社団法人　電子情報通信学会
THE INSTITUTE OF ELECTRONICS,
INFORMATION AND COMMUNICATION ENGINEERS

信学技報
TECHNICAL REPORT OF IEICE.
MBE2000-104 (2000-12)

Stochastic Analysis of Preemptive Priority Neural Signal Pulses.

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The neural signal that has arrived at the axon hillock excites the neural cell faster than the singal that has arrived at the distal dendrite of the neuron. Hence there is a preemptive resume priority in neural signal processing. The probabilistic states of the neural signal processing were described by the probabilistic differential equations. The equations were converted to the partial differential equations by introducing the generating functions for each probability. We introduced the Laplace transformed equations to analyze the time dependent changes of the probability. The present method will be available for evaluating the preemptive priority neural signal processing.

Neural signal processing, Axon Hillock, Dendrites, Probabilistic differential equations. Priority.

優先状態が内在する神経パルス列の確率的解析

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神経細胞の神経インパルス処理機構において、割り込優先が認められる場合の状態確率微分方程式を紹介し、それに対する解析的解法を報告した。神経細胞の軸索丘近傍に接続する入力は神経細胞体の樹状突起の末端に接続する入力よりも優先的に神経細胞を興奮させる。この場合、たとえ樹状突起入力が先行しても軸索丘入力のほうが優先されると解釈できる。状態確率微分方程式が有限収束することから確率母関数を設定し、それの満たす偏微分方程式を導出した。　時間経過を解析するため、同式に対してラプラス変換した場合の解析解の求め方も示した。　本研究は優先順位が内在する場合の神経インパルス処理を解析するうえで有用である。

神経インパルス処理.　軸索丘.　樹状突起末端.　状態確率微分方程式.　　割り込優先

1. Introduction

During neural processing of electrical signals, there may be interactions among the transmitted signals whether to be processed faster. The present work introduce a method for stochastic analysis of preemptive signal processing.

2. Method.

The priority and non priority units arrive at a processing neuron at Poisson streams with mean rates of $\lambda 1$ and $\lambda 2$ respectively.

Define the following probabilities.

1. **Pm,n(x,y,t)** dx dy for m≧1 and n≧1: the probability that at time y, there are m priority and n non priority units in the system. The processing time on the unit under processing (obviously a priority unit) lies between a and x + dx and the head of non priority units was prempted earlier when its processing time was lying between y and y +dy.
2. **Qm,n(x,t)** dx for m≧1 and n≧0. : The probability that at time t, there are m priority units and n non priority units in the system. The processing time on the unit under the processing lies between x and x + dx and non of the non priority units was preempted earlier. Obviously the non priority queue must have been built up during the processing g time of a priority.
3. **Un(y,t)** dy for n ≧1. : The probability that at time t, there are n non priority and no priority units in the system. The processing time of the unit under the processing lies between y and y + dy.

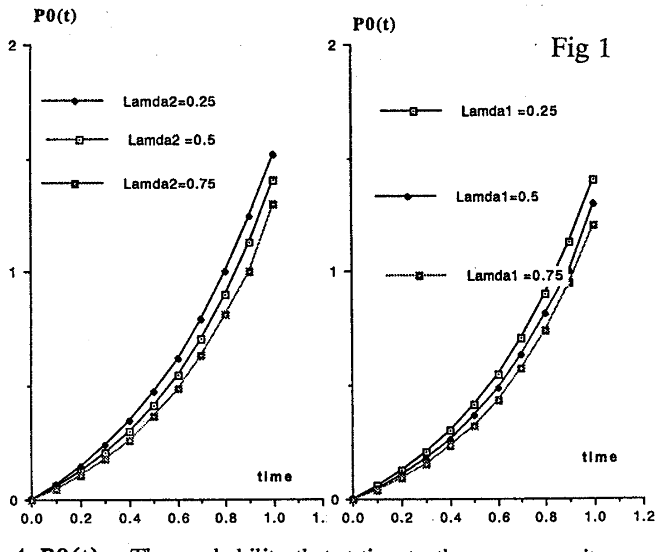

Fig 1

4. **P0(t).** : The probability that at time t, there are no units in the system.

The processing time in isolation of the two type of units is distributed according to general processing time distribution with the probability densities of S1(x) and S2(x) respectively. They are characterized by the means of $\eta 1$ and $\eta 2$ respectively. Let $\eta 1(x)\Delta$ and $\eta 2(x)\ \Delta$ be the first order probabilities that a priority and a non priority unit (in isolation) completes the processing in the interval (x, x + Δ) under the condition that the units have already completed a time x in processing so that following relations hold. Δ

$$S_1(x)=\eta_1(x)\exp\left\{-\int_0^x \eta_1(x)\,dx\right\},$$

$$S_2(x)=\eta_2(x)\exp\left\{-\int_0^x \eta_2(x)\,dx\right\}.$$

FORMULATION OF THE EQUATIONS

In order to derive the difference-differential equations between state probabilities, we follow Keilson and Kooharian,[6] and usual the probabilities at time $t+\Delta$ to those at time t. These lead to the following equations:

$$P_{m,n}(x+\Delta,y,t+\Delta)=P_{m,n}(x,y,t)\{1-[\lambda_1+\lambda_2+\eta_1(x)]\,\Delta\}$$
$$+P_{m-1,n}(x,y,t)\,\lambda_1\,\Delta+P_{m,n-1}(x,y,t)\,\lambda_2\,\Delta,\quad (m\geqq 1,\ n\geqq 1) \tag{1}$$

$$Q_{m,n}(x+\Delta,t+\Delta)=Q_{m,n}(x,t)\{1-[\lambda_1+\lambda_2+\eta_1(x)]\,\Delta\}$$
$$+Q_{m-1,n}(x,t)\,\lambda_1\,\Delta+Q_{m,n-1}(x,t)\,\lambda_2\,\Delta.\qquad (m\geqq 1,\ n\geqq 0) \tag{2}$$

For deriving difference-differential equation in $U_n(y,t)$ we define probability. $D_n(y,t)=\int_y^{y+\Delta+\epsilon} U_n(y,t)\,dy$ where ϵ is an arbitrary increment in y. With this definition and with a similar above, we get

$$D_n(y+\Delta,t+\Delta)=D_n(y,t)\{1-[\lambda_1+\lambda_2+\eta_2(y)]\,\Delta\}+D_{n-1}(y,t)\,\lambda_2\,\Delta$$
$$+\int_0^\infty P_{1,n}(x,y+\Delta,t)\,dx\,\epsilon\,\eta_1(x)\,\Delta.\quad (n\geqq 1) \tag{3}$$

Hence

$$U_n(y+\Delta,t+\Delta)=U_n(y,t)\{1-[\lambda_1+\lambda_2+\eta_2(y)]\,\Delta\}+U_{n-1}(y,t)\,\lambda_2\,\Delta$$
$$+\int_0^\infty P_{1,n}(x,y+\Delta,t)\,dx\,\frac{\epsilon}{\Delta+\epsilon}\,\eta_1(x)\,\Delta.\quad (n\geqq 1) \tag{4}$$

Equations (1), (2), and (4) become, when $\Delta\to 0$,

$$(\partial/\partial t)\,P_{m,n}(x,y,t)+(\partial/\partial x)\,P_{m,n}(x,y,t)+[\lambda_1+\lambda_2+\eta_1(x)]\,P_{m,n}(x,y,t)$$
$$=\lambda_1\,P_{m-1,n}(x,y,t)+\lambda_2\,P_{m,n-1}(x,y,t), \tag{5}$$

$$(\partial/\partial t)\,Q_{m,n}(x,t)+(\partial/\partial x)\,Q_{m,n}(x,t)+[\lambda_1+\lambda_2+\eta_1(x)]\,Q_{m,n}(x,t)$$
$$=\lambda_1\,Q_{m-1,n}(x,t)+\lambda_2\,Q_{m,n-1}(x,t), \tag{6}$$

$$\frac{\partial}{\partial t}\,U_n(y,t)+\frac{\partial}{\partial y}\,U_n(y,t)+[\lambda_1+\lambda_2+\eta_2(y)]\,U_n(y,t)$$
$$=\lambda_2\,U_{n-1}(y,t)+\int_0^\infty P_{1,n}(x,y,t)\,\eta_1(x)\,dx. \tag{7}$$

