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Introduction of Method for Micro Dynamics of Non Spherical Bio Molecular Particles.

--Method of Wu-Wang-Yi--

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A method was introduced for analyzing the creeping motions of non spherical particle in viscous fluid that has been proposed by Wu, Wang-yi (1984). To treat the arbitrary prolate axi-symmetrical biological particle, linear approximation of the line continuous distribution method of singularities was firstly proposed by Sampson, R,A (Phil, Trans vol 182, p 449-. 1081) where the spherical singularities distribute continuously over the nose and the tail of the axial body. Refined approach utilized the Legendre and Gegenbauer polynomials of n-order. This report explained the proposed mathematical approach in detail with verifying their mathematical formula. The present method will be available for analyzing the micro dynamics of bio molecular particles.

Non spherical particle, Prolate, Creeping motion, Linear distribution of discrete singularities.

形状を考慮した生体分子の挙動解析

-- 楕円体分子周囲の力学場解析 --

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非球状粒子の粘性流体中の遅い運動を解析する方法をWang-yi (1984) らの提唱した方法にそって解説した。 生体分子は通常非球状であり、楕円体とみなされる場合がある。そのような形状に対して特異点を連続的に、直線状に分布させて近似するやり方が Sampson, R,A (Phil, Trans vol 182, p 449-. 1081) らによってすでに提唱されており、Wang-yi (1984) らはそれを非球状粒子表面の摩擦力解析に応用している。本研報では、非球状生体分子にほん方法を応用すべく、その数学的過程および、彼等が発表した変形式を解説し、実践応用しやすいようにつとめた。粒子の速度、圧力は n 次の Legendre関数および Gegenbauer 関数による線形級数の形式で表現できた。数値計算で、解は収束することが確認されており、彼等の研究は非球状生体分子の挙動解析に有用である。

非球状粒子. 特異点. 摩擦力. Legendre関数. Gegenbauer 関数. 級数.速度、圧力

1. Introduction.

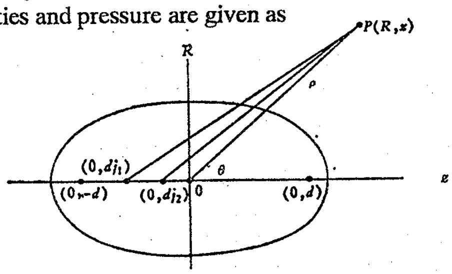
Biological molecules have lots of configurations and usually do not take spherical form. Hence,for analyzing biological particle interactions including fluid dynamical, diffusional and electrical ones, non spherical approach must be more adequate. Present paper introduces a method for linear approximation of the line continuous distribution of singularities in the creeping flow motion.

2. Mathematical formulation.

A uniform flow (velocity U) passes prolate body. The cylindrical coordinate R, z , θ is given in Fig 1. Z coincided with axisymmetrical axis of the body. The normalized stream function of the Stokes flow satisfies

$$D^2 (D^2 \psi) =0 \quad ---(1)$$

A segment (-d,d) inside the body is taken on the axis of symmetry. The centers of the curvature are A, B. The flow velocities and pressure are given as



$$v_Z = 1 + \sum_{n=2}^{\infty}\left[\int_{-d}^{d} C_n(\xi) F_n^{(1)}(R,Z-\xi)\,d\xi + \int_{-d}^{d} D_n(\xi) F_n^{(2)}(R,Z-\xi)\,d\xi\right]$$

$$v_R = \sum_{n=2}^{\infty}\left[\int_{-d}^{d} C_n(\xi) F_n^{(5)}(R,Z-\xi)\,d\xi + \int_{-d}^{d} D_n(\xi) F_n^{(6)}(R,Z-\xi)\,d\xi\right]$$

$$\psi = \frac{R^2}{2} + \sum_{n=2}^{\infty}\left[\int_{-d}^{d} C_n(\xi) F_n^{(3)}(R,Z-\xi)\,d\xi + \int_{-d}^{d} D_n(\xi) F_n^{(4)}(R,Z-\xi)\,d\xi\right]$$

$$p = p_{\infty} + \sum_{n=2}^{\infty}\int_{-d}^{d} D_n(\xi)\frac{4n-6}{n} F_{n-1}^{(1)}(R,Z-\xi)\,d\xi$$

