

Computation of Helmholtz Free Energy in the Channel Molecules on the Biological Membrane.

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A physical and mathematical method was introduced for analyzing the bio physical- chemical mechanism of transitional changes in opening and closing the channels on the biological membrane. For the first step, we computed the Helmholtz free energy of the channel cavity. In the present work, we assumed that the channel cavity is spherical one to which the spherical coordinates and the Legendre associated functions are easy to apply. The cavity was assumed to be composed of three layers, for the inner and the middle layer the potential satisfied the Laplace equation and the outer region, Poisson equation. The free energy decreased as the dielectric constant of the inner region increased. The present method is available for evaluating the thermodynamical function of the channel molecules when they transit between the open and the close states.

Biological membrane, Channel open and close transition. Helmholtz free energy. Poisson equation.

生体膜上のチャンネル分子の自由エネルギー-計算

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生体膜上に存在する各種チャンネルの開閉状態遷移の生物物理化学的メカニズムを解明する目的で、チャンネル分子がゆうするヘルムホルツ自由エネルギーを計算する方法を示した。まず最初のステップとして、球状のCavityを形成する場合のポテンシャル計算をDubey-Huckelの強電解液理論をもとに、3層構造モデルを考えた。内層および中間（境界層）層の電場はラプラス方程式で、外層のポテンシャルはポアソン方程式で記述した。球座標を用いて、3層のポテンシャルを計算する方法をしめした。また、離散的分布をする電荷の場合の自由エネルギーを単位電荷あたりの値で計算した。内層の相対誘電率が增加すると自由エネルギーは減少した。本研究を発展させると生体チャンネル分子が有する自由エネルギーを推定するのに有益である。

生体膜上. チャンネル開閉状態遷移. ヘルムホルツ自由エネルギー. ポアソン方程式

1. Introduction.

Molecular biological experiments disclosed minute structure of ion channel molecules. Almost all of them are composed of finite number of subunits (four to five) of macro molecules (Fig 1). When the transmitter or stimulant agents such as Ach have bound to the binding sites of the Ach channel or the change of membrane potential across the channel has been altered, the conformation of all the subunits of the channel molecules are altered at a time. Then activated forms of the channel perform their responsibility. The precise mechanism, why the molecular structure of the channel molecules changes is, however, still unknown. There may be some energy conversion mechanism so that the stable structure of the channel molecules are converted to another energy state.

In the present work, we propose a method to calculate the Helmholtz type free energy of the channel molecule. As the first step, we introduce the method for spherical cavity by Jayaram which was originally developed by Tanford 1961 and Kirkwood 1934.

2. Method.

The continuum approach is motivated by a separation of spatial scales involved in the channel molecules. For sufficiently large solute compared to solvent and counter ions, the the environment (solvent and counter ion) can be regarded as a homogeneous medium. Then, the continuum formalism is characterized mathematically by an adequate differential equation for the potential. For the electrostatic potential, ψ we treat with the Laplace equation for the solvent effects

$$\nabla^2 \psi = 0 \quad \text{---(1)}$$

and the Poisson Boltzmann equation for the ionic atmosphere.

$$\nabla^2 \psi = \kappa^2 \psi \quad \text{---(2)}$$

κ is the ionic strength which is the inverse of the Debye-Huckel parameter and its order is 10^8 . The general solution of (1) in the polar coordinates adequate for the symmetry of the problem of solute in a spherical cavity is (3) and the solution for (2) is the equation (4). $P_n^m(\cos \theta)$ is the associated Legendre function. Enm relate to the charge distribution.

By applying the concentric dielectric continua theory, there are three regions (Fig 2) with solvent treated as a polarizable dielectric continuum. Region A with a dielectric constant ϵ_i is the cavity of a radius a . It contains the solute represented as a discrete charge distribution. Region B involves solvent with ϵ_{loc} in a spherical shell of thickness $(b-a)$. Region C represents the bulk solvent of ϵ_{out} and extends radially from b to infinity.

$$\Phi = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(B_{nm} r^n + \frac{E_{nm}}{r^{n+1}} \right) P_n^m(\cos \theta) e^{im\phi} \quad (3) \quad A_1 = \frac{1}{2\epsilon_i} \left[\frac{2(1-\epsilon'_a)\mu^2}{(2\epsilon'_a+1)a^3} + \left(\frac{2Y_b + \epsilon_b b Y'_b}{Y_b - \epsilon_b b Y'_b} \right) \left(1 - \frac{(1-\epsilon'_a)\mu^2}{(2\epsilon'_a+1)b^3} \right) \right] \quad (38)$$

and for Eq. (2) it is⁵

$$\Phi = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left[\frac{C_{nm}}{r^{n+1}} e^{-\kappa r} X_n(\kappa r) \right] P_n^m(\cos \theta) e^{im\phi} \quad (4)$$

$$X_n(\kappa r) = \sum_{s=0}^n \left[\frac{2^s n! (2n-s)!}{s! (2n)! (n-s)!} \right] (\kappa r)^s \quad (5)$$

$$\Phi_i = \epsilon_i^{-1} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(B_{nm} r^n + F_{nm} r^n + \frac{E_{nm}}{r^{n+1}} \right) P_n^m(\cos \theta) e^{im\phi} \quad (6)$$

Enm contains the characteristics of the central charge distribution.

$$E_{nm} = \left[\frac{(n-|m|)!}{(n+|m|)!} \right] \sum_{k=1}^n q_k r_k^n P_n^m(\cos \theta_k) e^{-im\phi} \quad (7)$$

$$\Phi_R = \epsilon_i^{-1} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} (B_{nm} r^n + F_{nm} r^n) P_n^m(\cos \theta) e^{im\phi} \quad (8)$$

The Helmholtz free energy of polarisation is

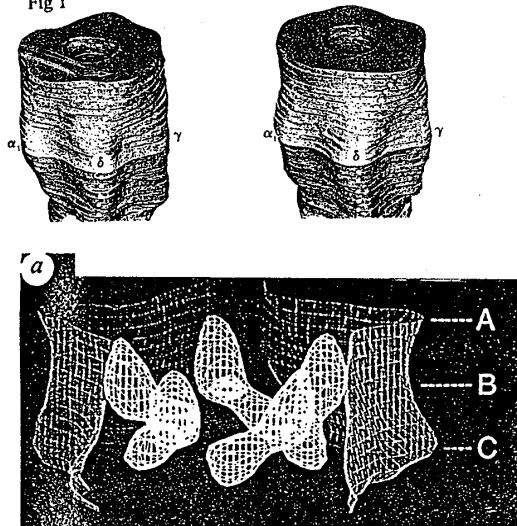
$$A = \frac{1}{2} \sum_k q_k \Phi_R(r_k) \quad (9)$$

$$= \frac{1}{2\epsilon_i} \sum_{n=0}^{\infty} \left\{ \left[\frac{(n+1)(1-\epsilon'_a)}{(n+1)\epsilon'_a + n} \right] \frac{Q_n}{a^{2n+1}} + \left[\frac{(n+1)(1-\epsilon_b)}{(n+1)\epsilon_b + n} \right] \left[1 - \frac{n(1-\epsilon'_a)}{(n+1)\epsilon'_a + n} \right] \frac{Q_n}{b^{2n+1}} \right\} \quad (10)$$

$$Q_n = \sum_k \sum_l q_k q_l r_k^n r_l^n P_n(\cos \theta_{kl}) \quad (11)$$

$$\epsilon'_a = \epsilon_a \left/ \left(1 + \frac{(n+1)(1-\epsilon_a)(1-\epsilon_b)}{[(n+1)\epsilon_b + n]} \frac{a^{2n+1}}{b^{2n+1}} \right) \right. \quad (12)$$

Fig 1



The electrostatic free energy is

$$A = \frac{1}{2\epsilon_i} \sum_{n=0}^{\infty} \left[\frac{(n+1)(1-\epsilon'_a)}{(n+1)\epsilon'_a + n} \right] \frac{Q_n}{a^{2n+1}} + \left[\frac{(n+1)Y_b + \epsilon_b b Y'_b}{nY_b - \epsilon_b b Y'_b} \right]$$

$$\times \left(1 - \frac{n(1-\epsilon'_a)}{(n+1)\epsilon'_a + n} \right) \frac{Q_n}{b^{2n+1}}$$

for $n=1$, $Q_1 = \mu^2$, and

$$\epsilon'_a = \epsilon_a \left/ \left(1 + \frac{a^3}{b^3} (1-\epsilon_a) \left(\frac{2Y_b + b\epsilon_b Y'_b}{Y_b - b\epsilon_b Y'_b} \right) \right) \right. \quad (39)$$

$$Y_b = (e^{-xb}/b^2)(1+xb) \quad (40)$$

$$Y'_b = (-e^{-xb}/b^3)(2+2xb+x^2b^2) \quad (41)$$

by setting $\epsilon_b = 1$, $\epsilon_i = 1$, and $x = 0$ in Eq. (38),

$$A_1 = \frac{(1-\epsilon_a)\mu^2}{(1+2\epsilon_a)a^3} \quad (42)$$

3. Results

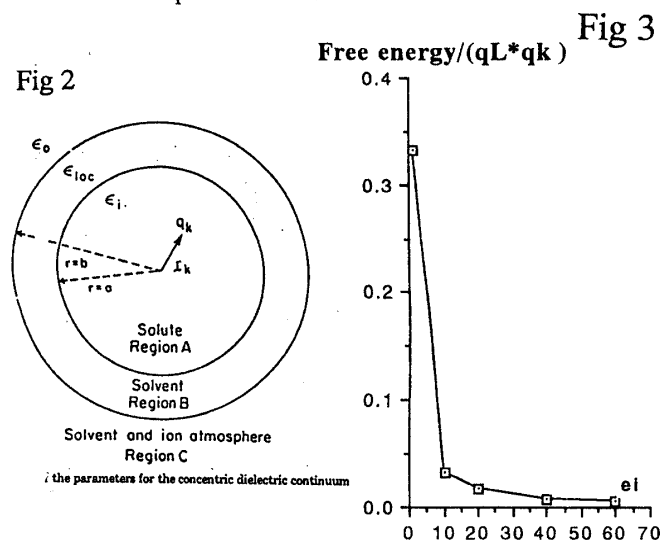
Fig 3 shows the calculated free energy for the parameters of physiological values of the channel on the membrane, $x = 7 \cdot 10^{-8}$, $a = 6 \cdot 10^{-8}$, $b = 10 \cdot 10^{-8}$. Dielectric constants are $\epsilon_i = 1$, $\epsilon_{loc} = 20$, $\epsilon_o = 80$ setting by $\epsilon_a = \epsilon_{loc}/\epsilon_i$ and $\epsilon_b = \epsilon_o/\epsilon_{loc}$. The free energy was insensitive to the changes in x nor the magnitude of Q_1 . With increase in the dielectric constant at the inner region, however, the potential decreased rapidly.

4. Conclusion.

For the first step of analyzing the mechanism of the molecular conformational change, computation of free energy is important

5. References

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APPENDIX 1

THE LEGENDRE'S ASSOCIATED FUNCTIONS

54. It has been shown in § 6 that Laplace's equation is satisfied by

$$\frac{r^n}{r^{n-1}} \cos m\phi \cdot u_n^m,$$

where u_n^m satisfies the differential equation

$$\frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{du}{d\mu} \right\} + \left\{ n(n+1) - \frac{m^2}{1 - \mu^2} \right\} u = 0 \quad \dots (1).$$

The arbitrary number n, m are subject to no restriction, but the most important case which shall for the present chiefly consider is when n and m are positive integers such that $n \geq m$. Some attention will, however, be paid to the case in which m and n are integers such that $m > n$, and also to that in which m is a negative integer.