Similarly for the empty state, we have

$$\frac{\partial P_0(t)}{\partial t}+(\lambda_1+\lambda_2)P_0(t)=\int_0^\infty Q_{1,0}(x,t)\eta_1(x)\,dx+\int_0^\infty U_1(y,t)\,\eta_2(y)\,dy. \tag{8}$$

Equations (5), (6), (7), and (8) are the basic equations and are solved subject to the following boundary conditions.

$$P_{m,n}(0,y,t)=\int_0^\infty P_{m+1,n}(x,y,t)\,\eta_1(x)\,dx, \tag{9}$$

$$P_{1,n}(0,y,t)=\int_0^\infty P_{2,n}(x,y,t)\,\eta_1(x)\,dx+\lambda_1\,U_n(y,t), \tag{10}$$

$$Q_{m,n}(0,t)=\int_0^\infty Q_{m+1,n}(x,t)\,\eta_1(x)\,dx, \tag{11}$$

$$Q_{1,0}(0,t)=\int_0^\infty Q_{2,0}(x,t)\,\eta_1(x)\,dx+\lambda_1 P_0(t), \tag{12}$$

$$U_n(0,t)=\int_0^\infty U_{n+1}(y,t)\,\eta_2(y)\,dy+\int_0^\infty Q_{1,n}(x,t)\,\eta_1(x)\,dx, \tag{13}$$

$$U_1(0,t)=\int_0^\infty U_2(y,t)\,\eta_2(y)\,dy+\int_0^\infty Q_{1,1}(x,t)\,\eta_1(x)\,dx+\lambda_2 P_0(t). \tag{14}$$

All these equations arise since the termination of the service give $x=0$ or $y=0$. Of course in formulating equation (10), similar to that used in deriving (4) has been used.

SOLUTION OF THE PROBLEM UNDER STEADY STATE

Under steady state, the probabilities become independent of t so $P_{m,n}(x,y,t), Q_{m,n}(x,t)$, etc., will be represented by $P_{m,n}(x,y), Q_{m,n}(x)$, We now define the following generating functions:

$$f_m(x,y,\alpha)=\sum_{n=1}^{n=\infty}\alpha^n P_{m,n}(x,y),\qquad F(x,y,\alpha,\beta)=\sum_{m=1}^{m=\infty}\beta^m f_m(x,y,\alpha)$$

$$g_m(x,\alpha)=\sum_{n=0}^{n=\infty}\alpha^n Q_{m,n}(x),\qquad G(x,\alpha,\beta)=\sum_{m=1}^{m=\infty}\beta^m g_m(x,\alpha)$$

and

$$H(y,\alpha)=\sum_{n=1}^{n=\infty}\alpha^n U_n(y).$$

Multiplying equations (5), (6), (7), (8) with appropriate powers of α and β we get,

$$\partial F/\partial x+[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+\eta_1(x)]F=0, \quad (15)$$

$$\partial G/\partial x+[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+\eta_1(x)]G=0, \quad (16)$$

$$\frac{\partial H}{\partial y}+[\lambda_1+\lambda_2(1-\alpha)+\eta_2(y)]H=\int_0^\infty f_1(x,y,\alpha)\ \eta_1(x)\ dx. \quad (17)$$

The solutions of (15) and (16) are

$$F(x,y,\alpha,\beta)=F(0,y,\alpha,\beta)\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x \eta_1(x)\ dx\right], \quad (18)$$

$$G(x,\alpha,\beta)=G(0,\alpha,\beta)\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x \eta_1(x)\ dx\right]. \quad (19)$$

Similarly equations (9) to (14) become with the help of (8),

$$F(0,y,\alpha,\beta)=\frac{1}{\beta}\int_0^\infty F(x,y,\alpha,\beta)\ \eta_1(x)\ dx-\int_0^\infty f_1(x,y,\alpha)\ \eta_1(x)\ dx+\lambda_1\beta H(y,\alpha), \quad (20)$$

$$G(0,\alpha,\beta)=\frac{1}{\beta}\int_0^\infty G(x,\alpha,\beta)\ \eta_1(x)\ dx-\int_0^\infty g_1(x,\alpha)\ \eta_1(x)\ dx+\lambda_1\beta P_0, \quad (21)$$

and

$$H(0,\alpha)=\frac{1}{\alpha}\int_0^\infty H(y,\alpha)\ \eta_2(y)\ dy+\int_0^\infty g_1(x,\alpha)\ \eta_1(x)\ dx-[\lambda_1+\lambda_2(1-\alpha)]P_0. \quad (22)$$

Substituting (18) and (19) in (20) and (21) we get

$$F(0,y,\alpha,\beta)\left\{1-\frac{1}{\beta}\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]\right\}=\lambda_1\beta H(y,\alpha)-\int_0^\infty f_1(x,y,\alpha)\ \eta_1(x)\ dx, \quad (23)$$

$$G(0,\alpha,\beta)\left\{1-\frac{1}{\beta}\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]\right\}=\lambda_1\beta P_0-\int_0^\infty g_1(x,\alpha)\ \eta_1(x)\ dx, \quad (24)$$

where $\bar{S}_1(s)$ is the Laplace transform of $S_1(x)$.

It can be easily shown by Rouché's theorem that the equation $\beta-\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]=0$, has one and only one root whose modulus is less than or equal to one for $|\alpha|\leqq 1$. Let it be $\beta(\alpha)$. Then $\beta(\alpha)$ is given by

$$\beta(\alpha)=\sum_{n=1}^\infty \frac{(-\lambda_1)^{n-1}}{n!}\frac{d^{n-1}}{dy^{n-1}}([\bar{S}_1(y)]^n)_{y=\lambda_1+\lambda_2(1-\alpha)}\ .$$

Now from (23) we get, on putting $\beta=\beta(\alpha)$,

$$\int_0^\infty f_1(x,y,\alpha)\ \eta_1(x)\ dx=\lambda_1\beta(\alpha)\ H(y,\alpha), \quad (25)$$

$$\int_0^\infty g_1(x,\alpha)\ \eta_1(x)\ dx=\lambda_1\beta(\alpha)\ P_0. \quad (26)$$

Substituting (25) in (17) we get

$$\partial H(y,\alpha)/\partial y+\{\lambda_1[1-\beta(\alpha)]+\lambda_2(1-\alpha)+\eta_2(y)\}H(y,\alpha)=0, \quad (27)$$

So that,

$$H(y,\alpha)=H(0,\alpha)\exp\left[-\{\lambda_1[1-\beta(\alpha)]+\lambda_2(1-\alpha)\}y-\int_0^y \eta_2(y)\ dy\right]. \quad (28)$$

Finally using (22), (26), and (28) we get

$$H(0,\alpha)=\frac{\{\lambda_1[\beta(\alpha)-1]+\lambda_2(\alpha-1)\}P_0}{1-(1/\alpha)\bar{S}_2\{\lambda_1[1-\beta(\alpha)]+\lambda_2(1-\alpha)\}}, \quad (29)$$

$\bar{S}_2(s)$ being the Laplace transform of $S_2(x)$. Hence

$$F(x,y,\alpha,\beta)=\frac{\lambda_1[\beta-\beta(\alpha)]H(y,\alpha)}{1-(1/\beta)\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]}\cdot\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x \eta_1(x)\ dx\right], \quad (30)$$

$$G(x,\alpha,\beta)=\frac{\lambda_1[\beta-\beta(\alpha)]P_0}{1-(1/\beta)\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]}\cdot\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x \eta_1(x)\ dx\right], \quad (31)$$

$$H(y,\alpha)=\frac{P_0\{\lambda_1[\beta(\alpha)-1]+\lambda_2(\alpha-1)\}}{1-(1/\alpha)\bar{S}_2\{\lambda_1[1-\beta(\alpha)]+\lambda_2(1-\alpha)\}}\cdot\exp\left[-\{\lambda_1[1-\beta(\alpha)]+\lambda_2(1-\alpha)\}y-\int_0^y \eta_2(y)\ dy\right]. \quad (32)$$

The probability generating function $\Pi(\alpha,\beta)$ is now given by

$$\Pi(\alpha,\beta)=\int_0^\infty\int_0^\infty F(x,y,\alpha,\beta)\ dx\ dy+\int_0^\infty G(x,\alpha,\beta)\ dx+\int_0^\infty H(y,\alpha)\ dy+P_0, \quad (33)$$

together with the condition $\Pi(1,1)=1$, so that $P_0=1-\lambda_1\eta_1-\lambda_2\eta_2$, $\eta_1=\int_0^\infty xS_1(x)\ dx$ and $\eta_2=\int_0^\infty xS_2(x)\ dx$ are the mean service time priority and nonpriority units.