where the functions are normalized by the characteristic length of the body L,. Cn(ξ) and Dn(ξ) are density distribution function. $F_n^{(k)}$ is given by $\zeta = Z(R^2 + Z^2)^{-1/2}$

$$F_n^{(1)}(R,Z) = (R^2+Z^2)^{-\frac{n+1}{2}} P_n(\zeta),$$

$$F_n^{(2)}(R,Z) = (R^2+Z^2)^{-\frac{n-1}{2}}[P_n(\zeta)+2J_n(\zeta)]$$

$$F_n^{(3)}(R,Z) = (R^2+Z^2)^{-\frac{n-1}{2}} J_n(\zeta),$$

$$F_n^{(4)}(R,Z) = (R^2+Z^2)^{-\frac{n-3}{2}} J_n(\zeta)$$

$$F_n^{(5)}(R,Z) = (n+1)(R^2+Z^2)^{-\frac{n}{2}}\frac{1}{R}J_{n+1}(\zeta)$$

$$F_n^{(6)}(R,Z) = (n+1)(R^2+Z^2)^{-\frac{n-2}{2}}\frac{1}{R}J_{n+1}(\zeta) - 2Z(R^2+Z^2)^{-\frac{n-1}{2}}\frac{1}{R}J_n(\zeta)$$

Divide AB into (M-1) segments. The density distribution on each segment is replaced by linear function($c_{nj2}\, d_{j1} - c_{nj1}\, d_{j2}$)/($d_{j2} - d_{j1}$) + ($c_{nj2} - c_{nj1}$) ξ / ($d_{j2} - d_{j1}$)

where d_{j1}, d_{j2}, c_{nj1}, c_{nj2} are coordinates of the end points and the corresponding values for density singularities.

$$v_Z = 1 + \sum_{n=2}^{N}\sum_{j=1}^{M-1}[S_n^{(1)}(R,Z)C_{nj} + T_n^{(1)}(R,Z)C_{n,j+1} + S_n^{(2)}(R,Z)D_{nj} + T_n^{(2)}(R,Z)D_{n,j+1}]$$

$$v_R = \sum_{n=2}^{N}\sum_{j=1}^{M-1}[S_n^{(5)}(R,Z)C_{nj} + T_n^{(5)}(R,Z)C_{n,j+1} + S_n^{(6)}(R,Z)D_{nj} + T_n^{(6)}(R,Z)D_{n,j+1}]$$

$$\psi = \frac{R^2}{2} + \sum_{n=2}^{N}\sum_{j=1}^{M-1}[S_n^{(3)}(R,Z)C_{nj} + T_n^{(3)}(R,Z)C_{n,j+1} + S_n^{(4)}(R,Z)D_{nj} + T_n^{(4)}(R,Z)D_{n,j+1}]$$

$$p = p_{\infty} + \sum_{n=2}^{N}\sum_{j=1}^{M-1}\frac{4n-6}{n}[S_{n-1}^{(1)}(R,Z)D_{nj} + T_{n-1}^{(1)}(R,Z)D_{n,j+1}]$$

$$S_n^{(k)}(R,Z) = G_n^{(k)}(R,Z-d_{j1}) + [H_n^{(k)}(R,Z-d_{j2}) - H_n^{(k)}(R,Z-d_{j1})]/(d_{j2}-d_{j1})$$

$$T_n^{(k)}(R,Z) = -G_n^{(k)}(R,Z-d_{j2}) - [H_n^{(k)}(R,Z-d_{j2}) - H_n^{(k)}(R,Z-d_{j1})]/(d_{j2}-d_{j1})$$

$$G_n^{(k)}(R,Z) = \int F_n^{(k)}(R,Z)\,dZ, \qquad L_n^{(k)}(R,Z) = \int Z F_n^{(k)}(R,Z)\,dZ$$