Let $u = (\mu^2 - 1)^{1/2} v$, then it will be found that v satisfies the equation

$$(1 - \mu^2) \frac{d^2 v}{d\mu^2} - 2(m+1)\mu \frac{dv}{d\mu} + (n-m)(n+m+1)v = 0 \quad \dots (2).$$

If, in Legendre's equation,

$$(1 - \mu^2) \frac{d^2 u}{d\mu^2} - 2\mu \frac{du}{d\mu} + n(n+1)u = 0$$

we differentiate m times, we find that

$$(1 - \mu^2) \frac{d^{m+2} u}{d\mu^{m+2}} - 2(m+1)\mu \frac{d^{m+1} u}{d\mu^{m+1}} + (n-m)(n+m+1) \frac{d^m u}{d\mu^m} = 0;$$

it follows that $\frac{d^m u}{d\mu^m}$ satisfies the equation (2); the complete integral of that equation is therefore

$$v = A \frac{d^m P_n(\mu)}{d\mu^m} + B \frac{d^m Q_n(\mu)}{d\mu^m},$$

where A, B are arbitrary constants. The complete solution of (1) is consequently

$$u = A (\mu^2 - 1)^{1/2} \frac{d^m P_n(\mu)}{d\mu^m} + B (\mu^2 - 1)^{1/2} \frac{d^m Q_n(\mu)}{d\mu^m} \quad \dots (3).$$

When μ has any value on the complex plane of μ , which is not real and between -1 and 1 , the functions

$$(\mu^2 - 1)^{1/2} \frac{d^m P_n(\mu)}{d\mu^m}, \quad (\mu^2 - 1)^{1/2} \frac{d^m Q_n(\mu)}{d\mu^m},$$

64. Suppose μ to be real and between ± 1 ; then if $h < 1$,

$$\frac{1}{\sqrt{1 - 2h\mu + h^2}} = \sum_{n=0}^{\infty} P_n(\mu) h^n;$$

on differentiating both sides m times with respect to μ , we have

$$\frac{1 \cdot 3 \cdot 5 \dots (2m-1) h^m}{(1 - 2h\mu + h^2)^{m+1/2}} = \sum_{n=m}^{\infty} h^n \frac{d^m P_n(\mu)}{d\mu^m}.$$

This term by term differentiation is justifiable because, in any interval interior to $(-1, 1)$, the series is uniformly convergent for $m = 1, 2, 3, \dots$

Hence the coefficient of h^{n-m} in the expansion of

$$\frac{1}{(1 - 2h\mu + h^2)^{m+1/2}}$$

in powers of h is $\frac{2^m m! (-1)^m}{(2m)!} (1 - \mu^2)^{-1/2} P_n^m(\mu)$.

Writing $h = r'/r$, $\mu = \cos \theta$, where $r' < r$, this becomes

$$\frac{1}{(\sqrt{r^2 + r'^2 - 2rr' \cos \theta})^{m+1/2}} = \sum_{n=m}^{\infty} \frac{2^m m! (-1)^m}{(2m)!} \frac{r'^{n-m}}{r^{n+m+1}} (1 - \mu^2)^{-1/2} P_n^m(\mu).$$

$$\text{Since } \frac{1}{(r^2 + r'^2 - 2rr' \cos \theta)^{m+1/2}} = \frac{1}{\{x^2 + y^2 + (z - r')^2\}^{m+1/2}} = \sum_{n=m}^{\infty} (-1)^{n-m} \frac{r'^{n-m}}{(n-m)!} \frac{\partial^{n-m}}{\partial z^{n-m}} \frac{1}{r^{2m+1}},$$

we have the formula

$$P_n^m(\cos \theta) = (-1)^n \frac{(2m)!}{2^m m! (n-m)!} \frac{\sin^m \theta}{\partial z^{n-m}} \frac{1}{r^{2m+1}} \quad \dots (36),$$

which is a generalization of the formula (13) of Chap. II, for $P_n(\cos \theta)$.

If the expression $\frac{1}{\sqrt{1 - 2h\mu + h^2}}$ be integrated m times with respect to μ between the limits 1 and μ , we have

$$(-1)^m \frac{2^m m!}{(2m)!} \frac{1}{h^m} (1 - 2h\mu + h^2)^{m-1/2},$$

together with a rational expression which involves only powers of h , the highest negative power of which is h^{-m} ; this expression must be equal to $\sum h^n P_n^m(\mu) (-1)^m (1 - \mu^2)^{1/2}$, hence $\sin^m \theta P_n^m(\cos \theta)$ is the coefficient of h^{n+m} in the expansion of $\frac{2^m m!}{(2m)!} (1 - 2h\mu + h^2)^{m-1/2}$. Writing $h = \frac{r'}{r}$,

we see that the coefficient of r'^{n+m} in the expansion of

$$(r^2 + r'^2 - 2rr' \cos \theta)^{m-1/2}$$

is

$$\frac{(-1)^{n+m}}{(n+m)!} \frac{\partial^{n+m}}{\partial z^{n+m}} \frac{1}{r^{2m-1}};$$

we thus obtain the formula

$$P_n^m(\cos \theta) = (-1)^{n+m} \frac{\sin^m \theta}{(2m)! (n+m)!} \frac{1}{r^{m-n-1}} \frac{\partial^{n+m}}{\partial z^{n+m}} \frac{1}{r^{2m-1}} \quad \dots (37).$$

65. If, in the theorem

$$\int_0^{\pi} \frac{\cos m\phi}{a + b \cos \phi} d\phi = \frac{\pi}{\sqrt{a^2 - b^2}} \left\{ \frac{-a + \sqrt{a^2 - b^2}}{b} \right\}^m,$$

we put $a = \mu - h$, $b = +\sqrt{\mu^2 - 1}$, we have

$$\int_0^{\pi} \frac{\cos m\phi}{\mu - h + \sqrt{\mu^2 - 1} \cos \phi} d\phi = \frac{\pi}{\sqrt{1 - 2h\mu + h^2}} \left\{ \frac{h - \mu + \sqrt{1 - 2h\mu + h^2}}{\sqrt{\mu^2 - 1}} \right\}^m,$$

which holds provided that $\frac{h - \mu}{\sqrt{\mu^2 - 1}}$ is not a positive real quantity less

than unity. Suppose that $h < 1$, and expand the integral on the left-hand side in powers of h ; we see then from (33) that

$$(-1)^m \frac{(n+m)!}{n!} P_n^m(\mu)$$

is equal to the coefficient of h^n in the expansion of

$$\frac{1}{\sqrt{1 - 2h\mu + h^2}} \left\{ \frac{h - \mu + \sqrt{1 - 2h\mu + h^2}}{\sqrt{\mu^2 - 1}} \right\}^m,$$

in powers of h . In this result, μ is not real and between ± 1 .

In the case of μ real and between ± 1 , let $h = \frac{r'}{r}$; in this case $P_n^m(\mu)$ must

be replaced by $e^{1/2 m \pi i} P_n^m(\mu)$, and we have for $\frac{(n+m)!}{n!} P_n^m(\mu)$, the

coefficient of $\left(\frac{r'}{r}\right)^n$ in the expansion of

$$\frac{r}{\sqrt{r^2 - 2\mu r r' + r'^2}} \left\{ \frac{z - r' - \sqrt{r^2 + r'^2 - 2\mu r r'}}{r \sqrt{1 - \mu^2}} \right\}^m,$$

which is
$$\frac{(-1)^{n+m}}{n!} \sin^m \theta \frac{1}{r^{m-1}} \frac{\partial^n}{\partial z^n} \left(\frac{r-z}{r} \right)^m;$$

hence
$$P_n^m(\cos \theta) = \frac{(-1)^n}{(n+m)!} \frac{\sin^m \theta}{r^{m-1}} \frac{\partial^n}{\partial z^n} \left(\frac{r-z}{r} \right)^m,$$

or, by (23),
$$P_n^m(\cos \theta) = \frac{(-1)^{n+m}}{(n-m)!} \frac{\sin^m \theta}{r^{m-1}} \frac{\partial^n}{\partial z^n} \left(\frac{r-z}{r} \right)^m = \frac{(-1)^{n+m}}{(n-m)!} r^{m+n+1} \sin^m \theta \frac{\partial^n}{\partial z^n} \frac{1}{(r+z)^{n+m}}.$$

Therefore
$$P_n^m(\cos \theta) = \frac{(-1)^{n+m}}{(n-m)!} r^{n+1} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r(z+z)} \right)^{n+m} \dots (38);$$

if z be changed into $-z$, the formula becomes

$$P_n^m(\cos \theta) = \frac{(-1)^n}{(n-m)!} r^{n+1} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r(r-z)} \right)^{n+m} \dots (39).$$

These formulae may also be deduced from (13) by means of Lagrange's theorem. Let $y = \mu + h \cdot \frac{y^2-1}{2}$, and let $f'(y) = \left(\frac{y+1}{y-1} \right)^m$. Then

$$f'(y) \frac{dy}{d\mu} = \sum \frac{h^n}{n!} \frac{d^n}{d\mu^n} \left\{ \left(\frac{\mu-1}{2} \right)^n \left(\frac{\mu+1}{\mu-1} \right)^m \right\}$$

or
$$\frac{1}{\sqrt{1-2h\mu+h^2}} \left(\frac{y+1}{y-1} \right)^m = \sum \frac{h^n}{n!} 2^n (n-m)! \left(\frac{1+\mu}{1-\mu} \right)^{n+m} P_n^m(\mu).$$

RECURRENT RELATIONS BETWEEN SUCCESSIVE FUNCTIONS

66. It has been shown in § 54 that $\frac{d^m P_n(\mu)}{d\mu^m}$ satisfies the equation

$$(\mu^2-1) \frac{d^{m+2}}{d\mu^{m+2}} P_n(\mu) + 2(m+1)\mu \frac{d^{m+1}}{d\mu^{m+1}} P_n(\mu) - (n-m)(n+m+1) \frac{d^m P_n(\mu)}{d\mu^m} = 0.$$

On writing $(\mu^2-1)^{1/2} \frac{d^m P_n(\mu)}{d\mu^m} = P_n^m(\mu)$,

we thus obtain the relation

$$P_n^{m+2}(\mu) + 2(m+1) \frac{\mu}{\sqrt{\mu^2-1}} P_n^{m+1}(\mu) - (n-m)(n+m+1) P_n^m(\mu) = 0 \dots (40),$$

which is the recurrent relation between the three functions $P_n^{m+2}(\mu)$, $P_n^{m+1}(\mu)$, $P_n^m(\mu)$, when μ is not real and < 1 . In the case $\mu = \cos \theta$, we write

$$(-1)^m (1-\mu^2)^{1/2} \frac{d^m P_n(\mu)}{d\mu^m} = P_n^m(\mu);$$

we have therefore, in that case,

$$P_n^{m+2}(\cos \theta) + 2(m+1) \cot \theta P_n^{m+1}(\cos \theta) + (n-m)(n+m+1) P_n^m(\cos \theta) = 0 \dots (41).$$

In order to obtain a recurrent relation between $P_{n+2}^m(\mu)$, $P_{n+1}^m(\mu)$, $P_n^m(\mu)$, we may make use of the relations

$$(2n+1)\mu P_n^m(\mu) - (n+1)P_{n+1}^m(\mu) - nP_{n-1}^m(\mu) = 0,$$

$$\frac{dP_{n+1}^m(\mu)}{d\mu} - \frac{dP_{n-1}^m(\mu)}{d\mu} = (2n+1)P_n^m(\mu),$$

Differentiating the first relation m times with respect to μ and multiplying, we have

$$(2n+1)\mu \frac{d^m P_n^m(\mu)}{d\mu^m} + (2n+1)m \frac{d^{m-1} P_n^m(\mu)}{d\mu^{m-1}} - (n+1) \frac{d^m P_{n+1}^m(\mu)}{d\mu^m} - n \frac{d^m P_{n-1}^m(\mu)}{d\mu^m} = 0.$$

On differentiating the second relation $m-1$ times with respect to μ , we have

$$\frac{d^m P_{n+1}^m(\mu)}{d\mu^m} - \frac{d^m P_{n-1}^m(\mu)}{d\mu^m} = (2n+1) \frac{d^{m-1} P_n^m(\mu)}{d\mu^{m-1}};$$

by eliminating $\frac{d^{m-1} P_n^m(\mu)}{d\mu^{m-1}}$, we find

$$(2n+1)\mu \frac{d^m P_n^m(\mu)}{d\mu^m} - (n-m+1) \frac{d^m P_{n+1}^m(\mu)}{d\mu^m} - (n+m) \frac{d^m P_{n-1}^m(\mu)}{d\mu^m} = 0,$$

or

$$(2n+1)\mu P_n^m(\mu) - (n-m+1)P_{n+1}^m(\mu) - (n+m)P_{n-1}^m(\mu) = 0 \dots (42),$$

or

$$(n-m+2)P_{n+2}^m(\mu) - (2n+3)\mu P_{n+1}^m(\mu) + (n+m+1)P_n^m(\mu) = 0 \dots (42)',$$

the required recurrent relation.