If there are no non priority units, $\Pi(\alpha\ \beta)$ reduce to the generating function of the queue length for the classical single server queue.

The moments of the queue length can be determined from (33).
mean number of priority units= (34)

$$\partial\Pi(1,\beta)/\partial\beta]_{\beta=1}=\rho_1+\tfrac{1}{2}[\rho_1^2+\lambda_1^2 v_1(x)]/[1-\rho_1],\ (\rho_1=\lambda_1\eta_1)$$

where v1(x) is the variance of the service time distribution of the priority units. This result is the same as for the priority queue alone. Because the priority units are not effected by the preemptive priority discipline. The mean number of non priority units in the queue is given by

$$\left.\frac{\partial\Pi(\alpha,1)}{\partial\alpha}\right]_{\alpha=1}=\frac{1}{1-\rho_1}\left[\rho_2+\tfrac{1}{2}\frac{\rho_2^2+\lambda_2^2 v_2(x)}{1-\rho_1-\rho_2}+\tfrac{1}{2}\frac{\lambda_1\lambda_2[v_1(x)+\eta_1^2]}{1-\rho_1-\rho_2}\right]. \quad (35)$$

Time dependent solution.

By applying the Laplace transform to Equation (5) to (14) with initial conditions, we have the time dependent probability of the system. By defining the following generating functions,

$$\bar{f}_m(x,y,\alpha,s)=\textstyle\sum_{n=1}^{n=\infty}\alpha^n P_{m,n}(x,y,s),$$

$$\bar{g}_m(x,\alpha,s)=\textstyle\sum_{n=0}^{n=\infty}\alpha^n Q_{m,n}(x,s),$$

$$\bar{F}(x,y,\alpha,\beta,s)=\textstyle\sum_{m=1}^{m=\infty}\beta^m \bar{f}_m(x,y,\alpha,s)$$

$$\bar{G}(x,\alpha,\beta,s)=\textstyle\sum_{m=1}^{m=\infty}\beta^m \bar{g}_m(x,\alpha,s)$$

and $\bar{H}(y,\alpha,s)=\sum_{n=1}^{n=\infty}\alpha^n \bar{U}_n(y,s)$

$$\bar{F}(x,y,\alpha,\beta,s)=\frac{\lambda_1[\beta-\beta(\alpha,s)]\bar{H}(y,\alpha,s)}{1-(1/\beta)\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s]}\cdot\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}x-\int_0^x \eta_1(x)\ dx\right], \quad (36)$$

$$\bar{G}(x,\alpha,\beta,s)=\frac{\lambda_1[\beta-\beta(\alpha,s)]\,\bar{P}_0(s)}{1-(1/\beta)\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s]} \cdot\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}\,x-\int_0^x \eta_1(x)\,dx\right], \tag{37}$$

$$\bar{H}(y,\alpha,s)=\frac{1-\bar{P}_0(s)\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}}{1-(1/\alpha)\,\bar{S}_2\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}} \cdot\exp\left[-\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}y-\int_0^y \eta_2(x)\,dx\right], \tag{38}$$

where $\beta(\alpha,s)$ is the root of the equation $\beta-\bar{S}_1[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s]=0$ lying inside the unit circle $|\beta|=1$. To determine $\bar{P}_0(s)$, we observe that the function $\bar{H}(y,\alpha,s)$ is regular inside the unit circle $|\alpha|=1$ and therefore

$$\bar{P}_0(s)=1/\{\lambda_1[1-\beta(\alpha_s,s)]+\lambda_2(1-\alpha)+s\}, \tag{39}$$

where $\alpha_s s$ is the unique root of the equation

$$\alpha-\bar{S}_2\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}=0$$

lying inside the unit circle $|\alpha|=1$.

Length of Busy Period.

The distribution of the length of busy periods for the priority queue is the same as for the classical queue. The problem of determining the distribution function of the length of busy period is

1. Starting from a single non priority unit which has just started service,
2. The probability density of the distribution of time to get the system empty is to be derived.
3. Since Qm,n(x,t) does not exist for this problem, we define Pm,n(x,t), Un(x,t) and Po(t) with the same meaning as stated before.
4. They are, however conditioned that the system stops when it attains the empty state for the first time. The required density function is γ = dPo(t)/dt. The system is

$$\frac{\partial P_{m,n}}{\partial t}(x,y,t)+\frac{\partial P_{m,n}}{\partial x}(x,y,t)+[\lambda_1+\lambda_2+\eta_1(x)]\,P_{m,n}(x,y,t) =\lambda_1\,P_{m-1,n}(x,y,t)+\lambda_2\,P_{m,n-1}(x,y,t), \tag{40}$$

$$\frac{\partial U_n(y,t)}{\partial t}+\frac{\partial U_n(y,t)}{\partial y}+[\lambda_1+\lambda_2+\eta_2(y)]\,U_n(y,t) =\lambda_2\,U_{n-1}(y,t)+\int_0^\infty P_{1,n}(x,y,t)\,\eta_1(x)\,dx, \tag{41}$$

$$\frac{\partial P_0(t)}{\partial t}=\int_0^\infty U_1(y,t)\,\eta_2(y)\,dy, \tag{42}$$

together with the boundary conditions

$$P_{m,n}(0,y,t)=\int_0^\infty P_{m+1,n}(x,y,t)\,\eta_1(x)\,dx, \tag{43}$$

$$P_{1,n}(0,y,t)=\int_0^\infty P_{2,n}(x,y,t)\,\eta_1(x)\,dx+\lambda_1\,U_n(y,t), \tag{44}$$

$$U_n(0,t)=\int_0^\infty U_{n+1}(y,t)\,\eta_2(y)\,dy, \tag{45}$$

and the initial condition $U_1(y,0)=\delta(y)$ where $\delta(y)$ is the Dirac delta function.

With similar notations as in the previous section, we get

$$\partial\bar{F}/\partial x+[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s+\eta_1(x)]\,\bar{F}=0, \tag{46}$$

$$\frac{\partial\bar{H}}{\partial y}+[\lambda_1+\lambda_2(1-\alpha)+s+\eta_2(y)]\,\bar{H}=\int_0^\infty \bar{f}_1(x,y,\alpha,s)\,\eta_1(x)\,dx+\alpha\delta(y), \tag{47}$$

giving

$$\bar{F}=\bar{F}(0,y,\alpha,\beta,s)\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}\,x-\int_0^x \eta_1(x)\,dx\right], \tag{48}$$

and $\bar{H}=[\bar{H}(0,\alpha,s)+\alpha]$

$$\cdot\exp\left[-\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}\,y-\int_0^y \eta_2(y)\,dy\right], \tag{49}$$

where $\beta(\alpha,s)$ is defined above. Finally substituting these values boundary conditions, we get

$$\bar{H}(0,\alpha,s)\{1-(1/\alpha)\bar{S}_2[\lambda_1\{1-\beta(\alpha,s)\}+\lambda_2(1-\alpha)+s]\} =\bar{S}_2\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}-s\bar{P}_0(s). \tag{50}$$

Hence $s\bar{P}_0(s)=\alpha_s$ where α_s is the root of the equation

$$\alpha-\bar{S}_2\{\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}=0,$$

so that $\gamma(t)=\mathcal{L}^{-1}\alpha_s$.

Example.