$$H_n^{(k)}(R,Z) = ZG_n^{(k)}(R,Z) - L_n^{(k)}(R,Z)$$

$$H_n^{(k)}(R,Z) = \frac{1}{n(n-1)} F_{n-2}^{(k)}(R,Z) \qquad (k=1,3,5)$$

$$H_n^{(2)}(R,Z) = \begin{cases} \dfrac{1}{n(n-1)} F_{n-2}^{(2)}(R,Z) + \dfrac{4}{n(n-1)(n-2)} ZF_{n-3}^{(1)}(R,Z) \\ \quad + \dfrac{6}{n(n-1)(n-2)(n-3)} F_{n-4}^{(1)}(R,Z) & (n\geqq 4) \\ \dfrac{Z}{2\sqrt{R^2+Z^2}} - \sinh^{-1}\dfrac{Z}{R} & (n=3) \\ -\dfrac{3}{2}\sqrt{R^2+Z^2} + Z\sinh^{-1}\dfrac{Z}{R} & (n=2) \end{cases}$$

$$H_n^{(4)}(R,Z) = \begin{cases} \dfrac{1}{n(n-1)} F_{n-2}^{(4)}(R,Z) + \dfrac{4}{n(n-1)(n-2)} ZF_{n-3}^{(3)}(R,Z) \\ \quad + \dfrac{6}{n(n-1)(n-2)(n-3)} F_{n-4}^{(3)}(R,Z) & (n\geqq 4) \\ -\dfrac{R^2}{2}\sinh^{-1}\dfrac{Z}{R} & (n=3) \\ \dfrac{R^2 Z}{2}\sinh^{-1}\dfrac{Z}{R} - \dfrac{R^2}{2}(R^2+Z^2)^{\frac{1}{2}} & (n=2) \end{cases}$$

$$H_n^{(6)}(R,Z) = \begin{cases} \dfrac{1}{n(n-1)} F_{n-2}^{(6)}(R,Z) + \dfrac{4}{n(n-1)(n-2)} ZF_{n-3}^{(6)} \\ \quad + \dfrac{2}{n(n-1)(n-2)(n-3)} F_{n-4}^{(6)}(R,Z) & (n\geqq 4) \\ \dfrac{R}{2}\dfrac{1}{\sqrt{R^2+Z^2}} & (n=3) \\ -\dfrac{R}{2}\sinh^{-1}\dfrac{Z}{R} & (n=2) \end{cases}$$

$$G_n^{(k)}(R,Z) = -\frac{1}{n} F_{n-1}^{(k)}(R,Z) \qquad (k=1,3,5)$$

$$G_n^{(2)}(R,Z) = \begin{cases} -\dfrac{1}{n} F_{n-1}^{(2)}(R,Z) - \dfrac{2}{n(n-1)} ZF_{n-2}^{(1)}(R,Z) - \dfrac{2}{n(n-1)(n-2)} F_{n-3}^{(1)}(R,Z) & (n\neq 2) \\ -\dfrac{1}{2}\dfrac{Z}{(R^2+Z^2)^{\frac{1}{2}}} + \sinh^{-1}\dfrac{Z}{R} & (n=2) \end{cases}$$

$$G_n^{(4)}(R,Z) = \begin{cases} -\dfrac{1}{n} F_{n-1}^{(4)}(R,Z) - \dfrac{2}{n(n-1)} ZF_{n-2}^{(3)}(R,Z) - \dfrac{2}{n(n-1)(n-2)} F_{n-3}^{(3)}(R,Z) & (n\neq 2) \\ \dfrac{R^2}{2}\sinh^{-1}\dfrac{Z}{R} & (n=2) \end{cases}$$

$$G_n^{(6)}(R,Z) = -\frac{1}{n} F_{n-1}^{(6)}(R,Z) - \frac{2}{n(n-1)} ZF_{n-2}^{(5)}(R,Z)$$

$$H_n^{(k)} = \frac{F_{n-2}^{(k)}}{n(n-1)} \qquad (2.2)$$

$$H_n^{(1)} = \frac{F_{n-2}^{(1)}}{n(n-1)} = \frac{1}{n(n-1)}\left(z^2+R^2\right)^{-\frac{(n-1)}{2}} P_{n-2}\left(\frac{z}{\sqrt{z^2+R^2}}\right)$$