THE FUNCTIONS $Q_n^m(\mu)$, $Q_n^m(\cos \theta)$

69. When μ is not real and between 1 and -1 , the function $Q_n^m(\mu)$ has been defined in § 54 as $(\mu^2-1)^{1/2} \frac{d^m Q_n(\mu)}{d\mu^m}$, which by Chap. II (59) is equivalent to

$$Q_n^m(\mu) = (-1)^n \frac{2^n n!}{(2n)!} (\mu^2-1)^{1/2} \frac{d^{n+m}}{d\mu^{n+m}} \left\{ (\mu^2-1)^n \int_{\mu}^{\infty} \frac{d\mu}{(\mu^2-1)^{n+1}} \right\},$$

where μ is not real and < 1 .

In order to obtain an expression for $Q_n^m(\mu)$ analogous to the expression (13) for $P_n^m(\mu)$, let $u = (\mu-1)^{n-m}(\mu+1)^{n+m}$; we then find by differentiation that

$$(1-\mu^2) \frac{du}{d\mu} + 2(n\mu-m)u = 0,$$

$$\text{and} \quad (1-\mu^2) \frac{d^2 u}{d\mu^2} + \{2(n-1)\mu-2m\} \frac{du}{d\mu} + 2nu = 0.$$

In order to find the complete primitive of this differential equation of the second order, we put $u = (\mu-1)^{n-m}(\mu+1)^{n+m}w$; we then find that w satisfies the equation

$$\frac{d^2 w}{d\mu^2} + 2 \frac{d}{d\mu} \left\{ \frac{(\mu-1)^{n-m}(\mu+1)^{n+m}}{(\mu-1)^{n-m}(\mu+1)^{n+m}} \right\} + \frac{2(n-1)\mu-2m}{1-\mu^2} w = 0,$$

hence

$$\frac{dw}{d\mu} (\mu-1)^{2n-2m} (\mu+1)^{2n+2m} (\mu^2-1)^{-n+1} (\mu+1)^{-m} (\mu-1)^m = C,$$

or

$$w = C \int \frac{d\mu}{(\mu-1)^{n-m+1}(\mu+1)^{n+m+1}}.$$

Thus

$$u = (\mu-1)^{n-m}(\mu+1)^{n+m} \left\{ A + B \int \frac{d\mu}{(\mu-1)^{n-m+1}(\mu+1)^{n+m+1}} \right\}$$

represents the complete primitive of the differential equation

$$(1-\mu^2) \frac{d^2 u}{d\mu^2} + \{2(n-1)\mu-2m\} \frac{du}{d\mu} + 2nu = 0.$$

On differentiating this equation n times with respect to μ , we find that

$$\frac{d}{d\mu} \left\{ (1-\mu^2) \frac{d^{n+1} u}{d\mu^{n+1}} \right\} - 2m \frac{d^{n+1} u}{d\mu^{n+1}} + n(n+1) \frac{d^n u}{d\mu^n} = 0;$$

this last equation is easily reducible to the equation satisfied by $P_n^m(\mu)$, $Q_n^m(\mu)$. In the equation (1) put $u = \left(\frac{\mu-1}{\mu+1} \right)^{1/2} U$; we then find that U satisfies the equation

$$\frac{d}{d\mu} \left\{ (1-\mu^2) \frac{dU}{d\mu} \right\} - 2m \frac{dU}{d\mu} + n(n+1)U = 0.$$

We thus see that the complete solution of (1) is of the form

$$u = \left(\frac{\mu-1}{\mu+1} \right)^{1/2} \frac{d^n}{d\mu^n} \left[(\mu-1)^{n-m} (\mu+1)^{n+m} \times \left\{ A + B \int \frac{d\mu}{(\mu-1)^{n-m+1}(\mu+1)^{n+m+1}} \right\} \right].$$

The first part of this solution leads to the formula (13), which expresses $P_n^m(\mu)$; the second part, which is infinite when $\mu = \pm 1$ and tends to 0 as $|\mu|$ becomes indefinitely great, gives us

$$Q_n^m(\mu) = K \left(\frac{\mu-1}{\mu+1} \right)^{1/2} \frac{d^n}{d\mu^n} \left\{ (\mu-1)^{n-m} (\mu+1)^{n+m} \int_{\mu}^{\infty} \frac{d\mu}{(\mu-1)^{n-m+1}(\mu+1)^{n+m+1}} \right\}.$$

The whole of this expression is algebraic with the exception of a part arising from a term $\log \frac{\mu+1}{\mu-1}$ in the integral.

To determine the constant K we may suppose $|\mu|$ to be large; then the leading term is

$$K \frac{d^n}{d\mu^n} \left\{ \mu^{2n} \cdot \frac{1}{2n+1} \cdot \frac{1}{\mu^{2n+1}} \right\},$$

or

$$\frac{K}{2n+1} \cdot \frac{n!(-1)^n}{\mu^{n+1}};$$

in accordance with the expression (43) this leading term is

$$(-1)^m \frac{2^n n! (n+m)!}{(2n+1)!} \frac{1}{\mu^{n+1}},$$

and it thus follows that

$$K = (-1)^{m-n} \frac{2^n (n+m)!}{(2n)!};$$

hence we obtain the formula

$$Q_n^m(\mu) = (-1)^{n-m} \frac{2^n (n+m)!}{(2n)!} \left(\frac{\mu-1}{\mu+1} \right)^{1/2} \frac{d^n}{d\mu^n} \left\{ (\mu-1)^{n-m} (\mu+1)^{n+m} \times \int_{\mu}^{\infty} \frac{d\mu}{(\mu-1)^{n-m+1}(\mu+1)^{n+m+1}} \right\} \dots (47),$$

which is the analogue of the expression (13), for $P_n^m(\mu)$.

EXPANSION OF $Q_n^m(\mu)$ AND $P_n^m(\mu)$ IN POWERS OF $\mu - \sqrt{\mu^2-1}$

71. If, in the equation (2), we make $(\mu - \sqrt{\mu^2-1})^2$, for which we shall write ξ , the independent variable, we find that the equation takes the form

$$\xi^2 (1-\xi) \frac{d^2 v}{d\xi^2} + \xi \left\{ \frac{1}{2} - m - (n+m+\frac{1}{2}) \xi \right\} \frac{dv}{d\xi} - \frac{1}{4} (n-m)(n+m+1)(1-\xi)v = 0;$$

if now we put $v = \xi^{1/2(n+m+1/2)} v'$, we find for v' the differential equation

$$\xi (1-\xi) \frac{d^2 v'}{d\xi^2} + \{ (n+\frac{1}{2}) - (n+2m+\frac{1}{2}) \xi \} \frac{dv'}{d\xi} - (n+m+1)(m+\frac{1}{2}) v' = 0.$$

Comparing this with the differential equation

$$\xi (1-\xi) \frac{d^2 v'}{d\xi^2} + \{ \gamma - (\alpha + \beta + 1) \xi \} \frac{dv'}{d\xi} - \alpha \beta v' = 0,$$

which is satisfied by $v' = F(\alpha, \beta; \gamma; \xi)$, we see that if $\alpha = n+m+1$, $\beta = m+\frac{1}{2}$, $\gamma = n+\frac{1}{2}$, the equations are identical. It follows that the fundamental equation (1) is satisfied by

$$u_1 = \xi^{1/2(n+m+1/2)} (\mu^2-1)^{1/2} F\left(\frac{1}{2}+m, n+m+1; n+\frac{1}{2}; \xi\right),$$

or by $u_2 = \xi^{\frac{1}{2}(m-n)} (\mu^2 - 1)^{\frac{1}{2}m} F\left(\frac{1}{2} + m, m - n; \frac{1}{2} - n; \xi\right)$.

By changing m into $-m$, we see that the equation (1) is also satisfied by

$$u_3 = \xi^{\frac{1}{2}(n-m+1)} (\mu^2 - 1)^{-\frac{1}{2}m} F\left(\frac{1}{2} - m, n - m + 1; n + \frac{3}{2}; \xi\right),$$

and by $u_4 = \xi^{-\frac{1}{2}(n+m)} (\mu^2 - 1)^{-\frac{1}{2}m} F\left(\frac{1}{2} - m, -m - n; \frac{1}{2} - n; \xi\right)$.

The series u_1 and u_2 are convergent for all values of μ which are not real and between ± 1 ; u_3, u_4 are convergent for all values of μ , since the real part of $\sqrt{\mu^2 - 1}$ has the same sign as the real part of μ . In order to obtain the expression for $Q_n^m(\mu)$, it is sufficient to suppose μ very great and to compare these solutions with (43), the principal part of which is

$$(-1)^m \frac{2^n n! (n+m)!}{(2n+1)!} (\mu^2 - 1)^{\frac{1}{2}m} \frac{1}{\mu^{n+m+1}};$$

we thus see that $Q_n^m(\mu)$ is expressible in terms of u_1 , the principal part of which is $\frac{1}{2^{n+m+1}} \frac{1}{\mu^{n+m+1}} (\mu^2 - 1)^{\frac{1}{2}m}$, and we obtain the formula

$$Q_n^m(\mu) = (-1)^m \frac{2^{2n+m+1} n! (n+m)!}{(2n+1)!} z^{-(n+m+1)} (\mu^2 - 1)^{\frac{1}{2}m} \times F\left(\frac{1}{2} + m, n + m + 1; n + \frac{3}{2}; \frac{1}{z}\right) \dots (40),$$

where $z = \mu + \sqrt{\mu^2 - 1}$.

APPENDIX 2

In the computation of potential in the prolate conformation, it is convenient to use The Neuman expansion for the distance between two points.

$$\begin{aligned} 1/r_{12} &= \sum_{n=0}^{\infty} (2n+1) Q_n(\xi >) P_n(\xi <) P_n(\eta_1) P_n(\eta_2) \\ &+ 2 \sum_{n=1}^{\infty} (2n+1) \sum_{m=1}^n (-1)^m \left[\frac{(n-m)!}{(n+m)!} \right] 2 \\ &\quad * Q_n^m(\xi >) P_n^m(\xi <) P_n^m(\eta_1) P_n^m(\eta_2) \\ &\quad * \cos(m(\phi_1 - \phi_2)) \end{aligned}$$

$$\xi_1, 2 = \cosh(u) \geq 1.$$

$$\eta = \cosh(v)$$

where $Q_n(\xi >)$ and $P_n(\xi <)$ the Legendre bi spherical functions defined by Hobson while $P_n^m(\eta_1) P_n^m(\eta_2)$ are those defined by Ferrer. For sufficiently large distance of $r \rightarrow \infty$ where $r >$ and $r <$ are very large, we have the following approximations

$$Q_n(r > / c) \doteq \sqrt{\pi} n! / [2^{n+1} \Gamma(n+3/2) (r > / c)^{n+1}]$$

$$Q_n^m(r > / c) \doteq \sqrt{\pi} (-1)^m (n+m)! / ((r > / c)^{2-1})^{m/2} / [2^{n+1} \Gamma(n+3/2) (r > / c)^{n+m+1}]$$

$$P_n(r < / c) = (2n)! / (2^n (n!)^2) (r < / c)^n +$$

$$P_n^m(r < / c) = (2n)! / (2^n (n!) (n-m)!) ((r < / c)^2 - 1)^{m/2} * (r < / c)^{(n-m)}$$

APPENDIX 3

=====

=====functions for the outer boundary values Lamda * z0 ===
differentiated type

$$Pn0m0z0 = 1$$

$$Pn1m0z0 = z0$$

$$\begin{aligned} Pn2m0z0 &= [(2*1+1)*z0*Pn1m0z0 - 1*Pn0m0z0]/2 \\ Pn3m0z0 &= [(2*2+1)*z0*Pn2m0z0 - 2*Pn1m0z0]/3 \\ Pn4m0z0 &= [(2*3+1)*z0*Pn3m0z0 - 3*Pn2m0z0]/4 \\ Pn5m0z0 &= [(2*4+1)*z0*Pn4m0z0 - 4*Pn3m0z0]/5 \\ Pn6m0z0 &= [(2*5+1)*z0*Pn5m0z0 - 5*Pn4m0z0]/6 \\ Pn7m0z0 &= [(2*6+1)*z0*Pn6m0z0 - 6*Pn5m0z0]/7 \\ Pn8m0z0 &= [(2*7+1)*z0*Pn7m0z0 - 7*Pn6m0z0]/8 \\ Pn9m0z0 &= [(2*8+1)*z0*Pn8m0z0 - 8*Pn7m0z0]/9 \\ Pn10m0z0 &= [(2*9+1)*z0*Pn9m0z0 - 9*Pn8m0z0]/10 \\ Pn11m0z0 &= [(2*10+1)*z0*Pn10m0z0 - 10*Pn9m0z0]/11 \\ Pn12m0z0 &= [(2*11+1)*z0*Pn11m0z0 - 11*Pn10m0z0]/12 \end{aligned}$$