1. Exponential service-time distribution.

If the service time distributions of the priority and non priority units are exponential with mean $1/\mu 1$ and $1/\mu 2$, the value of $\beta(\alpha,s)$ is

$$\beta(\alpha,s)=\frac{\mu_1+\lambda_1+\lambda_2(1-\alpha)+s-[\{\mu_1+\lambda_1+\lambda_2(1-\alpha)+s\}^2-4\lambda_1\mu_1]^{\frac{1}{2}}}{2\lambda_1},$$

and α_s is the root of the equation

$$\alpha\{\mu_2+\lambda_1[1-\beta(\alpha,s)]+\lambda_2(1-\alpha)+s\}-\mu_2=0.$$

by assuming equal service rates ($\mu_1=\mu_2=\mu$), so that α_s is now the solution of the quadratic $\alpha^2(\lambda_1+\lambda_2)-\alpha(\mu+s+\lambda_1+\lambda_2)+\mu=0$, which is given by

$$\alpha_s=\frac{\mu+\lambda_1+\lambda_2+s-\{(\mu+\lambda_1+\lambda_2+s)^2-4\mu(\lambda_1+\lambda_2)\}^{\frac{1}{2}}}{2(\lambda_1+\lambda_2)}.$$

Hence
$$\bar{P}_0(s)=\frac{\beta(\alpha_s,s)}{\mu\{1-\beta(\alpha_s,s)\}}=\frac{\alpha_s}{\mu(1-\alpha_s)}. \tag{28}$$

$$P_0(t)=\frac{e^{-t(\mu+\lambda_1+\lambda_2)}}{\mu t}\sum_{j=0}^{\infty}\left(\frac{\mu}{\lambda_1+\lambda_2}\right)^{\frac{1}{2}(j+1)}(j+1)\,I_{j+1}\{2t\mu^{\frac{1}{2}}(\lambda_1+\lambda_2)^{\frac{1}{2}}\}. \tag{29}$$

;, if $\Pi_1(\alpha,\beta,t)$ represent the generating function of the joint probability distribu- of the queue lengths at time t, its Laplace transform $\bar{\Pi}_1(\alpha,\beta,s)$ is given by

$$\bar{\Pi}_1(\alpha,\beta,s)=A_1(\alpha,\beta,s)\frac{[1-\bar{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}]}{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s} +A_2(\alpha,s)\frac{[1-\bar{S}_2\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}]}{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s}+\bar{P}_0(s), \tag{27}$$

$$A_2(\alpha,s)=\frac{1-\bar{P}_0(s)\,[\lambda_1\{1-\beta(\alpha,s)\}+\lambda_2(1-\alpha)+s]}{1-(1/\alpha)\,\bar{S}_2[\lambda_1\{1-\beta(\alpha,s)\}+\lambda_2(1-\alpha)+s]}, \tag{24}$$

$$A_1(\alpha,\beta,s)=\frac{1-\bar{P}_0(s)\,\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}}{1-(1/\beta)\,\bar{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}} -A_2(\alpha,s)\frac{[1-(1/\alpha)\,\bar{S}_2\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}]}{[1-(1/\beta)\,\bar{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}]}. \tag{25}$$

Now (27) becomes

$$\bar{\Pi}_1(\alpha,\beta,s)=\frac{1+\bar{P}_0(s)\,\mu[1-(1/\beta)]}{\mu+\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s-(\mu/\beta)} +\frac{\lambda_2\mu\bar{P}_0(s)\left(1-\frac{\alpha}{\beta}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda_1}{\mu}+\frac{\lambda_2}{\mu}\right)^n\beta^n(\alpha,s)\,\alpha_s^n}{\{\mu+\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s-(\mu/\beta)\}\{\mu+\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}}. \tag{30}$$

Inverting (30), we get $\Pi_1(\alpha,\beta,t)$, the generating function of the joint distribution of the queue lengths at time t. Hence

$$\Pi_1(\alpha,\beta,t)=e^{-at}+\mu\left(1-\frac{1}{\beta}\right)\int_0^t P_0(t-\tau)\,e^{-a\tau}\,d\tau +\lambda_2(\beta-\alpha)\int_0^t P_0(t-\tau)\{e^{-a\tau}-e^{-(a+(\mu/\beta))\tau}+X(\alpha,\beta,\tau)-Y(\alpha,\beta,\tau)\}\,d\tau, \tag{31}$$

where
$$X(\alpha,\beta,t)=\int_0^t e^{-a(t-\tau)}\,\phi(\alpha,\tau)\,d\tau,$$

$$Y(\alpha,\beta,t)=\int_0^t e^{-(a+(\mu/\beta))(t-\tau)}\,\phi(\alpha,\tau)\,d\tau,$$

$$\phi(\alpha,t)=\sum_{n=1}^{\infty}n^2\left(\frac{\mu}{\lambda_1}\right)^{\frac{1}{2}n}e^{-\{\mu+\lambda_1+\lambda_2(1-\alpha)\}t} \times\int_0^t\frac{e^{-\lambda_2\alpha\tau}}{\tau(t-\tau)}I_n\{2(\lambda_1\mu)^{\frac{1}{2}}(t-\tau)\}\,I_n\{2(\lambda_1+\lambda_2)^{\frac{1}{2}}\mu^{\frac{1}{2}}\tau\}\,d\tau,$$

and
$$a=\mu+\lambda_1(1-\beta)+\lambda_2(1-\alpha)-(\mu/\beta).$$

The Laplace transform of the mean number of priority units present in the sy at time t is given by

$$\frac{\lambda_1-\mu}{s^2}+\frac{\mu}{s}\bar{P}_0(s)+\frac{\lambda_2\mu}{s(\mu+s)}\frac{\bar{P}_0(s)}{1-\{(\lambda_1+\lambda_2)/\mu\}\beta(1,s)\,\alpha_s}. \tag{32}$$

Hence the mean number of the priority units at time t is

$$(\lambda_1-\mu)\,t+(\lambda_2+\mu)\int_0^t P_0(t)\,dt+\lambda_2\int_0^t P_0(t-\tau)\{X(1,1,\tau)-Y(1,1,\tau)-e^{-\mu\tau}\}\,d\tau. \quad (33)$$

Similarly the Laplace transform of the mean number of non-priority units present in the system is

$$\frac{\lambda_2}{s^2}-\frac{\lambda_2\mu}{s(\mu+s)}\frac{\bar{P}_0(s)}{1-\{(\lambda_1+\lambda_2)/\mu\}\beta(1,s)\,\alpha_s}. \quad (34)$$

Hence the mean number of the non-priority units present in the system at time t is

$$\lambda_2 t-\lambda_2\int_0^t P_0(\tau)\,d\tau-\lambda_2\int_0^t P_0(t-\tau)\{X(1,1,\tau)-Y(1,1,\tau)-e^{-\mu\tau}\}\,d\tau. \quad (35)$$

6. Classical Queuing

Equation (30) of Jaiswal (1961) with $\alpha=\beta$, the generating function of the total queue size is identical as it should be if $\lambda=\lambda 1+\lambda 2$ with Bailey's equation (8), The Equation (28) of Jaiswal (1961) is identical with Bailey's equation (11) with a =0. From equation (31) by putting $\alpha=\beta$ and $\lambda=\lambda 1+\lambda 2$. we obtain the generating function of the queue length in the classical queuing process M/M/1 in the form of

$$\exp[-\{\lambda+\mu-\lambda\beta-(\mu/\beta)\}t]+\mu\left(1-\frac{1}{\beta}\right)\int_0^t P_0(t-\tau)\exp[-\{\mu+\lambda-\lambda\beta-(\mu/\beta)\}\tau]\,d\tau, \quad (36)$$

which can be written as

$$e^{-(\mu+\mu)t}\sum_{n=-\infty}^{\infty}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}n}\beta^n I_n\{2(\lambda\mu)^{\frac{1}{2}t}\}$$
$$+\mu\left(1-\frac{1}{\beta}\right)\int_0^t P_0(t-\tau)\,e^{-(\mu+\lambda)\tau}\sum_{n=-\infty}^{\infty}\beta^n I_n\{2(\lambda\mu)^{\frac{1}{2}}\tau\}\,d\tau, \quad (37)$$

where $I_n(x)$ is the modified Bessel function of the first kind. Now from (28), putting $\lambda=\lambda_1+\lambda_2$, we have

$$\frac{\mu\bar{P}_0(s)\,\alpha_s^n}{\{(\lambda+\mu+s)^2-4\lambda\mu\}^{\frac{1}{2}}}=\sum_{m=n+1}^{\infty}\frac{\alpha_s^m}{\{(\mu+\lambda+s)^2-4\lambda\mu\}^{\frac{1}{2}}}. \quad (38)$$

Inverting (38), we get

$$\mu\int_0^t P_0(t-\tau)\,e^{-(\mu+\lambda)\tau}I_n\{2(\lambda\mu)^{\frac{1}{2}}\tau\}=\sum_{m=n+1}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}(m-n)}e^{-(\mu+\lambda)t}I_m\{2(\lambda\mu)^{\frac{1}{2}}t\}. \quad (39)$$

By using (39) it can be easily shown that the coefficient of β^{-n} in (37) vanishes, and the coefficient of β^n, i.e. the probability that there are n units present in the system at time t, is given by

$$e^{-(\mu+\mu)t}\left[\left(\frac{\mu}{\lambda}\right)^{-\frac{1}{2}n}I_{-n}\{2(\lambda\mu)^{\frac{1}{2}}t\}+\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}(1-n)}I_{n+1}\{2(\lambda\mu)^{\frac{1}{2}}t\}+\sum_{m=n+2}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}m}I_m\{2(\lambda\mu)^{\frac{1}{2}}t\}\right] \quad (40)$$