differentiate z $n(n-1)$ *tor out*

$$\frac{\partial}{\partial z}\left(F_{n-2}^{(1)}\right) = P_{n-2}\left(\frac{z}{\sqrt{z^2+R^2}}\right)\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-1)}{2}}}{\partial z} + \left(z^2+R^2\right)^{-\frac{(n-1)}{2}}\frac{\partial}{\partial z}\left(P_{n-2}\right)$$

$$\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-1)}{2}}}{\partial z} = -\frac{(n-1)}{2}\cdot\left(z^2+R^2\right)^{-\frac{(n-1)}{2}-1}\cdot 2z$$

$$= -(n-1)\cdot z\cdot\left(z^2+R^2\right)^{-\frac{(n+1)}{2}}$$

$$\frac{\partial}{\partial z}P_{n-2}\left(\frac{z}{\sqrt{z^2+R^2}}\right) = \frac{\partial P_{n-2}(q)}{\partial q}\frac{\partial q}{\partial z} \qquad \left(q = \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{z^2+R^2}}\right) = \frac{\left(z^2+R^2\right)^{1/2} - \frac{1}{2}\left(z^2+R^2\right)^{-1/2}\cdot 2z^2}{z^2+R^2}$$

$$= \left(z^2+R^2\right)^{-1/2} - z^2\cdot\left(z^2+R^2\right)^{-3/2}$$

putting $r = \left(z^2+R^2\right)$

$$\frac{\partial}{\partial z}F_{n-2}^{(1)} = P_{n-2}(q)(-(n-1))z\cdot r^{-\frac{(n+1)}{2}} + r^{-\frac{(n-1)}{2}}P'_{n-2}\left(r^{-1/2} - z^2 r^{-3/2}\right)$$

$$= r^{-n/2}\left[-(n-1)\cdot z\cdot r^{-1/2}\cdot P_{n-2}(q) + \left(1-z^2r^{-1}\right)\cdot P'_{n-2}\right]$$

$$= r^{-n/2}\left[-(n-1)\cdot q\cdot P_{n-2}(q) + \left(1-q^2\right)\cdot P'_{n-2}(q)\right]$$

Change $n-2 \rightarrow n-1$

Legendre

$$P'_{n+1} - q\cdot P'_n = (n+1)\cdot P_n$$

$$n \rightarrow n-2$$

$$P'_{n-1} - q\cdot P'_{n-2} = (n-1)\cdot P_{n-2}$$

$$q\cdot P'_{n-1} - q^2\cdot P'_{n-2} = (n-1)\cdot q\cdot P_{n-2}$$

Hence

$$r^{-n/2}\left[-q\cdot P'_{n-1} + P'_{n-2}\right]$$

Moreover

$$P'_{n-1} - q\cdot P'_n = -n\cdot P_n \qquad put \quad n \rightarrow n-1$$

$$P'_{n-2} - q\cdot P'_{n-1} = -(n-1)\cdot P_{n-1}$$

Thus

$$r^{-n/2}\cdot(-(n-1))\cdot P_{n-1}(q)$$

Thus

$$\frac{\partial}{\partial z}F_{n-2}^{(1)} = -r^{-n/2}\cdot(n-1)\cdot P_{n-1} = -(n-1)F_{n-1}^{(1)}$$

Formual Beltrami $\rightarrow$ $\left(1-q^2\right)P'_n = (n+1)\left(qP_n - P_{n+1}\right)$

we have $\left(1-q^2\right)P'_{n-2} = (n-1)\left(qP_{n-2} - P_{n-1}\right)$

Thus

$$-(n-1)\cdot q\cdot P_{n-2} + \left(1-q^2\right)P'_{n-2} = -(n-1)qP_{n-2} + (n-1)qP_{n-2} - (n-1)P_{n-1}$$

$$H_n^{(3)} = \frac{F_{n-2}^{(3)}}{n(n-1)} = \frac{1}{n(n-1)}\left(z^2+R^2\right)^{-\frac{(n-3)}{2}}C_{n-2}\left(\frac{z}{\sqrt{z^2+R^2}}\right)$$

z differentiate $n(n-1)$ by *factor out*

$$\frac{\partial}{\partial z}F_{n-2}^{(3)} = C_{n-2}(q)\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-3)}{2}}}{\partial z} + \left(z^2+R^2\right)^{-\frac{(n-3)}{2}}\frac{\partial C_{n-2}(q)}{\partial z}$$