===== Associated Legendre functions ===
===== Differentiated type

$$Pn0m1z0 = 0$$

$$\begin{aligned} Pn1m1z0 &= [(2*n+1)*z0*Pn,m(z0) - (n+m)*Pn-1,m(z0)]/(n-m+1) \\ Pn2m1z0 &= [(2*z0*Pn2m0z0 - (2*1-1)*Pn0m0z0)/(z0^2-1)]^0.5 \\ Pn2m2z0 &= [(2*z0*Pn2m0z0 - (2*1-1)*Pn1m0z0)/(z0^2-1)]^0.5 \end{aligned}$$

=====

$$dPn2m1z0 = 1/(1-z0^2)*((2+1)*Pn1m1z0 - 2*z0*Pn2m1z0)$$

$$dPn1m1z0 = 1/(z0^2-1)*(Pn2m1z0 - 2*z0*Pn1m1z0)$$

=====

$$\begin{aligned} Pn1m1m1z0 &= 1/2 * Pn1m1z0 \\ Pn2m1m1z0 &= 1/((2+1)*2) * Pn2m1z0 \\ Pn2m2m1z0 &= 1/((2+2)*3*2) * Pn2m2z0 \end{aligned}$$

=====

$$\begin{aligned} dPn1m1m1z0 &= 1/2 * dPn1m1z0 \\ dPn2m1m1z0 &= \text{gamma}(2) / \text{gamma}(4) * dPn2m1z0 \end{aligned}$$

=====

$$\begin{aligned} Pn3m1z0 &= ((2*2+1)*z0*Pn2m1z0 - (2+1)*Pn1m1z0)/(2-1+1) \\ Pn3m2z0 &= ((2*z0*Pn3m1z0 - 2*2*Pn2m1z0)/(z0^2-1))^0.5 \\ Pn3m3z0 &= ((2*z0*Pn3m2z0 - (2*3-1)*Pn2m2z0)/(z0^2-1))^0.5 \end{aligned}$$

=====

$$\begin{aligned} dPn3m1z0 &= 1/(1-z0^2)*((3+1)*Pn2m1z0 - 3*z0*Pn3m1z0) \\ dPn3m2z0 &= 1/(1-z0^2)*((3+2)*Pn2m2z0 - 3*z0*Pn3m2z0) \end{aligned}$$

$$Pn3m1m1z0 = 2/\text{gamma}(5) * Pn3m1z0$$

$$Pn3m2m1z0 = 1/\text{gamma}(6) * Pn3m2z0$$

$$Pn3m3m1z0 = 1/\text{gamma}(7) * Pn3m3z0$$

=====

$$\begin{aligned} dPn3m1m1z0 &= \text{gamma}(3) / \text{gamma}(5) * dPn3m1z0 \\ dPn3m2m1z0 &= \text{gamma}(2) / \text{gamma}(6) * dPn3m2z0 \end{aligned}$$

=====

$$\begin{aligned} Pn4m1z0 &= ((2*3+1)*z0*Pn3m1z0 - (3+1)*Pn2m1z0)/(3-1+1) \\ Pn4m2z0 &= ((2*3+1)*z0*Pn3m2z0 - (3+2)*Pn2m2z0)/(3-2+1) \\ Pn4m3z0 &= ((2*z0*Pn4m2z0 - 2*3*Pn3m2z0)/(z0^2-1))^0.5 \\ Pn4m4z0 &= ((z0*Pn4m3z0 - (2*4-1)*Pn3m3z0)/(z0^2-1))^0.5 \end{aligned}$$

=====

$$\begin{aligned} dPn4m1z0 &= 1/(1-z0^2)*((4+1)*Pn3m1z0 - 4*z0*Pn4m1z0) \\ dPn4m2z0 &= 1/(1-z0^2)*((4+2)*Pn3m2z0 - 4*z0*Pn4m2z0) \\ dPn4m3z0 &= 1/(1-z0^2)*((4+3)*Pn3m3z0 - 4*z0*Pn4m3z0) \end{aligned}$$

$$Pn4m1m1z0 = 3*2/\text{gamma}(6) * Pn4m1z0$$

$$Pn4m2m1z0 = 2/\text{gamma}(7) * Pn4m2z0$$

$$Pn4m3m1z0 = 1/\text{gamma}(8) * Pn4m3z0$$

$$Pn4m4m1z0 = 1/\text{gamma}(9) * Pn4m4z0$$

=====

$$\begin{aligned} dPn4m1m1z0 &= \text{gamma}(4) / \text{gamma}(6) * dPn4m1z0 \\ dPn4m2m1z0 &= \text{gamma}(3) / \text{gamma}(7) * dPn4m2z0 \\ dPn4m3m1z0 &= \text{gamma}(2) / \text{gamma}(8) * dPn4m3z0 \end{aligned}$$

=====

$$\begin{aligned} Pn5m1z0 &= ((2*4+1)*z0*Pn4m1z0 - (4+1)*Pn3m1z0)/(4-1+1) \\ Pn5m2z0 &= ((2*4+1)*z0*Pn4m2z0 - (4+2)*Pn3m2z0)/(4-2+1) \\ Pn5m3z0 &= ((2*4+1)*z0*Pn4m3z0 - (4+3)*Pn3m3z0)/(4-3+1) \\ Pn5m4z0 &= ((2*z0*Pn5m3z0 - 2*4*Pn4m3z0)/(z0^2-1))^0.5 \\ Pn5m5z0 &= ((z0*Pn5m4z0 - (2*5-1)*Pn4m4z0)/(z0^2-1))^0.5 \end{aligned}$$

=====

$$\begin{aligned} dPn5m1z0 &= 1/(1-z0^2)*((5+1)*Pn4m1z0 - 5*z0*Pn5m1z0) \\ dPn5m2z0 &= 1/(1-z0^2)*((5+2)*Pn4m2z0 - 5*z0*Pn5m2z0) \\ dPn5m3z0 &= 1/(1-z0^2)*((5+3)*Pn4m3z0 - 5*z0*Pn5m3z0) \\ dPn5m4z0 &= 1/(1-z0^2)*((5+4)*Pn4m4z0 - 5*z0*Pn5m4z0) \end{aligned}$$

$$Pn5m1m1z0 = \text{gamma}(5) / \text{gamma}(7) * Pn5m1z0$$

$$Pn5m2m1z0 = 3*2/\text{gamma}(8) * Pn5m2z0$$

$$Pn5m3m1z0 = 2/\text{gamma}(9) * Pn5m3z0$$

$$Pn5m4m1z0 = 1/\text{gamma}(10) * Pn5m4z0$$

$$Pn5m5m1z0 = 1/\text{gamma}(11) * Pn5m5z0$$

=====

$$\begin{aligned} dPn5m1m1z0 &= \text{gamma}(5) / \text{gamma}(7) * dPn5m1z0 \\ dPn5m2m1z0 &= \text{gamma}(4) / \text{gamma}(8) * dPn5m2z0 \\ dPn5m3m1z0 &= \text{gamma}(3) / \text{gamma}(9) * dPn5m3z0 \\ dPn5m4m1z0 &= \text{gamma}(2) / \text{gamma}(10) * dPn5m4z0 \end{aligned}$$

=====

$$\begin{aligned} Pn6m1z0 &= ((2*5+1)*z0*Pn5m1z0 - (5+1)*Pn4m1z0)/(5-1+1) \\ Pn6m2z0 &= ((2*5+1)*z0*Pn5m2z0 - (5+2)*Pn4m2z0)/(5-2+1) \\ Pn6m3z0 &= ((2*5+1)*z0*Pn5m3z0 - (5+3)*Pn4m3z0)/(5-3+1) \\ Pn6m4z0 &= ((2*5+1)*z0*Pn5m4z0 - (5+4)*Pn4m4z0)/(5-4+1) \\ Pn6m5z0 &= ((2*z0*Pn6m4z0 - 2*5*Pn5m4z0)/(z0^2-1))^0.5 \\ Pn6m6z0 &= ((z0*Pn6m5z0 - (2*6-1)*Pn5m5z0)/(z0^2-1))^0.5 \end{aligned}$$

=====

$$\begin{aligned} dPn6m1z0 &= 1/(1-z0^2)*((6+1)*Pn5m1z0 - 6*z0*Pn6m1z0) \\ dPn6m2z0 &= 1/(1-z0^2)*((6+2)*Pn5m2z0 - 6*z0*Pn6m2z0) \\ dPn6m3z0 &= 1/(1-z0^2)*((6+3)*Pn5m3z0 - 6*z0*Pn6m3z0) \\ dPn6m4z0 &= 1/(1-z0^2)*((6+4)*Pn5m4z0 - 6*z0*Pn6m4z0) \\ dPn6m5z0 &= 1/(1-z0^2)*((6+5)*Pn5m5z0 - 6*z0*Pn6m5z0) \end{aligned}$$

```

Pn6m1z0 = gamma(6)/gamma(8)*Pn6m1z0
Pn6m2z0 = gamma(5)/gamma(9)*Pn6m2z0
Pn6m3z0 = 3*2 /gamma(10)*Pn6m3z0
Pn6m4z0 = 2 /gamma(11)*Pn6m4z0
Pn6m5z0 = 1 /gamma(12)*Pn6m5z0
Pn6m6z0 = 1 /gamma(13)*Pn6m6z0

```

```

%===
dPn6m1z0 = gamma(6) /gamma(8)*dPn6m1z0
dPn6m2z0 = gamma(5) /gamma(9)*dPn6m2z0
dPn6m3z0 = gamma(4) /gamma(10)*dPn6m3z0
dPn6m4z0 = gamma(3) /gamma(11)*dPn6m4z0
dPn6m5z0 = gamma(2) /gamma(12)*dPn6m5z0

```

```

%Pn+1,m = [(2*n+1)*z0*Pn,m(z0) - (n+m)*Pn-1,m(z0)]/(n-m+1)
%---n=7
Pn7m1z0 = ((2*6+1)*z0*Pn6m1z0 - (6+1)*Pn5m1z0)/(6-1+1)
Pn7m2z0 = ((2*6+1)*z0*Pn6m2z0 - (6+2)*Pn5m2z0)/(6-2+1)
Pn7m3z0 = ((2*6+1)*z0*Pn6m3z0 - (6+3)*Pn5m3z0)/(6-3+1)
Pn7m4z0 = ((2*6+1)*z0*Pn6m4z0 - (6+4)*Pn5m4z0)/(6-4+1)
Pn7m5z0 = ((2*6+1)*z0*Pn6m5z0 - (6+5)*Pn5m5z0)/(6-5+1)
Pn7m6z0 = ((2*z0*Pn7m5z0 - 2*6*Pn6m5z0)/(z0^2-1))^0.5
Pn7m7z0 = ((z0*Pn7m6z0 - (2*7-1)*Pn6m6z0)/(z0^2-1))^0.5

```

```

%=====
%=====
dPn7m1z0 = 1/(1-z0^2)*((7+1)*Pn6m1z0 - 7*z0*Pn7m1z0)
dPn7m2z0 = 1/(1-z0^2)*((7+2)*Pn6m2z0 - 7*z0*Pn7m2z0)
dPn7m3z0 = 1/(1-z0^2)*((7+3)*Pn6m3z0 - 7*z0*Pn7m3z0)
dPn7m4z0 = 1/(1-z0^2)*((7+4)*Pn6m4z0 - 7*z0*Pn7m4z0)
dPn7m5z0 = 1/(1-z0^2)*((7+5)*Pn6m5z0 - 7*z0*Pn7m5z0)
dPn7m6z0 = 1/(1-z0^2)*((7+6)*Pn6m6z0 - 7*z0*Pn7m6z0)

```

```

Pn7mm1z0 = gamma(7) /gamma(9)*Pn7m1z0
Pn7mm2z0 = gamma(6) /gamma(10)*Pn7m2z0
Pn7mm3z0 = gamma(5) /gamma(11)*Pn7m3z0
Pn7mm4z0 = 3*2 /gamma(12)*Pn7m4z0
Pn7mm5z0 = 2 /gamma(13)*Pn7m5z0
Pn7mm6z0 = 1 /gamma(14)*Pn7m6z0
Pn7mm7z0 = 1 /gamma(15)*Pn7m7z0