7. Steady-state Solution

The steady-state solution can be easily obtained either directly or by usin well-known corollary of Abel's theorem

$$\lim_{s\to 0}s\bar{\Pi}_1(\alpha,\beta,s)=\lim_{t\to\infty}\Pi_1(\alpha,\beta,t),$$

provided the limit on the right-hand side exists. Assuming the limit to exist dropping t from the notation used in earlier sections, to denote the steady-state condition, we have

$$\Pi_1(\alpha,\beta)=A_2(\alpha)\left[1-Q(\alpha,\beta)-\frac{\{1-(1/\alpha)\,Q(\alpha,\beta)\}\{1-P(\alpha,\beta)\}}{1-(1/\beta)\,P(\alpha,\beta)}\right]+\frac{P_0\{1-(1/\beta)\}P(\alpha,\beta)}{1-(1/\beta)\,P(\alpha,\beta)} \quad (41)$$

where

$$A_2(\alpha)=\frac{P_0[\lambda_1\{\beta(\alpha)-1\}+\lambda_2(\alpha-1)]}{1-(1/\alpha)\,\bar{S}_2[\lambda_1\{1-\beta(\alpha)\}+\lambda_2(1-\alpha)]},$$
$$P(\alpha,\beta)=\bar{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\},$$
$$Q(\alpha,\beta)=\bar{S}_2\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\},$$

and $\beta(\alpha)$ is the smallest root of the equation

$$\beta-\bar{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}=0.$$

Since $\Pi_1(1,1)=1$, we have $P_0=1-\lambda_1\eta_1-\lambda_2\eta_2$, where

$$\eta_1=\int_0^\infty xS_1(x)\,dx \quad\text{and}\quad \eta_2=\int_0^\infty xS_2(x)\,dx$$

If the service-time distributions are exponential with mean $1/\mu_1$ and $1/\mu_2$, the function $\Pi_1(\alpha,\beta)$ becomes

$$\frac{P_0[\lambda_1\{\beta(\alpha)-1\}+\lambda_2(\alpha-1)]\,[\mu_2+\lambda_1\{1-\beta(\alpha)\}+\lambda_2(1-\alpha)]\times[\mu_1\{1-(1/\beta)\}-\mu_2\{1-(1/\alpha)\}]}{[\mu_2\{1-(1/\alpha)\}+\lambda_1\{1-\beta(\alpha)\}+\lambda_2(1-\alpha)]\,[\mu_2+\lambda_1(1-\beta)+\lambda_2(1-\alpha)]\times[\mu_1\{1-(1/\beta)\}+\lambda_1(1-\beta)+\lambda_2(1-\alpha)]}$$
$$+\frac{\mu_1 P_0\{1-(1/\beta)\}}{\mu_1\{1-(1/\beta)\}+\lambda_1(1-\beta)+\lambda_2(1-\alpha)}. \quad (42)$$

From (41), the moments of the queue lengths can easily be calculated. example, the mean number of priority units present in the system is given by

$$\rho_1+\frac{\lambda_1^2E_1(x^2)+\lambda_1\lambda_2E_2(x^2)}{2(1-\rho_1)} \quad (\rho_1=\lambda_1\eta_1),$$
$$E_1(x^2)=\int_0^\infty x^2S_1(x)\,dx \quad\text{and}\quad E_2(x^2)=\int_0^\infty x^2S_2(x)\,dx. \quad (43)$$

the mean number of non-priority units present in the system is given by

$$\rho_2+\frac{\lambda_2\{\lambda_2E_2(x^2)+\lambda_1E_1(x^2)\}}{2(1-\rho_1)(1-\rho_1-\rho_2)} \quad (\rho_2=\lambda_2\eta_2). \quad (44)$$

8. Distribution of Queue lengths at Epochs at which impulse departs.

Above limiting probability distribution equations were obtained by considering the evolution of the process in continuous time. Following gives the limiting distribution of the queue lengths at epochs at which units depart. This can be also obtained by the embedded Markov chain technique. Let pm,n the probability that under the steady state, a priority or non priority unit departs leaving m priority and n non priority units in the system.

$$p_{m,n}=\int_0^\infty P_{m+1,n}(x)\,\eta_1(x)\,dx+\int_0^\infty Q_{m,n+1}(x)\,\eta_2(x)\,dx. \quad (45)$$

Multiplying (45) by α^n and β^m and summing over m and n from 0 to ∞, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^n\beta^m p_{m,n}=\frac{1}{\beta}A_1(\alpha,\beta)P(\alpha,\beta)+\frac{1}{\alpha}A_2(\alpha)\,Q(\alpha,\beta) \quad (46)$$

$$=\frac{P_0}{\{1-(1/\beta)P(\alpha,\beta)\}}\left[\lambda_1P(\alpha,\beta)+\lambda_2Q(\alpha,\beta)-\frac{\lambda_1+\lambda_2}{\alpha}Q(\alpha,\beta)+\frac{(1/\alpha)\,Q(\alpha,\beta)-(1/\beta)\,P(\alpha,\beta)}{1-(1/\alpha)\,R(\alpha)}\left\{\lambda_1\beta(\alpha)+\lambda_2R(\alpha)-\frac{\lambda_1+\lambda_2}{\alpha}R(\alpha)\right\}\right] \quad (47)$$

where

$$R(\alpha)=\bar{S}_2[\lambda_1\{1-\beta(\alpha)\}+\lambda_2(1-\alpha)].$$

Now, since

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_{m,n}=\lambda_1+\lambda_2,$$

we normalize (47) and finally obtain $\Pi_2(\alpha,\beta)$, the generating function of the joint probability distribution of queue lengths at epochs at which customers depart. Hence,

$$\Pi_2(\alpha,\beta)=P_0\{1-\beta^{-1}P(\alpha,\beta)\}\left[C_1P(\alpha,\beta)+C_2Q(\alpha,\beta)-\frac{1}{\alpha}Q(\alpha,\beta)+\frac{(1/\alpha)\,Q(\alpha,\beta)-(1/\beta)\,P(\alpha,\beta)}{1-(1/\alpha)\,R(\alpha)}\left\{C_1\beta(\alpha)+C_2R(\alpha)-\frac{1}{\alpha}R(\alpha)\right\}\right] \quad (48)$$

where $C_1=\lambda_1/(\lambda_1+\lambda_2)$, $C_2=\lambda_2/(\lambda_1+\lambda_2)$.

This result is the same as obtained by Miller (1960), since obviously

$$\beta(\alpha)=\bar{B}\{\lambda_2(1-\alpha)\},$$

$\bar{B}(s)$ is the Laplace transform of the length of the busy period of the priority in isolation.

Conclusion.

The present method will be available for evaluating the neural signal processing under the priority principle.

7.. Reference

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APPENDIX

The equations (1), (2) and (4) are

$$\partial P_{m,n}(x,y,t)/\partial t+\partial\cdot P_{m,n}(x,y,t)/\partial x+\left[\lambda_1+\lambda_2+\eta_1(x)\right]P_{m,n}(x,y,t)$$
$$=\lambda_1\cdot P_{m-1,n}(x,y,t)+\lambda_2\cdot P_{m,n-1}(x,y,t) \qquad (5)$$

$$\partial Q_{m,n}(x,t)/\partial t+\partial\cdot Q_{m,n}(x,t)/\partial x+\left[\lambda_1+\lambda_2+\eta_1(x)\right]Q_{m,n}(x,t)$$
$$=\lambda_1\cdot Q_{m-1,n}(x,t)+\lambda_2\cdot Q_{m,n-1}(x,t) \qquad (6)$$

$$\partial U_n(y,t)/\partial t+\partial\cdot U_n(y,t)/\partial y+\left[\lambda_1+\lambda_2+\eta_1(y)\right]U_n(y,t)$$
$$=\lambda_2\cdot U_{n-1}(y,t)+\int_0^\infty P_{1,n}(x,y,t)\cdot\eta_1(x)dx \qquad (7)$$

F o r the empty state

$$\frac{\partial P_0(t)}{\partial t}+(\lambda_1+\lambda_2)\cdot P_0(t)=\int_0^\infty Q_{1,0}(x,t)\eta_1(x)dx+\int_0^\infty U_1(y,t)\eta_2(y)dy \qquad (8)$$

1) The steady state

In this case, the operation for t is all null.

