2 + (z - \xi)^2)^{1/2} \cdot (R^2 + (z - \xi)^2)^{3/2}} d\xi \right] \\ &= \frac{-1}{2R} \left[ \int \frac{\xi^3 (\xi - z)}{((z - \xi)^2 + R^2)^{1/2}} d\xi - \int \frac{\xi^3 (\xi - z)^3}{(R^2 + (z - \xi)^2)^{1/2} \cdot (R^2 + (z - \xi)^2)^{3/2}} d\xi \right] \\ &= \frac{-1}{2R} \int \frac{(\xi - z)^4 + 3\xi^2 \cdot z (\xi - z) - 3\xi \cdot z^2 (\xi - z) + z^3 (\xi - z)}{((z - \xi)^2 + R^2)^{1/2}} d\xi \end{aligned}$$

$$- \int \frac{(\xi - z)^6 + 3\xi^2 z \cdot (\xi - z)^3 - 3\xi \cdot z^2 (\xi - z)^3 + z^3 (\xi - z)^3}{((z - \xi)^2 + R^2)^{3/2}} d\xi$$

2nd + 6th

$$\begin{aligned} &= \frac{-1}{2R} \int 3 \cdot z \left[ \frac{(\xi - z) \cdot \xi^2}{[(z - \xi)^2 + R^2]^{1/2}} - \frac{(\xi - z)^3 \cdot \xi^2}{[(z - \xi)^2 + R^2]^{3/2}} \right] d\xi \\ &= 3 \cdot z \cdot G_{2j3}^{(4)} \end{aligned}$$

3rd + 7th

$$\begin{aligned} &= \frac{-1}{2R} (-3z^2) \int \left[ \frac{\xi \cdot (\xi - z)}{[(z - \xi)^2 + R^2]^{1/2}} - \frac{\xi \cdot (\xi - z)^3}{[(z - \xi)^2 + R^2]^{3/2}} \right] d\xi \\ &= -3z^2 \cdot G_{2j2}^{(4)} \end{aligned}$$

4th + 8th

$$\begin{aligned} &= \frac{-1}{2R} (z^3) \int \left[ \frac{(\xi - z)}{[(z - \xi)^2 + R^2]^{1/2}} - \frac{(\xi - z)^3}{[(z - \xi)^2 + R^2]^{3/2}} \right] d\xi \\ &= z^3 \cdot G_{2j1}^{(4)} \end{aligned}$$

5th, for, the, integral

$$\int \frac{(\xi - z)^6}{[(z - \xi)^2 + R^2]^{3/2}} d\xi$$

Putting,  $x = \xi - z$ , and using

*Gradshteyn* p87 2.274.4,  $c = 1$   $a = R^2$

$$= \frac{x^5}{4(x^2 + R^2)^{1/2}} - \frac{5R^2 x^3}{8(x^2 + R^2)^{1/2}} - \frac{15R^4 x}{8(x^2 + R^2)^{1/2}} + \frac{15R^4}{8} I_F$$

The, 1st, term

$$\int \frac{(\xi - z)^4}{[(z - \xi)^2 + R^2]^{1/2}} d\xi$$

*Gradshteyn* p87 2.274.4  $c = 1$   $a = R^2$

$$\int \frac{x^4}{(x^2 + R^2)^{1/2}} dx = \frac{x^3}{4} (x^2 + R^2)^{1/2} - \frac{3}{8} R^2 x \cdot (x^2 + R^2)^{1/2} + \frac{3}{8} R^4 \cdot I_F$$

Associating, these

$$\begin{aligned} G_{2j4}^{(4)} &= \frac{-1}{2R} \left[ \frac{(z - \xi)^3 \cdot ((z - \xi)^2 + R^2)^{1/2}}{4} \right. \\ &\quad \left. - \frac{3 \cdot R^2 (\xi - z) \cdot ((\xi - z)^2 + R^2)^{1/2}}{8} \right. \\ &\quad \left. - \frac{(z - \xi)^5}{4((z - \xi)^2 + R^2)^{1/2}} - \frac{5R^2 (z - \xi)^3}{8((z - \xi)^2 + R^2)^{1/2}} \right. \\ &\quad \left. - \frac{15R^4 (z - \xi)}{8((z - \xi)^2 + R^2)^{1/2}} + \frac{(3 - 15)R^4}{8} I_F \right] \\ &+ 3 \cdot z \cdot G_{2j3}^{(4)} - 3 \cdot z^2 \cdot G_{2j2}^{(4)} + z^3 \cdot G_{2j1}^{(4)} \end{aligned}$$