```

```

%===
dPn7mm1z0 = gamma(7) /gamma(9)*dPn7m1z0
dPn7mm2z0 = gamma(6) /gamma(10)*dPn7m2z0
dPn7mm3z0 = gamma(5) /gamma(11)*dPn7m3z0
dPn7mm4z0 = gamma(4) /gamma(12)*dPn7m4z0
dPn7mm5z0 = gamma(3) /gamma(13)*dPn7m5z0
dPn7mm6z0 = gamma(2) /gamma(14)*dPn7m6z0

```

```

%---n=8
Pn8m1z0 = ((2*7+1)*z0*Pn7m1z0 - (7+1)*Pn6m1z0)/(7-1+1)
Pn8m2z0 = ((2*7+1)*z0*Pn7m2z0 - (7+2)*Pn6m2z0)/(7-2+1)
Pn8m3z0 = ((2*7+1)*z0*Pn7m3z0 - (7+3)*Pn6m3z0)/(7-3+1)
Pn8m4z0 = ((2*7+1)*z0*Pn7m4z0 - (7+4)*Pn6m4z0)/(7-4+1)
Pn8m5z0 = ((2*7+1)*z0*Pn7m5z0 - (7+5)*Pn6m5z0)/(7-5+1)
Pn8m6z0 = ((2*7+1)*z0*Pn7m6z0 - (7+6)*Pn6m6z0)/(7-6+1)
Pn8m7z0 = ((2*z0*Pn8m6z0 - 2*7*Pn7m6z0)/(z0^2-1))^0.5
Pn8m8z0 = ((z0*Pn8m7z0 - (2*8-1)*Pn7m7z0)/(z0^2-1))^0.5

```

```

%=====
%=====
%=====
dPn8m1z0 = 1/(1-z0^2)*((8+1)*Pn7m1z0 - 8*z0*Pn8m1z0)
dPn8m2z0 = 1/(1-z0^2)*((8+2)*Pn7m2z0 - 8*z0*Pn8m2z0)
dPn8m3z0 = 1/(1-z0^2)*((8+3)*Pn7m3z0 - 8*z0*Pn8m3z0)
dPn8m4z0 = 1/(1-z0^2)*((8+4)*Pn7m4z0 - 8*z0*Pn8m4z0)
dPn8m5z0 = 1/(1-z0^2)*((8+5)*Pn7m5z0 - 8*z0*Pn8m5z0)
dPn8m6z0 = 1/(1-z0^2)*((8+6)*Pn7m6z0 - 8*z0*Pn8m6z0)
dPn8m7z0 = 1/(1-z0^2)*((8+7)*Pn7m7z0 - 8*z0*Pn8m7z0)

```

```

%===
Pn8mm1z0 = gamma(8) /gamma(10)*Pn8m1z0
Pn8mm2z0 = gamma(7) /gamma(11)*Pn8m2z0
Pn8mm3z0 = gamma(6) /gamma(12)*Pn8m3z0
Pn8mm4z0 = gamma(5) /gamma(13)*Pn8m4z0
Pn8mm5z0 = 3*2 /gamma(14)*Pn8m5z0
Pn8mm6z0 = 2 /gamma(15)*Pn8m6z0
Pn8mm7z0 = 1 /gamma(16)*Pn8m7z0
Pn8mm8z0 = 1 /gamma(17)*Pn8m8z0

```

```

%===
%=====
dPn8mm1z0 = gamma(8) /gamma(10)*dPn8m1z0
dPn8mm2z0 = gamma(7) /gamma(11)*dPn8m2z0
dPn8mm3z0 = gamma(6) /gamma(12)*dPn8m3z0
dPn8mm4z0 = gamma(5) /gamma(13)*dPn8m4z0
dPn8mm5z0 = gamma(4) /gamma(14)*dPn8m5z0
dPn8mm6z0 = gamma(3) /gamma(15)*dPn8m6z0
dPn8mm7z0 = gamma(2) /gamma(16)*dPn8m7z0

```

```

%---n=9
Pn9m1z0 = ((2*8+1)*z0*Pn8m1z0 - (8+1)*Pn7m1z0)/(8-1+1)
Pn9m2z0 = ((2*8+1)*z0*Pn8m2z0 - (8+2)*Pn7m2z0)/(8-2+1)
Pn9m3z0 = ((2*8+1)*z0*Pn8m3z0 - (8+3)*Pn7m3z0)/(8-3+1)
Pn9m4z0 = ((2*8+1)*z0*Pn8m4z0 - (8+4)*Pn7m4z0)/(8-4+1)
Pn9m5z0 = ((2*8+1)*z0*Pn8m5z0 - (8+5)*Pn7m5z0)/(8-5+1)
Pn9m6z0 = ((2*8+1)*z0*Pn8m6z0 - (8+6)*Pn7m6z0)/(8-6+1)
Pn9m7z0 = ((2*8+1)*z0*Pn8m7z0 - (8+7)*Pn7m7z0)/(8-7+1)
Pn9m8z0 = ((2*z0*Pn9m7z0 - 2*8*Pn8m7z0)/(z0^2-1))^0.5
Pn9m9z0 = ((z0*Pn9m8z0 - (2*9-1)*Pn8m8z0)/(z0^2-1))^0.5

```

```

%=====
%=====
%=====
dPn9m1z0 = 1/(1-z0^2)*((9+1)*Pn8m1z0 - 9*z0*Pn9m1z0)
dPn9m2z0 = 1/(1-z0^2)*((9+2)*Pn8m2z0 - 9*z0*Pn9m2z0)
dPn9m3z0 = 1/(1-z0^2)*((9+3)*Pn8m3z0 - 9*z0*Pn9m3z0)
dPn9m4z0 = 1/(1-z0^2)*((9+4)*Pn8m4z0 - 9*z0*Pn9m4z0)
dPn9m5z0 = 1/(1-z0^2)*((9+5)*Pn8m5z0 - 9*z0*Pn9m5z0)
dPn9m6z0 = 1/(1-z0^2)*((9+6)*Pn8m6z0 - 9*z0*Pn9m6z0)
dPn9m7z0 = 1/(1-z0^2)*((9+7)*Pn8m7z0 - 9*z0*Pn9m7z0)
dPn9m8z0 = 1/(1-z0^2)*((9+8)*Pn8m8z0 - 9*z0*Pn9m8z0)

```

```

Pn9mm1z0 = gamma(9)/gamma(11)*Pn9m1z0
Pn9mm2z0 = gamma(8)/gamma(12)*Pn9m2z0
Pn9mm3z0 = gamma(7)/gamma(13)*Pn9m3z0
Pn9mm4z0 = gamma(6)/gamma(14)*Pn9m4z0
Pn9mm5z0 = gamma(5)/gamma(15)*Pn9m5z0
Pn9mm6z0 = 3*2 /gamma(16)*Pn9m6z0
Pn9mm7z0 = 2 /gamma(17)*Pn9m7z0
Pn9mm8z0 = 1 /gamma(18)*Pn9m8z0
Pn9mm9z0 = 1 /gamma(19)*Pn9m9z0

```

```

dPn9mm1z0 = gamma(9) /gamma(11)*dPn9m1z0
dPn9mm2z0 = gamma(8) /gamma(12)*dPn9m2z0
dPn9mm3z0 = gamma(7) /gamma(13)*dPn9m3z0
dPn9mm4z0 = gamma(6) /gamma(14)*dPn9m4z0
dPn9mm5z0 = gamma(5) /gamma(15)*dPn9m5z0
dPn9mm6z0 = gamma(4) /gamma(16)*dPn9m6z0
dPn9mm7z0 = gamma(3) /gamma(17)*dPn9m7z0
dPn9mm8z0 = gamma(2) /gamma(18)*dPn9m8z0

```

```

%=====
%---n=10
Pn10m1z0 = ((2*9+1)*z0*Pn9m1z0 - (9+1)*Pn8m1z0)/(9-1+1)
Pn10m2z0 = ((2*9+1)*z0*Pn9m2z0 - (9+2)*Pn8m2z0)/(9-2+1)
Pn10m3z0 = ((2*9+1)*z0*Pn9m3z0 - (9+3)*Pn8m3z0)/(9-3+1)
Pn10m4z0 = ((2*9+1)*z0*Pn9m4z0 - (9+4)*Pn8m4z0)/(9-4+1)
Pn10m5z0 = ((2*9+1)*z0*Pn9m5z0 - (9+5)*Pn8m5z0)/(9-5+1)
Pn10m6z0 = ((2*9+1)*z0*Pn9m6z0 - (9+6)*Pn8m6z0)/(9-6+1)
Pn10m7z0 = ((2*9+1)*z0*Pn9m7z0 - (9+7)*Pn8m7z0)/(9-7+1)
Pn10m8z0 = ((2*9+1)*z0*Pn9m8z0 - (9+8)*Pn8m8z0)/(9-8+1)
Pn10m9z0 = ((2*z0*Pn10m8z0 - 2*9*Pn9m8z0)/(z0^2-1))^0.5
Pn10m10z0 = ((z0*Pn10m9z0 - (2*10-1)*Pn9m9z0)/(z0^2-1))^0.5

```

```

%=====
dPn10m1z0 = 1/(1-z0^2)*((10+1)*Pn9m1z0 - 10*z0*Pn10m1z0)
dPn10m2z0 = 1/(1-z0^2)*((10+2)*Pn9m2z0 - 10*z0*Pn10m2z0)
dPn10m3z0 = 1/(1-z0^2)*((10+3)*Pn9m3z0 - 10*z0*Pn10m3z0)
dPn10m4z0 = 1/(1-z0^2)*((10+4)*Pn9m4z0 - 10*z0*Pn10m4z0)
dPn10m5z0 = 1/(1-z0^2)*((10+5)*Pn9m5z0 - 10*z0*Pn10m5z0)
dPn10m6z0 = 1/(1-z0^2)*((10+6)*Pn9m6z0 - 10*z0*Pn10m6z0)
dPn10m7z0 = 1/(1-z0^2)*((10+7)*Pn9m7z0 - 10*z0*Pn10m7z0)
dPn10m8z0 = 1/(1-z0^2)*((10+8)*Pn9m8z0 - 10*z0*Pn10m8z0)
dPn10m9z0 = 1/(1-z0^2)*((10+9)*Pn9m9z0 - 10*z0*Pn10m9z0)

```

```

Pn10mm1z0 = gamma(10)/gamma(12)*Pn10m1z0
Pn10mm2z0 = gamma(9)/gamma(13)*Pn10m2z0
Pn10mm3z0 = gamma(8)/gamma(14)*Pn10m3z0
Pn10mm4z0 = gamma(7)/gamma(15)*Pn10m4z0
Pn10mm5z0 = gamma(6)/gamma(16)*Pn10m5z0
Pn10mm6z0 = gamma(5)/gamma(17)*Pn10m6z0
Pn10mm7z0 = gamma(4)/gamma(18)*Pn10m7z0
Pn10mm8z0 = 2 /gamma(19)*Pn10m8z0
Pn10mm9z0 = 1 /gamma(20)*Pn10m9z0
Pn10mm10z0 = 1 /gamma(21)*Pn10m10z0

```

```

%===
dPn10mm1z0 = gamma(10)/gamma(12)*dPn10m1z0
dPn10mm2z0 = gamma(9)/gamma(13)*dPn10m2z0
dPn10mm3z0 = gamma(8)/gamma(14)*dPn10m3z0
dPn10mm4z0 = gamma(7)/gamma(15)*dPn10m4z0
dPn10mm5z0 = gamma(6)/gamma(16)*dPn10m5z0
dPn10mm6z0 = gamma(5)/gamma(17)*dPn10m6z0
dPn10mm7z0 = gamma(4)/gamma(18)*dPn10m7z0
dPn10mm8z0 = gamma(3)/gamma(19)*dPn10m8z0
dPn10mm9z0 = gamma(2)/gamma(20)*dPn10m9z0