$P_{m,n}(x,y,t)=P_{m,n}(x,y)$

1) multiple $\sum_{n=1}\alpha^n$ on the both sides of (5)

$$\partial/\partial x\cdot\sum_{n=1}\alpha^n\cdot P_{m,n}+(\lambda_1+\lambda_2+\eta_1(x))\sum_{n=1}\alpha^n\cdot P_{m,n}$$

*(1) $$=\lambda_1\cdot\sum_{n=1}\alpha^n\cdot P_{m-1,n}+\lambda_2\cdot\sum_{n=1}\alpha^n\cdot P_{m,n-1}$$

Introducing partial generating function defined as

$$f_m(x,y,\alpha)=\sum_{n=1}^{\infty}\alpha^n\cdot P_{m,n}(x,y)$$

Then,

$$\partial/\partial x\cdot f_m+(\lambda_1+\lambda_2+\eta_1(x))\cdot f_m=\lambda_1\cdot f_{m-1}+\lambda_2\cdot\alpha(f_m+P_{m,0}) \qquad -(5)$$

Multiply this on β^m , we set

$\sum_{m=1}\beta^m\cdot(5)'$ where

$$\partial/\partial x\cdot\sum_{m=1}\beta^m\cdot f_m+(\lambda_1+\lambda_2+\eta_1(x))\cdot\sum_{m=1}\beta^m\cdot f_m$$
$$=\lambda_1\cdot\sum_{m=1}\beta^m\cdot f_{m-1}+\lambda_2\cdot\alpha\cdot\sum_{m=1}\beta^m\cdot f_m+\lambda_2\cdot\alpha\cdot\sum_{m=1}\beta^m\cdot P_{m,0}$$

here, we pu$F(x,y,\alpha,\beta)=\sum_{m=1}^{\infty}\beta^m\cdot f_m(x,y,\alpha)$

we have

$$\partial/\partial x\cdot F+(\lambda_1+\lambda_2+\eta_1(x))\cdot F=\lambda_1\cdot\sum_{m=1}\beta^m\cdot f_{m-1}$$

*(2) $$+\lambda_2\cdot\alpha\cdot F+\lambda_2\cdot\alpha\cdot\sum_{m=1}\beta^m\cdot P_{m,0}$$

$$\therefore\partial F/\partial x+(\lambda_1+\lambda_2+\eta_1(x)-\lambda_1\cdot\beta-\lambda_2\cdot\alpha)\cdot F$$
$$=\lambda_1\cdot\beta\cdot f_0+\lambda_2\cdot\sum_{m=1}\beta^m\cdot P_{m,0}$$

The defined region of $P_{m,n}$ is $m\geq 1, n\geq 1$

$f_0=P_{m,0}=0$

$$\therefore\partial F/\partial x+\left[\lambda_1(1-\beta)+\lambda_2(1-\alpha)+\eta_1(x)\right]F=0 \qquad -(15)$$

2) Multiply α^n on the both sides of (6), we formulate

$\sum_{n=0}\alpha^n\cdot(6)$

$$\partial/\partial x\cdot\sum_{n=0}\alpha^n\cdot Q_{m,n}+(\lambda_1+\lambda_2+\eta_1(x))\sum_{n=0}\alpha^n\cdot Q_{m,n}$$

*(3) $$=\lambda_1\cdot\sum_{n=0}\alpha^n\cdot Q_{m-1,n}+\lambda_2\cdot\sum_{n=0}\alpha^n\cdot Q_{m,n-1}$$

Setting partial generating function

$$g_m(x,\alpha)=\sum_{n=0}\alpha^n\cdot Q_{m,n}(x)$$

We, have

$$\partial g_m/\partial x+(\lambda_1+\lambda_2+\eta_1(x))\cdot g_m=\lambda_2\cdot\alpha\cdot g_m+\lambda_1\cdot g_{m-1} \qquad -(6)$$

Moreover, setting the partial generating function.

$$G(x,\alpha,\beta)=\sum_{m=1}\beta^m\cdot g_m(x,\alpha)$$

We operate the following factor

$\sum_{m=1}\beta^m\cdot(6)$

$$\partial/\partial x\cdot\sum_{m=1}\beta^m\cdot g_m+(\lambda_1+\lambda_2+\eta_1(x))\cdot\sum_{m=1}\beta^m\cdot g_m$$

*(4) $$=\lambda_2\cdot\alpha\cdot\sum_{m=1}\beta^m\cdot g_m+\lambda_1\cdot\sum_{m=1}\beta^m\cdot g_{m-1}$$

$$\partial G/\partial x+\left[\lambda_1+\lambda_2+\eta_1(x)-\lambda_2\cdot\alpha-\lambda_1\cdot\beta\right]G=0 \qquad -(16)$$

(15) and (16) are the first order linear differential equations with variables coefficients, which solutions are

$$F(x,y,\alpha,\beta)=F(0,y,\alpha,\beta)\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x\eta_1(x)\cdot dx\right] \qquad (18)$$

$$G(x,\alpha,\beta)=G(0,\alpha,\beta)\exp\left[-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x-\int_0^x\eta_1(x)\cdot dx\right] \qquad (19)$$

For the part (1).
The second term in the right side
$=\lambda_2(\alpha P_{m,0}+\alpha^2 P_{m,1}+\alpha^3 P_{m,3}+{*}{*}{*})$
$=\lambda_2\alpha(P_{m,0}+\alpha P_{m,1}+\alpha^2 P_{m,3}+{*}{*}{*})$
$=\lambda_2\alpha[\sum_{n=1}\alpha^n P_{m,n}+P_{m,0}]$

-%==============================-----

For the part (2).
The first term in the right side
$=\lambda_1(\beta f_0+\beta^2 f_1+\beta^3 f_2+{*}{*}{*})$
$=\lambda_1\beta(f_0+\beta f_1+\beta^2 f_2+{*}{*}{*})$
$=\lambda_1\beta(f_0+F)$

$$\partial/\partial x F+(\lambda_1+\lambda_2+\eta_1(x))F=\lambda_1\beta f_0+\lambda_1\beta F+\lambda_2\alpha F+\lambda_2\alpha\sum_{m=1}\beta^m P_{m,0}$$

%=======

For the part (3).
$=\lambda_2(Q_{m,-1}+\alpha Q_{m,0}+\alpha^2 P_{m,1}+{*}{*}{*})$
Since $Q_{m,n}$ is defined at n=0 and n>0,
$=\lambda_2\alpha[\sum_{n=0}\alpha^n Q_{m,n}]$

%++++ For the part (4)
The second term in the right
$=\lambda_1(\beta g_0+\beta^2 g_1+\beta^3 g_2+{*}{*}{*})$
since go =0
$=\lambda_1\beta(\beta g_1+\beta^2 g_2+\beta^3 g_3+{*}{*}{*})$
$=\lambda_1\beta\sum_{m=1}\beta^m g_m=\lambda_1\beta G$

Thus,
$$\partial/\partial x G+(\lambda_1+\lambda_2+\eta_1(x))G=\lambda_2\alpha G+\lambda_1\beta G$$
This leads to (16)

[I]) Multiply the factor α^n on the both sides of (7)

To produce $\sum_{n=1}\alpha^n\cdot u_n$

$$\partial/\partial y\cdot\sum_{n=1}\alpha^n\cdot u_n+(\lambda_1+\lambda_2+\eta_2(y))\cdot\sum_{n=1}\alpha^n\cdot u_n$$

$$=\lambda_2\cdot\sum_{n=1}\alpha^n\cdot u_{n-1}+\sum_{n=1}\alpha^n\cdot\int_0^\infty P_{1,n}(x,y)\cdot\eta_1(x)dx$$

※ The 2nd term of right $=\lambda_2\cdot(\alpha\cdot u_0+\alpha^2\cdot u_1+\alpha^3\cdot u_2+\cdots)$

since $u_0=0$

$$=\lambda_2\cdot\alpha\cdot\sum_{n=1}\alpha^n\cdot u_n$$

Generating function is

$$H(y,\alpha)=\sum_{n=1}\alpha^n\cdot u_n(y)$$

Σ n=1

$$\partial H/\partial y+(\lambda_1+\lambda_2+\eta_2(y))\cdot H=\lambda_2\cdot\alpha\cdot H+\int_0^\infty\alpha^n\cdot P_{1,n}(x,y)\cdot\eta_1(x)dx$$