$$G_{2j5}^{(4)} = \int \xi^4 \cdot F_2^{(4)}(z - \xi, R) d\xi$$

$$= \frac{-1}{2R} \left[ \int \frac{\xi^4(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi - \int \frac{\xi^4(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$\xi^4 = (\xi-z+z)^4 = (\xi-z)^4 + 4\xi^3z - 6\xi^2z^2 + 4\xi z^3 - z^4$$

$$= \frac{-1}{2R} \left[ \int \frac{(\xi-z)^5 + 4\xi^3z(\xi-z) - 6\xi^2z^2(\xi-z) - z^4(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi \right.$$

$$\left. - \int \frac{(\xi-z)^7 + 4\xi^3z(\xi-z)^3 - 6\xi^2z^2(\xi-z)^3 + 4\xi z^3(\xi-z)^3 - z^4(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$2\text{nd} + 7\text{th} = \frac{-1}{2R} \cdot 4z \cdot \left[ \int \frac{\xi^3(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi - \int \frac{\xi^3(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$= 4 \cdot z \cdot G_{2j4}^{(4)}$$

$$3\text{rd} + 8\text{th} = \frac{-1}{2R} \cdot (-6z^2) \cdot \left[ \int \frac{\xi^2(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi - \int \frac{\xi^2(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$= (-6 \cdot z^2) \cdot G_{2j3}^{(4)}$$

$$4\text{th} + 9\text{th} = \frac{-1}{2R} \cdot 4z^3 \cdot \left[ \int \frac{\xi(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi - \int \frac{\xi(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$= 4z^3 \cdot G_{2j2}^{(4)}$$

$$5\text{th} + 10\text{th} = \frac{-1}{2R} \cdot (-z^4) \cdot \left[ \int \frac{(\xi-z)}{((\xi-z)^2 + R^2)^{3/2}} d\xi - \int \frac{(\xi-z)^3}{((\xi-z)^2 + R^2)^{5/2}} d\xi \right]$$

$$= -z^4 \cdot G_{2j1}^{(4)}$$

$$6\text{th} = \int \frac{(\xi-z)^7}{((\xi-z)^2 + R^2)^{5/2}} d\xi \quad \text{putting } \xi-z = x$$

From, the, table, of, Gradshteyn p87 2.274.9

$$\int \frac{x^7}{(cx^2 + a)^{3/2}} dx = \frac{(x^2 + R^2)^{3/2}}{5} - R^2 \cdot (x^2 + R^2)^{1/2}$$

$$+ 3R^4(x^2 + R^2)^{1/2} + R^6(x^2 + R^2)^{-1/2}$$

The, 1st, term

$$\int \frac{x^5}{(x^2 + R^2)^{5/2}} dx \quad \text{Gradshteyn p87 2.273.8}$$

$$= \frac{(x^2 + R^2)^{3/2}}{5} - \frac{2R^2 \cdot (x^2 + R^2)^{1/2}}{3} + R^4(x^2 + R^2)^{-1/2}$$

associating, these

$$G_{2j5}^{(4)} = -\frac{1}{2R} \left[ \frac{U^5}{5} - \frac{2R^2U^3}{3} + R^4U - \left( \frac{U^5}{5} - R^2U^3 + 3R^4U + \frac{R^6}{U} \right) \right]$$

$$+ 4 \cdot z \cdot G_{2j4}^{(4)} - 6 \cdot z^2 G_{2j3}^{(4)} + 4z^3 G_{2j2}^{(4)} - z^4 \cdot G_{2j1}^{(4)}$$

Induction, for, equation, (4.17)

From, p.24

$$T_{njk}^{(i)} = \int_{d_j}^{d_{j+1}} \xi^{k-1} \cdot S_n^{(i)}(R, z, \xi) d\xi$$

$$S_n^{(i)}(R, z, \xi) \text{ can, be, given,, from (3.14)}$$

$$1), \dots, T_{njk}^{(1)} = \int \xi^{k-1} \cdot S_n^{(1)}(R, z, \xi) d\xi$$

$$= \int \xi^{k-1} \cdot \left[ F_n^{(1)}(R, z - \xi) - F_n^{(1)}(R, 2d - z - \xi) \right] d\xi$$