```

%===

```

%Pn+1,m = [(2*n+1)*z0*Pn,m(z0) - (n+m)*Pn-1,m(z0)]/(n-m+1)
%---n=10
Pn11m1z0 = ((2*10+1)*z0*Pn10m1z0 - (10+1)*Pn9m1z0)/(10-1+1)
Pn11m2z0 = ((2*10+1)*z0*Pn10m2z0 - (10+2)*Pn9m2z0)/(10-2+1)
Pn11m3z0 = ((2*10+1)*z0*Pn10m3z0 - (10+3)*Pn9m3z0)/(10-3+1)
Pn11m4z0 = ((2*10+1)*z0*Pn10m4z0 - (10+4)*Pn9m4z0)/(10-4+1)
Pn11m5z0 = ((2*10+1)*z0*Pn10m5z0 - (10+5)*Pn9m5z0)/(10-5+1)
Pn11m6z0 = ((2*10+1)*z0*Pn10m6z0 - (10+6)*Pn9m6z0)/(10-6+1)
Pn11m7z0 = ((2*10+1)*z0*Pn10m7z0 - (10+7)*Pn9m7z0)/(10-7+1)
Pn11m8z0 = ((2*10+1)*z0*Pn10m8z0 - (10+8)*Pn9m8z0)/(10-8+1)
Pn11m9z0 = ((2*10+1)*z0*Pn10m9z0 - (10+9)*Pn9m9z0)/(10-9+1)
Pn11m10z0 = ((2*z0*Pn11m9z0 - 2*10*Pn10m9z0)/(z0^2-1))^0.5
Pn11m11z0 = ((z0*Pn11m10z0 - (2*11-1)*Pn10m10z0)/(z0^2-1))^0.5

```

```

dPn11m1z0 = 1/(z0^2-1)*((Pn2m1z0 - 2*z0*Pn1m1z0)
dPn2m2z0 = 1/(z0^2-1)*((Pn3m2z0 - 3*z0*Pn2m2z0)
dPn3m3z0 = 1/(z0^2-1)*((Pn4m3z0 - 4*z0*Pn3m3z0)
dPn4m4z0 = 1/(z0^2-1)*((Pn5m4z0 - 5*z0*Pn4m4z0)
dPn5m5z0 = 1/(z0^2-1)*((Pn6m5z0 - 6*z0*Pn5m5z0)
dPn6m6z0 = 1/(z0^2-1)*((Pn7m6z0 - 7*z0*Pn6m6z0)
dPn7m7z0 = 1/(z0^2-1)*((Pn8m7z0 - 8*z0*Pn7m7z0)
dPn8m8z0 = 1/(z0^2-1)*((Pn9m8z0 - 9*z0*Pn8m8z0)
dPn9m9z0 = 1/(z0^2-1)*((Pn10m9z0 - 10*z0*Pn9m9z0)
dPn10m10z0 = 1/(z0^2-1)*((Pn11m10z0 - 11*z0*Pn10m10z0)

```

```

%=====
dPn1m1z0 = gamma(1)/gamma(3)*dPn1m1z0
dPn2m2z0 = gamma(1)/gamma(5)*dPn2m2z0
dPn3m3z0 = gamma(1)/gamma(7)*dPn3m3z0
dPn4m4z0 = gamma(1)/gamma(9)*dPn4m4z0
dPn5m5z0 = gamma(1)/gamma(11)*dPn5m5z0
dPn6m6z0 = gamma(1)/gamma(13)*dPn6m6z0
dPn7m7z0 = gamma(1)/gamma(15)*dPn7m7z0
dPn8m8z0 = gamma(1)/gamma(17)*dPn8m8z0
dPn9m9z0 = gamma(1)/gamma(19)*dPn9m9z0
dPn10m10z0 = gamma(1)/gamma(21)*dPn10m10z0

```

%=====The second type associated Legendre functions

Qn0m0z0 = 1/2*log((z0+1)/(z0-1))

Qn1m0z0 = z0*Qn0m0z0 - 1

```

Qn2m0z0 = [(2+1)*z0*Qn1m0z0 - 1*Qn0m0z0]/2
Qn3m0z0 = [(2+2)*z0*Qn2m0z0 - 2*Qn1m0z0]/3
Qn4m0z0 = [(2+3)*z0*Qn3m0z0 - 3*Qn2m0z0]/4
Qn5m0z0 = [(2+4)*z0*Qn4m0z0 - 4*Qn3m0z0]/5
Qn6m0z0 = [(2+5)*z0*Qn5m0z0 - 5*Qn4m0z0]/6
Qn7m0z0 = [(2+6)*z0*Qn6m0z0 - 6*Qn5m0z0]/7
Qn8m0z0 = [(2+7)*z0*Qn7m0z0 - 7*Qn6m0z0]/8
Qn9m0z0 = [(2+8)*z0*Qn8m0z0 - 8*Qn7m0z0]/9
Qn10m0z0 = [(2+9)*z0*Qn9m0z0 - 9*Qn8m0z0]/10
Qn11m0z0 = [(2+10)*z0*Qn10m0z0 - 10*Qn9m0z0]/11
Qn12m0z0 = [(2+11)*z0*Qn11m0z0 - 11*Qn10m0z0]/12
Qn13m0z0 = [(2+12)*z0*Qn12m0z0 - 12*Qn11m0z0]/13
Qn14m0z0 = [(2+13)*z0*Qn13m0z0 - 13*Qn12m0z0]/14
Qn15m0z0 = [(2+14)*z0*Qn14m0z0 - 14*Qn13m0z0]/15

```

%Qn,m = ((n+m)*z0*Qn-1,m + (z0^2-1)^0.5*Qn-1,m+1)/(n-m) =====

Qn1m1z0 = (z0*Qn1m0z0 - (2+0)*Qn0m0z0)/(z0^2-1)^0.5

Qn2m1z0 = (Qn1m1z0 + 2*(z0^2-1)^0.5*Qn2m0z0)/z0

Qn2m2z0 = (z0*Qn2m1z0 - (2+1)*Qn1m1z0)/(z0^2-1)^0.5

%==

%Qn,m = ((n+m)*z0*Qn-1,m + (z0^2-1)^0.5*Qn-1,m+1)/(n-m) =====

Qn3m1z0 = ((3+1)*z0*Qn2m1z0 + (z0^2-1)^0.5*Qn2m2z0)/(3-1)

Qn3m2z0 = (Qn2m2z0 + 2*(z0^2-1)^0.5*Qn3m1z0)/z0

Qn3m3z0 = (z0*Qn3m2z0 - (2+2)*Qn2m2z0)/(z0^2-1)^0.5

%=====

Qn4m1z0 = ((4+1)*z0*Qn3m1z0 + (z0^2-1)^0.5*Qn3m2z0)/(4-1)

Qn4m2z0 = ((4+2)*z0*Qn3m2z0 + (z0^2-1)^0.5*Qn3m3z0)/(4-2)

Qn4m3z0 = (Qn3m3z0 + 2*(z0^2-1)^0.5*Qn4m2z0)/z0

Qn4m4z0 = (z0*Qn4m3z0 - (2+3)*Qn3m3z0)/(z0^2-1)^0.5

%==

Qn5m1z0 = ((5+1)*z0*Qn4m1z0 + (z0^2-1)^0.5*Qn4m2z0)/(5-1)

Qn5m2z0 = ((5+2)*z0*Qn4m2z0 + (z0^2-1)^0.5*Qn4m3z0)/(5-2)

Qn5m3z0 = ((5+3)*z0*Qn4m3z0 + (z0^2-1)^0.5*Qn4m4z0)/(5-3)

Qn5m4z0 = (Qn4m4z0 + 2*(z0^2-1)^0.5*Qn5m3z0)/z0

Qn5m5z0 = (z0*Qn5m4z0 - (2+4)*Qn4m4z0)/(z0^2-1)^0.5

%=====

Qn6m1z0 = ((6+1)*z0*Qn5m1z0 + (z0^2-1)^0.5*Qn5m2z0)/(6-1)

Qn6m2z0 = ((6+2)*z0*Qn5m2z0 + (z0^2-1)^0.5*Qn5m3z0)/(6-2)

Qn6m3z0 = ((6+3)*z0*Qn5m3z0 + (z0^2-1)^0.5*Qn5m4z0)/(6-3)

Qn6m4z0 = ((6+4)*z0*Qn5m4z0 + (z0^2-1)^0.5*Qn5m5z0)/(6-4)

Qn6m5z0 = (Qn5m5z0 + 2*(z0^2-1)^0.5*Qn6m4z0)/z0

Qn6m6z0 = (z0*Qn6m5z0 - (2+5)*Qn5m5z0)/(z0^2-1)^0.5

%=====

Qn7m1z0 = ((7+1)*z0*Qn6m1z0 + (z0^2-1)^0.5*Qn6m2z0)/(7-1)

Qn7m2z0 = ((7+2)*z0*Qn6m2z0 + (z0^2-1)^0.5*Qn6m3z0)/(7-2)

Qn7m3z0 = ((7+3)*z0*Qn6m3z0 + (z0^2-1)^0.5*Qn6m4z0)/(7-3)

Qn7m4z0 = ((7+4)*z0*Qn6m4z0 + (z0^2-1)^0.5*Qn6m5z0)/(7-4)

Qn7m5z0 = ((7+5)*z0*Qn6m5z0 + (z0^2-1)^0.5*Qn6m6z0)/(7-5)

Qn7m6z0 = (Qn6m6z0 + 2*(z0^2-1)^0.5*Qn7m5z0)/z0

Qn7m7z0 = (z0*Qn7m6z0 - (2+6)*Qn6m6z0)/(z0^2-1)^0.5

%=====

%Qn,m = ((n+m)*z0*Qn-1,m + (z0^2-1)^0.5*Qn-1,m+1)/(n-m) =====

Qn8m1z0 = ((8+1)*z0*Qn7m1z0 + (z0^2-1)^0.5*Qn7m2z0)/(8-1)

Qn8m2z0 = ((8+2)*z0*Qn7m2z0 + (z0^2-1)^0.5*Qn7m3z0)/(8-2)

Qn8m3z0 = ((8+3)*z0*Qn7m3z0 + (z0^2-1)^0.5*Qn7m4z0)/(8-3)

Qn8m4z0 = ((8+4)*z0*Qn7m4z0 + (z0^2-1)^0.5*Qn7m5z0)/(8-4)

Qn8m5z0 = ((8+5)*z0*Qn7m5z0 + (z0^2-1)^0.5*Qn7m6z0)/(8-5)

Qn8m6z0 = ((8+6)*z0*Qn7m6z0 + (z0^2-1)^0.5*Qn7m7z0)/(8-6)

Qn8m7z0 = (Qn7m7z0 + 2*(z0^2-1)^0.5*Qn8m6z0)/z0

Qn8m8z0 = (z0*Qn8m7z0 - (2+7)*Qn7m7z0)/(z0^2-1)^0.5

%=====

Qn9m1z0 = ((9+1)*z0*Qn8m1z0 + (z0^2-1)^0.5*Qn8m2z0)/(9-1)

Qn9m2z0 = ((9+2)*z0*Qn8m2z0 + (z0^2-1)^0.5*Qn8m3z0)/(9-2)

Qn9m3z0 = ((9+3)*z0*Qn8m3z0 + (z0^2-1)^0.5*Qn8m4z0)/(9-3)

Qn9m4z0 = ((9+4)*z0*Qn8m4z0 + (z0^2-1)^0.5*Qn8m5z0)/(9-4)

Qn9m5z0 = ((9+5)*z0*Qn8m5z0 + (z0^2-1)^0.5*Qn8m6z0)/(9-5)

Qn9m6z0 = ((9+6)*z0*Qn8m6z0 + (z0^2-1)^0.5*Qn8m7z0)/(9-6)

Qn9m7z0 = ((9+7)*z0*Qn8m7z0 + (z0^2-1)^0.5*Qn8m8z0)/(9-7)

Qn9m8z0 = (Qn8m8z0 + 2*(z0^2-1)^0.5*Qn9m7z0)/z0

Qn9m9z0 = (z0*Qn9m8z0 - (2+8)*Qn8m8z0)/(z0^2-1)^0.5

%=====

Qn10m1z0 = ((10+1)*z0*Qn9m1z0 + (z0^2-1)^0.5*Qn9m2z0)/(10-1)

Qn10m2z0 = ((10+2)*z0*Qn9m2z0 + (z0^2-1)^0.5*Qn9m3z0)/(10-2)

```

Qn10m3z0 = ((10+3)*z0*Qn9m3z0 + (z0^2-1)^0.5*Qn9m4z0)/(10-3)
Qn10m4z0 = ((10+4)*z0*Qn9m4z0 + (z0^2-1)^0.5*Qn9m5z0)/(10-4)
Qn10m5z0 = ((10+5)*z0*Qn9m5z0 + (z0^2-1)^0.5*Qn9m6z0)/(10-5)
Qn10m6z0 = ((10+6)*z0*Qn9m6z0 + (z0^2-1)^0.5*Qn9m7z0)/(10-6)
Qn10m7z0 = ((10+7)*z0*Qn9m7z0 + (z0^2-1)^0.5*Qn9m8z0)/(10-7)
Qn10m8z0 = ((10+8)*z0*Qn9m8z0 + (z0^2-1)^0.5*Qn9m9z0)/(10-8)
Qn10m9z0 = (Qn9m9z0 + 2*(z0^2-1)^0.5*Qn10m8z0)/z0
Qn10m10z0 = (z0*Qn10m9z0 - (2+9)*Qn9m9z0)/(z0^2-1)^0.5
%=====