∴ $-\lambda_2\alpha H]$ — (17)

$$\partial H/\partial y+(\lambda_1+\lambda_2+\eta_2(y))\cdot H=\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx$$

IV] About (9) and (10)

Produce $\sum_{n=1}\alpha^n\cdot(9)$

$$\sum_{n=1}\alpha^n\cdot P_{m,n}(0,y,t)=\int_0^\infty\sum_{n=1}\alpha^n\cdot P_{m+1,n}(x,y,t)\eta_1(x)dx$$

$f_m(0,y,\alpha)=\int_0^\infty f_{m+1}(x,y,t)\cdot\eta_1(x)dx$ — (9)'

produce $\sum_{n=1}\alpha^n\cdot(10)$

$$\sum_{n=1}\alpha^n\cdot P_{1,n}(0,y,t)=\int_0^\infty\sum_{n=1}\alpha^n\cdot P_{2,n}(x,y,t)\eta_1(x)dx+\lambda_1\cdot\sum_{n=1}\alpha^n\cdot u_n(y,t)$$

$f_1(0,y,\alpha)=\int_0^\infty f_2(x,y,t)\cdot\eta_1(x)dx+\lambda_1\cdot H$ — (10)'

For (9)', we operate

$\sum_{m=2}\beta^m$ $\Sigma\beta^m$ m=2

$$\sum_{m=2}\beta^m\cdot f_m(0,y,\alpha)=\int\beta^m\cdot f_{m+1}(x,y,t)\cdot\eta_1(x)dx$$

$$\beta\cdot f_1(0,y,\alpha)=\int_0^\infty\beta\cdot f_2(x,y,t)\cdot\eta_1(x)dx+\beta\cdot\lambda_1\cdot H$$

Sum side by side

$$\sum_{m=1}\beta^m f_m(0,y\ \ \alpha)=\int_0^\infty\sum_{m=1}\beta^m\cdot f_{m+1}(x,y,t)\cdot\eta_1(x)dx+\beta\cdot\lambda_1\cdot H$$

※The 1st term in the right side

$$=\beta\cdot f_2+\beta^2\cdot f_3+\cdots$$
$$=\beta^{-1}\cdot(\pm\beta\cdot f_1+\beta^2\cdot f_2+\beta^3\cdot f_3+\cdots)$$
$$=\beta^{-1}\cdot\left(-\beta\cdot f+\sum_{m=1}\beta^m\cdot f_m\right)$$
$$=-f_1+\beta^{-1}\cdot F(x,y,\ \alpha\cdot\beta)$$

Hence

$$F(0,y,\alpha,\beta)=\int_0^\infty\left[-f_1(x,y,\alpha)+\beta^{-1}\cdot F(x,y,\alpha,\beta)\right]\eta_1(x)dx+\beta\cdot\lambda_1\cdot H$$
$$=1/\beta\cdot\int_0^\infty F(x,y,\alpha,\beta)\cdot\eta_1(x)dx-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx+\beta\cdot\lambda_1\cdot H(y,\alpha)\quad\text{— (20)}$$

V] For (11) and (12)

Produce $\sum_{n=0}\alpha^n\cdot(11)$

$$\sum_n\alpha^n\cdot Q_{m,n}(0,t)=\int_0^\infty\sum_{n=0}\alpha^n\cdot Q_{m+1,n}(x,t)\eta_1(x)dx$$

Portal generating function

$$g_m(x,\alpha)=\sum_{n=0}\alpha^n\cdot Q_{m,n}(x)$$

$$g_m(0,\alpha)=\int_0^\infty g_{m+1}(x,\alpha)\eta_1(x)dx\quad(11)'$$

Similar operation for (12)

$$\sum_{n=0}\alpha^n\cdot Q_{1,0}(0,t)=\sum_{n=0}\alpha^n\cdot\int_0^\infty Q_{2,0}(x,t)\cdot\eta_1(x)dx+\lambda_1\sum_{n=0}\alpha^n P_0(t)$$

$$g_1(0,\alpha)=\int_0^\infty g_2(x,\alpha)\eta_1(x)dx+\lambda_1\sum_{n=0}\alpha^n\cdot P_0(t)\quad(12)'$$

For (11)' produce $\sum_{m=2}\beta^m\cdot g_m(0,\alpha)$

$$\sum_{m=2}g_m(0,\alpha)\cdot\beta^m=\int_0^\infty\sum_{m=2}\beta^m\cdot g_{m+1}(x,\alpha)\cdot\eta_1(x)dx$$

$(12)'\times\beta'$

$$\beta\cdot g_1(0,\alpha)=\int_0^\infty\beta\cdot g_2(x,\alpha)\cdot\eta_1(x)dx+\lambda_1\cdot\beta\cdot\sum_{n=0}\alpha^n P_0(t)$$

Sum these terms side by sides

$$\sum_{m=1}\beta^m\cdot g_m(0,\alpha)=\int_0^\infty\sum_{m=1}\beta^m\cdot g_{m+1}(x,\alpha)\cdot\eta_1(x)dx+\lambda_1\cdot\beta\cdot P_0(t)\ \sum_{n=0}\alpha^m\ P0(t)$$

$$G(0,\alpha,\beta)=\int_0^\infty(\beta\cdot g_2+\beta^2\cdot g_3+\beta^3\cdot g_4+\cdots)\eta_1(x)dx+\lambda_1\cdot\beta\cdot P_0$$
$$=\beta^{-1}\int_0^\infty(\pm\beta\cdot g_1+\beta^2\cdot g_2+\beta^3\cdot g_3+\beta^4\cdot g_4+\cdots)\eta_1(x)dx+\lambda_1\cdot\beta\cdot P_0$$
$$=\beta^{-1}\int_0^\infty\left(-\beta\cdot g_1+\sum_{m=1}\beta^{m2}\cdot g_m\right)\eta_1(x)dx+\lambda_1\cdot\beta\cdot P_0$$
$$=-\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx+1/\beta\int_0^\infty G(x,\alpha,\beta)\eta_1(x)dx+\lambda_1\cdot\beta\cdot P_0$$

.21

VI].For (13)and(14)

Produce. $\sum_{n=1}\alpha^n U_n(y)$

Operate. $\sum_{n=2}\alpha^n\cdot(13)$

$$\sum_{n=2}\alpha^n U_n(0,t)=\int_0^\infty\sum_{n=2}\alpha^n U_{n+1}(y,t)\cdot\eta_2(y)dy+\int_0^\infty\sum_{n=2}\alpha^n Q_{1,n}(x,t)\cdot\eta_1(x)dx\quad\text{--(13)'}$$

Produce,(14)·α'

$$\alpha\cdot U_1(0,t)=\int^\infty\alpha\cdot U_2(y,t)\cdot\eta_2(y)dy+\int_0^\infty\alpha^n Q_{1,1}(x,t)\cdot\eta_1(x)dx+\alpha\cdot\lambda_2\cdot P_0(t)\quad(14)'$$

Sum, these with side by sides

$$\sum_{n=1}\alpha^n U_n(0,t)=\int_0^\infty\sum_{n=1}\alpha^n U_{n+1}(y,t)\cdot\eta_2(y)dy+\sum_{n=1}\int_0^\infty\alpha^n Q_{1,1}(x,t)\cdot\eta_1(x)dx+\alpha\cdot\lambda_2\cdot P_0(t)$$

Since, the generating, function $g_m(x,\alpha)=\sum_{n=0}\alpha^n Q_{m,n}(x)$

$$H(0,\alpha)=\int_0^\infty(\alpha\cdot U_2+\alpha^2\cdot U_3+\cdots)\cdot\eta_2(y)dy+\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx+\alpha\cdot\lambda_2 P_0-\int_0^\infty Q_{1,0}(x)\cdot\eta_1(x)dx$$

The 1st term $=\int_0^\infty\alpha^{-1}(\alpha^2U_2+\alpha^3U_3+\cdots)\eta_2(y)dy$

$$=\alpha^{-1}\int_0^\infty(\pm\alpha U_1+\alpha^2U_2+\alpha^3U_3+\cdots)\eta_2(y)dy$$
$$=\alpha^{-1}\int_0^\infty\left(-\alpha U_1+\sum_{n=1}\alpha^n U_n\right)\eta_2(y)dy$$
$$=-\int_0^\infty U_1\cdot\eta_2(y)dy+1/\alpha\cdot\int_0^\infty H(y,\alpha)\cdot\eta_2(y)dy$$