$$+ 2(z-d)(n+1)F_{n+1}^{(1)}(R, 2d - z - \xi) d\xi$$

Since, from, (4.18)

$$G_{njk}^{(i)} = \int \xi^{k-1} \cdot F_n^{(i)}(z - \xi, R) d\xi$$

Apply, this, to, above,

$$= G_{njk}^{(1)} - G_{njk}^{(1)}(2d - z) + 2(n+1)(z-d)G_{n+1jk}^{(1)}(2d - z)$$

$$(2), \dots, T_{njk}^{(2)} = \int \xi^{k-1} \cdot S_n^{(2)}(R, z, \xi) d\xi$$

$$= \int \xi^{k-1} \cdot \left[ F_n^{(2)}(z - \xi) - F_n^{(2)}(2d - z - \xi) \right] d\xi$$

$$+ 2(n-2)(z-d)F_{n-1}^{(1)}(2d - z - \xi) d\xi$$

$$+ 2 \cdot (2n-3)(z-d)(d-\xi)F_n^{(1)}(2d - z - \xi) d\xi$$

$$G_{njk}^{(i)} = \int \xi^{k-1} \cdot F_n^{(i)}(z - \xi, R) d\xi$$

$$= G_{njk}^{(2)} - G_{njk}^{(2)}(2d - z) - 2(n-2)(z-d)G_{n+1jk}^{(1)}(2d - z)$$

$$+ 2 \cdot (2n-3)(z-d) \int d \cdot \xi^{k-1} \cdot F_n^{(1)}(2d - z - \xi) d\xi$$

$$- 2 \cdot (2n-3)(z-d) \int \xi^k \cdot F_n^{(1)}(2d - z - \xi) d\xi$$

$$= G_{njk}^{(2)} - G_{njk}^{(2)}(2d - z) - 2(n-2)(z-d)G_{n+1jk}^{(1)}(2d - z)$$

$$+ 2 \cdot (2n-3)(z-d) \left[ d \cdot G_{njk}^{(1)}(2d - z) - G_{njk+1}^{(1)}(2d - z) \right]$$

$$(3), \dots, T_{njk}^{(3)} = \int \xi^{k-1} \cdot S_n^{(3)} d\xi$$

$$= \int \xi^{k-1} \cdot \left[ F_n^{(3)}(z - \xi) - F_n^{(3)}(2d - z - \xi) \right] d\xi$$

$$- 2(n+1)(z-d)F_{n+1}^{(3)}(2d - z - \xi) d\xi$$

$$G_{njk}^{(i)} = \int \xi^{k-1} \cdot F_n^{(i)}(z - \xi, R) d\xi$$

$$= G_{njk}^{(3)} - G_{njk}^{(3)}(2d - z) - 2(n+1)(z-d)G_{n+1jk}^{(3)}(2d - z)$$

$$(4), \dots, T_{njk}^{(4)} = \int \xi^{k-1} \cdot S_n^{(4)} d\xi$$

$$= \int \xi^{k-1} \cdot \left[ F_n^{(4)} - F_n^{(4)}(2d - z - \xi) - 2(2n-3) \cdot \right.$$

$$\left. \cdot (z-d)(d-\xi)F_n^{(3)}(2d - z - \xi) \right] d\xi$$

$$+ \frac{2(n-1)(n-3)}{n} (z-d) \cdot F_{n-1}^{(3)}(2d - z - \xi) d\xi$$

$$G_{njk}^{(i)} = \int \xi^{k-1} \cdot F_n^{(i)}(z - \xi) d\xi$$

$$= G_{njk}^{(4)} - G_{njk}^{(4)}(2d - z) - \frac{2(n-1)(n-3)}{n} \cdot$$

$$\cdot (z-d)G_{n-1jk}^{(3)}(2d - z - \xi)$$

$$- 2(2n-3)(z-d) \left[ \int d \cdot \xi^{k-1} \cdot F_n^{(3)}(2d - z - \xi) d\xi \right]$$

$$\left[ - \int \xi^k \cdot F_n^{(3)}(2d - z - \xi) d\xi \right]$$

$$= G_{njk}^{(4)} - G_{njk}^{(4)}(2d - z) + \frac{2(n-1)(n-3)}{n}$$

$$\cdot (z-d)G_{n-1jk}^{(3)}(2d - z - \xi)$$

$$- 2(2n-3)(z-d) \left[ dG_{njk}^{(3)}(2d - z) - G_{njk+1}^{(3)}(2d - z) \right]$$