```

%=====

Qn11m1z0 = ((11+1)*z0*Qn10m1z0 + (z0^2-1)^0.5*Qn10m2z0)/(11-1)

Qn11m2z0 = ((11+2)*z0*Qn10m2z0 + (z0^2-1)^0.5*Qn10m3z0)/(11-2)

Qn11m3z0 = ((11+3)*z0*Qn10m3z0 + (z0^2-1)^0.5*Qn10m4z0)/(11-3)

Qn11m4z0 = ((11+4)*z0*Qn10m4z0 + (z0^2-1)^0.5*Qn10m5z0)/(11-4)

Qn11m5z0 = ((11+5)*z0*Qn10m5z0 + (z0^2-1)^0.5*Qn10m6z0)/(11-5)

Qn11m6z0 = ((11+6)*z0*Qn10m6z0 + (z0^2-1)^0.5*Qn10m7z0)/(11-6)

Qn11m7z0 = ((11+7)*z0*Qn10m7z0 + (z0^2-1)^0.5*Qn10m8z0)/(11-7)

Qn11m8z0 = ((11+8)*z0*Qn10m8z0 + (z0^2-1)^0.5*Qn10m9z0)/(11-8)

Qn11m9z0 = ((11+9)*z0*Qn10m9z0 + (z0^2-1)^0.5*Qn10m10z0)/(11-9)

Qn11m10z0 = (Qn10m10z0 + 2*(z0^2-1)^0.5*Qn11m9z0)/z0

Qn11m11z0 = (z0*Qn11m10z0 - (2+10)*Qn10m10z0)/(z0^2-1)^0.5

%=====

%=====

Qn12m1z0 = 1/(1-z0^2)*(1+1)*z0*Qn11m1z0 - (1-1+1)*Qn12m1z0

dQn2m1z0 = 1/(1-z0^2)*(2+1)*z0*Qn2m1z0 - (2-1+1)*Qn3m1z0

dQn2m2z0 = 1/(1-z0^2)*(2+1)*z0*Qn2m2z0 - (2-2+1)*Qn3m2z0

%=====

Qn3m1z0 = 1/(1-z0^2)*(3+1)*z0*Qn3m1z0 - (3-1+1)*Qn4m1z0

dQn3m2z0 = 1/(1-z0^2)*(3+1)*z0*Qn3m2z0 - (3-2+1)*Qn4m2z0

dQn3m3z0 = 1/(1-z0^2)*(3+1)*z0*Qn3m3z0 - (3-3+1)*Qn4m3z0

%=====

Qn4m1z0 = 1/(1-z0^2)*(4+1)*z0*Qn4m1z0 - (4-1+1)*Qn5m1z0

dQn4m2z0 = 1/(1-z0^2)*(4+1)*z0*Qn4m2z0 - (4-2+1)*Qn5m2z0

dQn4m3z0 = 1/(1-z0^2)*(4+1)*z0*Qn4m3z0 - (4-3+1)*Qn5m3z0

dQn4m4z0 = 1/(1-z0^2)*(4+1)*z0*Qn4m4z0 - (4-4+1)*Qn5m4z0

%=====

Qn5m1z0 = 1/(1-z0^2)*(5+1)*z0*Qn5m1z0 - (5-1+1)*Qn6m1z0

dQn5m2z0 = 1/(1-z0^2)*(5+1)*z0*Qn5m2z0 - (5-2+1)*Qn6m2z0

dQn5m3z0 = 1/(1-z0^2)*(5+1)*z0*Qn5m3z0 - (5-3+1)*Qn6m3z0

dQn5m4z0 = 1/(1-z0^2)*(5+1)*z0*Qn5m4z0 - (5-4+1)*Qn6m4z0

dQn5m5z0 = 1/(1-z0^2)*(5+1)*z0*Qn5m5z0 - (5-5+1)*Qn6m5z0

%=====

Qn6m1z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m1z0 - (6-1+1)*Qn7m1z0

dQn6m2z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m2z0 - (6-2+1)*Qn7m2z0

dQn6m3z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m3z0 - (6-3+1)*Qn7m3z0

dQn6m4z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m4z0 - (6-4+1)*Qn7m4z0

dQn6m5z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m5z0 - (6-5+1)*Qn7m5z0

dQn6m6z0 = 1/(1-z0^2)*(6+1)*z0*Qn6m6z0 - (6-6+1)*Qn7m6z0

%=====

dQn7m1z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m1z0 - (7-1+1)*Qn8m1z0

dQn7m2z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m2z0 - (7-2+1)*Qn8m2z0

dQn7m3z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m3z0 - (7-3+1)*Qn8m3z0

dQn7m4z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m4z0 - (7-4+1)*Qn8m4z0

dQn7m5z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m5z0 - (7-5+1)*Qn8m5z0

dQn7m6z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m6z0 - (7-6+1)*Qn8m6z0

dQn7m7z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m7z0 - (7-7+1)*Qn8m7z0

dQn7m8z0 = 1/(1-z0^2)*(7+1)*z0*Qn7m8z0 - (7-8+1)*Qn8m8z0

%=====

Qn8m1z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m1z0 - (8-1+1)*Qn9m1z0

uQn8m2z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m2z0 - (8-2+1)*Qn9m2z0

dQn8m3z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m3z0 - (8-3+1)*Qn9m3z0

dQn8m4z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m4z0 - (8-4+1)*Qn9m4z0

dQn8m5z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m5z0 - (8-5+1)*Qn9m5z0

dQn8m6z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m6z0 - (8-6+1)*Qn9m6z0

dQn8m7z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m7z0 - (8-7+1)*Qn9m7z0

dQn8m8z0 = 1/(1-z0^2)*(8+1)*z0*Qn8m8z0 - (8-8+1)*Qn9m8z0

%=====

dQn9m1z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m1z0 - (9-1+1)*Qn10m1z0

dQn9m2z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m2z0 - (9-2+1)*Qn10m2z0

dQn9m3z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m3z0 - (9-3+1)*Qn10m3z0

dQn9m4z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m4z0 - (9-4+1)*Qn10m4z0

dQn9m5z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m5z0 - (9-5+1)*Qn10m5z0

dQn9m6z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m6z0 - (9-6+1)*Qn10m6z0

dQn9m7z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m7z0 - (9-7+1)*Qn10m7z0

dQn9m8z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m8z0 - (9-8+1)*Qn10m8z0

dQn9m9z0 = 1/(1-z0^2)*(9+1)*z0*Qn9m9z0 - (9-9+1)*Qn10m9z0

%=====

%=====

Qn1m1z0 = 1/2*Qn1m1z0

Qn2m1z0 = 1/(2+1)*2*Qn2m1z0

Qn2m2z0 = 1/(2+2)*3*2*Qn2m2z0

%=====

dQn1m1z0 = 1/2*dQn1m1z0

dQn2m1z0 = 1/(2+1)*2*dQn2m1z0

dQn2m2z0 = 1/(2+2)*3*2*dQn2m2z0

%=====

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%=====

```

Qn3mm1z0 = 2/gamma(5)*Qn3m1z0
Qn3mm2z0 = 1/gamma(6)*Qn3m2z0
Qn3mm3z0 = 1/gamma(7)*Qn3m3z0
%====
dQn3mm1z0 = 2/gamma(5)*dQn3m1z0
dQn3mm2z0 = 1/gamma(6)*dQn3m2z0
dQn3mm3z0 = 1/gamma(7)*dQn3m3z0

%====
Qn4mm1z0 = 3*2/gamma(6)*Qn4m1z0
Qn4mm2z0 = 2 /gamma(7)*Qn4m2z0
Qn4mm3z0 = 1 /gamma(8)*Qn4m3z0
Qn4mm4z0 = 1 /gamma(9)*Qn4m4z0
%====
dQn4mm1z0 = 3*2/gamma(6)*dQn4m1z0
dQn4mm2z0 = 2 /gamma(7)*dQn4m2z0
dQn4mm3z0 = 1 /gamma(8)*dQn4m3z0
dQn4mm4z0 = 1 /gamma(9)*dQn4m4z0

%====
Qn5mm1z0 = gamma(5)/gamma(7)*Qn5m1z0
Qn5mm2z0 = 3*2 /gamma(8)*Qn5m2z0
Qn5mm3z0 = 2 /gamma(9)*Qn5m3z0
Qn5mm4z0 = 1 /gamma(10)*Qn5m4z0
Qn5mm5z0 = 1 /gamma(11)*Qn5m5z0
%====
dQn5mm1z0 = gamma(5)/gamma(7)*dQn5m1z0
dQn5mm2z0 = 3*2 /gamma(8)*dQn5m2z0
dQn5mm3z0 = 2 /gamma(9)*dQn5m3z0
dQn5mm4z0 = 1 /gamma(10)*dQn5m4z0
dQn5mm5z0 = 1 /gamma(11)*dQn5m5z0

%====
Qn6mm1z0 = gamma(6)/gamma(8)*Qn6m1z0
Qn6mm2z0 = gamma(5)/gamma(9)*Qn6m2z0
Qn6mm3z0 = 3*2 /gamma(10)*Qn6m3z0
Qn6mm4z0 = 2 /gamma(11)*Qn6m4z0
Qn6mm5z0 = 1 /gamma(12)*Qn6m5z0
Qn6mm6z0 = 1 /gamma(13)*Qn6m6z0
%====
dQn6mm1z0 = gamma(6)/gamma(8)*dQn6m1z0
dQn6mm2z0 = gamma(5)/gamma(9)*dQn6m2z0
dQn6mm3z0 = 3*2 /gamma(10)*dQn6m3z0
dQn6mm4z0 = 2 /gamma(11)*dQn6m4z0
dQn6mm5z0 = 1 /gamma(12)*dQn6m5z0
dQn6mm6z0 = 1 /gamma(13)*dQn6m6z0

%====
Qn7mm1z0 = gamma(7) /gamma(9)*Qn7m1z0
Qn7mm2z0 = gamma(6) /gamma(10)*Qn7m2z0
Qn7mm3z0 = gamma(5) /gamma(11)*Qn7m3z0
Qn7mm4z0 = 3*2 /gamma(12)*Qn7m4z0
Qn7mm5z0 = 2 /gamma(13)*Qn7m5z0
Qn7mm6z0 = 1 /gamma(14)*Qn7m6z0
Qn7mm7z0 = 1 /gamma(15)*Qn7m7z0
%====
dQn7mm1z0 = gamma(7) /gamma(9)*dQn7m1z0
dQn7mm2z0 = gamma(6) /gamma(10)*dQn7m2z0
dQn7mm3z0 = gamma(5) /gamma(11)*dQn7m3z0
dQn7mm4z0 = 3*2 /gamma(12)*dQn7m4z0
dQn7mm5z0 = 2 /gamma(13)*dQn7m5z0
dQn7mm6z0 = 1 /gamma(14)*dQn7m6z0
dQn7mm7z0 = 1 /gamma(15)*dQn7m7z0

%====
Qn8mm1z0 = gamma(8) /gamma(10)*Qn8m1z0
Qn8mm2z0 = gamma(7) /gamma(11)*Qn8m2z0
Qn8mm3z0 = gamma(6) /gamma(12)*Qn8m3z0
Qn8mm4z0 = gamma(5) /gamma(13)*Qn8m4z0
Qn8mm5z0 = 3*2 /gamma(14)*Qn8m5z0
Qn8mm6z0 = 2 /gamma(15)*Qn8m6z0
Qn8mm7z0 = 1 /gamma(16)*Qn8m7z0
Qn8mm8z0 = 1 /gamma(17)*Qn8m8z0
%====
dQn8mm1z0 = gamma(8) /gamma(10)*dQn8m1z0
dQn8mm2z0 = gamma(7) /gamma(11)*dQn8m2z0
dQn8mm3z0 = gamma(6) /gamma(12)*dQn8m3z0
dQn8mm4z0 = gamma(5) /gamma(13)*dQn8m4z0
dQn8mm5z0 = 3*2 /gamma(14)*dQn8m5z0
dQn8mm6z0 = 2 /gamma(15)*dQn8m6z0
dQn8mm7z0 = 1 /gamma(16)*dQn8m7z0
dQn8mm8z0 = 1 /gamma(17)*dQn8m8z0

%====
Qn9mm1z0 = gamma(9)/gamma(11)*Qn9m1z0
Qn9mm2z0 = gamma(8)/gamma(12)*Qn9m2z0
Qn9mm3z0 = gamma(7)/gamma(13)*Qn9m3z0
Qn9mm4z0 = gamma(6) /gamma(14)*Qn9m4z0
Qn9mm5z0 = gamma(5) /gamma(15)*Qn9m5z0
Qn9mm6z0 = 3*2 /gamma(16)*Qn9m6z0
Qn9mm7z0 = 2 /gamma(17)*Qn9m7z0
Qn9mm8z0 = 1 /gamma(18)*Qn9m8z0
Qn9mm9z0 = 1 /gamma(19)*Qn9m9z0
%====
dQn9mm1z0 = gamma(9)/gamma(11)*dQn9m1z0
dQn9mm2z0 = gamma(8)/gamma(12)*dQn9m2z0
dQn9mm3z0 = gamma(7)/gamma(13)*dQn9m3z0
dQn9mm4z0 = gamma(6) /gamma(14)*dQn9m4z0
dQn9mm5z0 = gamma(5) /gamma(15)*dQn9m5z0
dQn9mm6z0 = 3*2 /gamma(16)*dQn9m6z0
dQn9mm7z0 = 2 /gamma(17)*dQn9m7z0
dQn9mm8z0 = 1 /gamma(18)*dQn9m8z0
dQn9mm9z0 = 1 /gamma(19)*dQn9m9z0

%====