By them

$$H(0,\alpha)=\frac{1}{\alpha}\cdot\int_0^\infty H(y,\alpha)\cdot\eta_2(y)dy-\int_0^\infty U_1(y)\cdot\eta_2(y)dy$$
$$+\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx+\alpha\cdot\lambda_2\cdot P_0-\int_0^\infty Q_{1,0}(x)\cdot\eta_1(x)dx$$

For(8).at.the Steady State ??

$$(\lambda_1+\lambda_2)\cdot P_0=\int_0^\infty Q_{1,0}(x,t)\cdot\eta_1(x)dx+\int_0^\infty U_1(y,t)\cdot\eta_2(y)dy$$

Hence

$$H(0,\alpha)=\frac{1}{\alpha}\cdot\int_0^\infty H(y,\alpha)\cdot\eta_2(y)dy+\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx$$
$$+(\alpha\cdot\lambda_2-\lambda_1-\lambda_2)\cdot P_0 \qquad (22)$$

Substitute.(18),(19) *into* (20),(21)

$$F(0,y,\alpha,\beta)=\frac{1}{\beta}\cdot\int_0^\infty\eta_1(x)^{*}F(0,y,\alpha,\beta)e^{-x\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}-\int_0^x\eta_1(y)dy}dx$$
$$-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx+\lambda_1\cdot\beta\cdot H(y,\alpha)$$

Since

$$S_1(x)=\eta_1(x)e^{-\int_0^x\eta_1(x)dt}$$

1st term in the right $=\frac{F(0,y,\alpha,\beta)}{\beta}\int_0^\infty S_1(x)\cdot e^{-\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x}dx$

while Laplace transformation

$$\int_0^\infty e^{-sx}\cdot f(x)dx=f^{*}(s)$$

Putting $s=\lambda_1(1-\beta)+\lambda_2(1-\alpha)$

1st term in the right $=\frac{F(0,y,\alpha,\beta)}{\beta}S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]$

Thus

$$F(0,y,\alpha,\beta)=\frac{1}{\beta}\cdot F(0,y,\alpha,\beta)\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]$$
$$-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx+\lambda_1\cdot\beta\cdot H(y,\alpha)$$

Hence,

$$\{1-\beta S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]\}F(0,y,\alpha,\beta)$$
$$=-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx+\lambda_1\cdot\beta\cdot H(y,\alpha)$$

By the same process

$$\frac{\{1-S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]\}G(0,\alpha,\beta)}{\beta}=\lambda_1\cdot\beta\cdot Po_0--\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx$$

while

The.eigen.values of (23),(24) in $F(0,y,\alpha,\beta)$ $G(0,\alpha,\beta)$

$$\beta-S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]=0$$

There is only one root that satidfies $|\alpha|\leq 1$

Put the sole root for $\beta(\alpha)$ *and using Lagrange* expansion

$$\beta(\alpha)=\sum_{n=1}^{\infty}\frac{(-\lambda_1)^{n-1}}{n!}\frac{d^{n-1}[S_1^{*}(y)]^n}{dy^{n-1}} \qquad y=\lambda_1+\lambda_2(1-\alpha)$$

In (23) putting $\beta=\beta(\alpha)$

$$0=\lambda_1\cdot\beta(\alpha)\cdot H(y,\alpha)-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx \qquad 25$$

In(24)

$$0=\lambda_1\cdot\beta(\alpha)\cdot P_0-\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx \qquad ?26$$

*Substitute.*int *o*(25)*from*(17)

$$\frac{\partial H}{\partial y}+[\lambda_1+\lambda_2(1-\alpha)+\eta_2(y)]\cdot H=\lambda_1\cdot\beta(\alpha)\cdot H(y,\alpha)$$

This can be

$$H'+[\lambda_1+\lambda_2(1-\alpha)-\lambda_1\cdot\beta(\alpha)]\cdot H=-\eta_2(y)\cdot H$$

From, this

$$H(y,\alpha)=H(0,\alpha)e^{-[\lambda_1+\lambda_2(1-\alpha)-\lambda_1\cdot\beta(\alpha)]y-\int_0^y\eta_2(t)dt} \qquad (28)$$

*To.*det *er* min *e.* $H(0,\alpha)$ *,use* (22),(26)

*substitute.*int*o.the*1*st.*int*egral*(22) from the (28)

$$H(0,\alpha)=\frac{1}{\alpha}\cdot\int_0^\infty\eta_2(y)\cdot H(0,\alpha)e^{-[\lambda_1+\lambda_2(1-\alpha)-\lambda_1\cdot\beta(\alpha)]y-\int_0^y\eta_2(t)dt}dy$$
$$+\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx-[\lambda_1+\lambda_2(1-\alpha)]\cdot P_0$$

In.the.first.integral

$$\eta_2(y)\cdot e^{-\int_0^y\eta_2(t)dt}=S_2(y)$$

Hence,

1st.term $=\frac{1}{\alpha}\cdot H(0,\alpha)\cdot\int_0^\infty S_2(y)\cdot e^{-[\lambda_1+\lambda_2(1-\alpha)-\lambda_1\cdot\beta(\alpha)]y}dy$

Laplace transformation

$$\int_0^\infty e^{-sx}\cdot f(x)dx=f^{*}(s)$$

Putting $s=\lambda_1+\lambda_2(1-\alpha)-\lambda_1\cdot\beta(\alpha)$

$x\to y$

$f(x)=S(y)$

The 1st term $=H(0,\alpha)/\alpha\cdot S_2^{*}(\lambda_1+\lambda_2(1-\alpha)-\lambda_1\beta(\alpha))$

Hence,

$$H(0,\alpha)=H(0,\alpha)/\alpha\cdot S_2^{*}(\lambda_1+\lambda_2(1-\alpha)-\lambda_1\beta(\alpha))$$
$$+\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx-[\lambda_1+\lambda_2(1-\alpha)]\cdot P_0$$

Substitute 26.int *o,the*sec *ond.term*

$$H(0,\alpha)=H(0,\alpha)/\alpha\cdot S_2^{*}(\lambda_1+\lambda_2(1-\alpha)-\lambda_1\beta(\alpha))$$
$$+\lambda_1\cdot\beta(\alpha)\cdot P_0-[\lambda_1+\lambda_2(1-\alpha)]\cdot P_0$$

Multiply.the *α.on.the.both.sodes*

$$H(y,\alpha)=\frac{\alpha\cdot P_0\cdot[\lambda_1\beta(\alpha)-\lambda_1-\lambda_2(1-\alpha)]}{[\alpha-S_2^{*}(\lambda_1(1-\beta(\alpha))+\lambda_2(1-\alpha))]}$$
$$e^{-[\lambda_1(1-\beta(\alpha))+\lambda_2(1-\alpha)]y}\cdot e^{-\int_0^y\eta_2(t)dt}$$

F(x,y,α,β)of F(0,y,α,β) can.be.obtained.from/ (23)

$$F(0,y,\alpha,\beta)=\frac{\lambda_1\cdot\beta\cdot H(y,\alpha)-\int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$

$$(25) \qquad \int_0^\infty f_1(x,y,\alpha)\cdot\eta_1(x)dx=\lambda_1\cdot\beta(\alpha)H(y,\alpha)$$

$$=\frac{\lambda_1\cdot H(y,\alpha)[\beta-\beta(\alpha)]}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$

Hence

$$F(x,y,\alpha,\beta)=\frac{\lambda_1\cdot H(y,\alpha)[\beta-\beta(\alpha)]\cdot e^{-[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]x}\cdot e^{-\int_0^x\eta_1(t)dt}}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$

(24)

$$G(0,\alpha,\beta)=\frac{\lambda_1\cdot\beta\cdot P_0-\int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$

$$(26) \qquad \int_0^\infty g_1(x,\alpha)\cdot\eta_1(x)dx=\lambda_1\cdot\beta(\alpha)\cdot P_0$$

$$=\frac{\lambda_1\cdot P_0(\beta-\beta(\alpha))}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$

$$G(x,\alpha,\beta)=\frac{\lambda_1\cdot P_0(\beta-\beta(\alpha))\cdot e^{-[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]x}\cdot e^{-\int_0^x\eta_1(t)dt}}{[1-\frac{1}{\beta}\cdot S_1^{*}[\lambda_1(1-\beta)+\lambda_2(1-\alpha)]]}$$