```

```

Qn10mm1z0 = gamma(10)/gamma(12)*Qn10m1z0
Qn10mm2z0 = gamma(9) /gamma(13)*Qn10m2z0
Qn10mm3z0 = gamma(8) /gamma(14)*Qn10m3z0
Qn10mm4z0 = gamma(7) /gamma(15)*Qn10m4z0
Qn10mm5z0 = gamma(6) /gamma(16)*Qn10m5z0
Qn10mm6z0 = gamma(5) /gamma(17)*Qn10m6z0
Qn10mm7z0 = gamma(4) /gamma(18)*Qn10m7z0
Qn10mm8z0 = 2 /gamma(19)*Qn10m8z0
Qn10mm9z0 = 1 /gamma(20)*Qn10m9z0
Qn10mm10z0 = 1 /gamma(21)*Qn10m10z0

```

```

dQn10mm1z0 = gamma(10)/gamma(12)*dQn10m1z0
dQn10mm2z0 = gamma(9) /gamma(13)*dQn10m2z0
dQn10mm3z0 = gamma(8) /gamma(14)*dQn10m3z0
dQn10mm4z0 = gamma(7) /gamma(15)*dQn10m4z0
dQn10mm5z0 = gamma(6) /gamma(16)*dQn10m5z0
dQn10mm6z0 = gamma(5) /gamma(17)*dQn10m6z0
dQn10mm7z0 = gamma(4) /gamma(18)*dQn10m7z0
dQn10mm8z0 = 2 /gamma(19)*dQn10m8z0
dQn10mm9z0 = 1 /gamma(20)*dQn10m9z0
dQn10mm10z0 = 1 /gamma(21)*dQn10m10z0

```

```

%=====
Qn11m1z0 = ( ( 11+1)*z0*Qn10m1z0 + ( z0^2 -1 )^0.5*Qn10m2z0 )/( 11 -1)
Qn11m2z0 = ( ( 11+2)*z0*Qn10m2z0 + ( z0^2 -1 )^0.5*Qn10m3z0 )/( 11 -2)
Qn11m3z0 = ( ( 11+3)*z0*Qn10m3z0 + ( z0^2 -1 )^0.5*Qn10m4z0 )/( 11 -3)
Qn11m4z0 = ( ( 11+4)*z0*Qn10m4z0 + ( z0^2 -1 )^0.5*Qn10m5z0 )/( 11 -4)
Qn11m5z0 = ( ( 11+5)*z0*Qn10m5z0 + ( z0^2 -1 )^0.5*Qn10m6z0 )/( 11 -5)
Qn11m6z0 = ( ( 11+6)*z0*Qn10m6z0 + ( z0^2 -1 )^0.5*Qn10m7z0 )/( 11 -6)
Qn11m7z0 = ( ( 11+7)*z0*Qn10m7z0 + ( z0^2 -1 )^0.5*Qn10m8z0 )/( 11 -7)
Qn11m8z0 = ( ( 11+8)*z0*Qn10m8z0 + ( z0^2 -1 )^0.5*Qn10m9z0 )/( 11 -8)
Qn11m9z0 = ( ( 11+9)*z0*Qn10m9z0 + ( z0^2 -1 )^0.5*Qn10m10z0 )/( 11 -9)
Qn11m10z0 = ( Qn10m10z0 + 2*( z0^2 -1 )^0.5*Qn11m9z0 )/z0
Qn11m11z0 = ( z0*Qn11m10z0 - ( 2*10+1)*Qn10m10z0 )/( z0^2 -1 )^0.5

```

```

%Qn11mm1z0 = gamma(11)/gamma(13)*Qn11m1z0

```

```

%=====
%Qn,m = ( ( n+m)*z0*Qn-1,m + ( z0^2 -1 )^0.5*Qn-1,m+1 )/( n-m) =====
Qn12m1z0 = ( ( 12+1)*z0*Qn11m1z0 + ( z0^2 -1 )^0.5*Qn11m2z0 )/( 12 -1)
Qn12m2z0 = ( ( 12+2)*z0*Qn11m2z0 + ( z0^2 -1 )^0.5*Qn11m3z0 )/( 12 -2)
Qn12m3z0 = ( ( 12+3)*z0*Qn11m3z0 + ( z0^2 -1 )^0.5*Qn11m4z0 )/( 12 -3)
Qn12m4z0 = ( ( 12+4)*z0*Qn11m4z0 + ( z0^2 -1 )^0.5*Qn11m5z0 )/( 12 -4)
Qn12m5z0 = ( ( 12+5)*z0*Qn11m5z0 + ( z0^2 -1 )^0.5*Qn11m6z0 )/( 12 -5)
Qn12m6z0 = ( ( 12+6)*z0*Qn11m6z0 + ( z0^2 -1 )^0.5*Qn11m7z0 )/( 12 -6)
Qn12m7z0 = ( ( 12+7)*z0*Qn11m7z0 + ( z0^2 -1 )^0.5*Qn11m8z0 )/( 12 -7)
Qn12m8z0 = ( ( 12+8)*z0*Qn11m8z0 + ( z0^2 -1 )^0.5*Qn11m9z0 )/( 12 -8)
Qn12m9z0 = ( ( 12+9)*z0*Qn11m9z0 + ( z0^2 -1 )^0.5*Qn11m10z0 )/( 12 -9)
Qn12m10z0 = ( ( 12+10)*z0*Qn11m10z0 + ( z0^2 -1 )^0.5*Qn11m11z0 )/( 12 -10)
Qn12m11z0 = ( Qn11m11z0 + 2*( z0^2 -1 )^0.5*Qn12m10z0 )/z0
Qn12m12z0 = ( z0*Qn12m11z0 - ( 2*11+1)*Qn11m11z0 )/( z0^2 -1 )^0.5

```

```

%---
%Qn12mm1z0 = gamma(12)/gamma(14)*Qn12m1z0

```

```

%Qn+1,n+1=(z0*Qn+1,n - ( 2*n+1)*Qn,n )/( z0^2 -1 )^0.5
Qn1m1z0 = ( z0*Qn1m0z0 - ( 2*0+1)*Qn0m0z0 )/( z0^2 -1 )^0.5
Qn2m2z0 = ( z0*Qn2m1z0 - ( 2*1+1)*Qn1m1z0 )/( z0^2 -1 )^0.5
Qn3m3z0 = ( z0*Qn3m2z0 - ( 2*2+1)*Qn2m2z0 )/( z0^2 -1 )^0.5
Qn4m4z0 = ( z0*Qn4m3z0 - ( 2*3+1)*Qn3m3z0 )/( z0^2 -1 )^0.5
Qn5m5z0 = ( z0*Qn5m4z0 - ( 2*4+1)*Qn4m4z0 )/( z0^2 -1 )^0.5
Qn6m6z0 = ( z0*Qn6m5z0 - ( 2*5+1)*Qn5m5z0 )/( z0^2 -1 )^0.5
Qn7m7z0 = ( z0*Qn7m6z0 - ( 2*6+1)*Qn6m6z0 )/( z0^2 -1 )^0.5
Qn8m8z0 = ( z0*Qn8m7z0 - ( 2*7+1)*Qn7m7z0 )/( z0^2 -1 )^0.5
Qn9m9z0 = ( z0*Qn9m8z0 - ( 2*8+1)*Qn8m8z0 )/( z0^2 -1 )^0.5
Qn10m10z0 = ( z0*Qn10m9z0 - ( 2*9+1)*Qn9m9z0 )/( z0^2 -1 )^0.5
Qn11m11z0 = ( z0*Qn11m10z0 - ( 2*10+1)*Qn10m10z0 )/( z0^2 -1 )^0.5
Qn12m12z0 = ( z0*Qn12m11z0 - ( 2*11+1)*Qn11m11z0 )/( z0^2 -1 )^0.5

```

```

%Qn,n-1 = ( Qn-1,n-1 + 2 ( z0^2 -1 )^0.5*Qn,n-2 )/z0
Qn2m1z0 = ( Qn1m1z0 + 2*( z0^2 -1 )^0.5*Qn2m0z0 )/z0
Qn3m2z0 = ( Qn2m2z0 + 2*( z0^2 -1 )^0.5*Qn3m1z0 )/z0
Qn4m3z0 = ( Qn3m3z0 + 2*( z0^2 -1 )^0.5*Qn4m2z0 )/z0
Qn5m4z0 = ( Qn4m4z0 + 2*( z0^2 -1 )^0.5*Qn5m3z0 )/z0
Qn6m5z0 = ( Qn5m5z0 + 2*( z0^2 -1 )^0.5*Qn6m4z0 )/z0
Qn7m6z0 = ( Qn6m6z0 + 2*( z0^2 -1 )^0.5*Qn7m5z0 )/z0
Qn8m7z0 = ( Qn7m7z0 + 2*( z0^2 -1 )^0.5*Qn8m6z0 )/z0
Qn9m8z0 = ( Qn8m8z0 + 2*( z0^2 -1 )^0.5*Qn9m7z0 )/z0
Qn10m9z0 = ( Qn9m9z0 + 2*( z0^2 -1 )^0.5*Qn10m8z0 )/z0
Qn11m10z0 = ( Qn10m10z0 + 2*( z0^2 -1 )^0.5*Qn11m9z0 )/z0
Qn12m11z0 = ( Qn11m11z0 + 2*( z0^2 -1 )^0.5*Qn12m10z0 )/z0

```

```

%===

```

```

%=====functions for the inner boundary values Lamda0 * zi ===
differentiated type

```

```

Pn0m0zi = 1
Pn1m0zi = zi

```

```

Pn2m0zi = [ ( 2*1+1)*zi*Pn1m0zi - 1*Pn0m0zi ]/2
Pn3m0zi = [ ( 2*2+1)*zi*Pn2m0zi - 2*Pn1m0zi ]/3
Pn4m0zi = [ ( 2*3+1)*zi*Pn3m0zi - 3*Pn2m0zi ]/4
Pn5m0zi = [ ( 2*4+1)*zi*Pn4m0zi - 4*Pn3m0zi ]/5
Pn6m0zi = [ ( 2*5+1)*zi*Pn5m0zi - 5*Pn4m0zi ]/6
Pn7m0zi = [ ( 2*6+1)*zi*Pn6m0zi - 6*Pn5m0zi ]/7
Pn8m0zi = [ ( 2*7+1)*zi*Pn7m0zi - 7*Pn6m0zi ]/8
Pn9m0zi = [ ( 2*8+1)*zi*Pn8m0zi - 8*Pn7m0zi ]/9
Pn10m0zi = [ ( 2*9+1)*zi*Pn9m0zi - 9*Pn8m0zi ]/10
Pn11m0zi = [ ( 2*10+1)*zi*Pn10m0zi - 10*Pn9m0zi ]/11
Pn12m0zi = [ ( 2*11+1)*zi*Pn11m0zi - 11*Pn10m0zi ]/12

```