$$\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-3)}{2}}}{\partial z} = -\frac{(n-3)}{2}\left(z^2+R^2\right)^{-\frac{(n-3)}{2}-1}\cdot 2\cdot z$$

$$= -(n-3)\cdot z\cdot r^{-n/2+1/2} \qquad = -(n-3)\cdot z\cdot r^{-(n-1)/2}$$

$$= -(n-3)\cdot z\cdot r^{-\frac{(n-1)}{2}}\cdot C_{n-2} + r^{-\frac{(n-3)}{2}}\cdot C'_{n-2}\left(r^{-1/2} - z^2r^{-3/2}\right)$$

$$= r^{-\frac{(n-2)}{2}}\left[-(n-3)\cdot z\cdot r^{-1/2}\cdot C_{n-2} + C'_{n-2}\left(1-z^2r^{-1}\right)\right]$$

$$= r^{-\frac{(n-2)}{2}}\left[-(n-3)\cdot q\cdot C_{n-2} + \left(1-q^2\right)C'_{n-2}\right]$$

1.*First Convertto Legendre* function

Since $C_n = \left(P_{n-2} - P_n\right)/(2n-1)$ *we, have*

$$C_{n-2} = \left(P_{n-4} - P_{n-2}\right)/(2n-5)$$

$$C'_{n-2} = -P_{n-3}$$

$$[\ ] = \frac{-(n-3)\cdot q\cdot\left(P_{n-4} - P_{n-2}\right)}{(2n-5)} + \left(1-q^2\right)\cdot\left(-P_{n-3}\right)$$

For, the first, term, apply *Beltrami formula*

$$\left(1-q^2\right)\cdot P'_n = \frac{n(n+1)}{(2n+1)}\cdot\left(P_{n-1} - P_{n+1}\right)$$

$$\left(1-q^2\right)\cdot P'_{n-3} = \frac{(n-3)(n-2)\left(P_{n-4} - P_{n-2}\right)}{(2n-5)}$$

Then,

$$= -(n-3)\cdot q\cdot\frac{\left(1-q^2\right)\cdot P'_{n-3}}{(n-3)(n-2)} - \left(1-q^2\right)\cdot P_{n-3}$$

$$= -\left(1-q^2\right)\left[\frac{q\cdot P'_{n-3}}{(n-2)} + P_{n-3}\right]$$

$$= -\frac{\left(1-q^2\right)}{(n-2)}\left[q\cdot P'_{n-3} + (n-2)P_{n-3}\right]$$

From, therecurrent, form, ofLegendre

$$P'_{n+1} - q\cdot P'_n = (n+1)\cdot P_n$$

$$P'_{n-2} - q\cdot P'_{n-3} = (n-2)\cdot P_{n-3}$$

$$= \frac{-\left(1-q^2\right)\cdot P'_{n-2}}{(n-2)}$$

while

$$\left(1-q^2\right)P'_{n-1} = n(n-1)C_n$$

$$\left(1-q^2\right)P'_{n-2} = (n-1)(n-2)C_{n-1}$$

$$[\ ] = -\frac{(n-1)(n-2)C_{n-1}}{(n-2)}$$

$$= -(n-1)\cdot C_{n-1}$$

Therefore

$$\frac{\partial}{\partial z}F_{n-2}^{(3)} = -r^{-\frac{(n-2)}{2}}(n-1)\cdot C_{n-1}$$

$$= -(n-1)\cdot F_{n-1}^{(3)}$$

2. The, second, method, by using, the recurrent, formula of, the Genebauer, function $C_n^{-1/2}$

$$(1-n)\cdot q\cdot C_n + \left(1-q^2\right)C'_n = -(n+1)C_{n+1}$$

changing, $n \rightarrow n-2$ *, substitute*

$$(1-n+2)\cdot q\cdot C_{n-2} + \left(1-q^2\right)C'_{n-2} = -(n-1)C_{n-1}$$

Directly, $[\ ] = -(n-1)\cdot C_{n-1}$

$$H_n^{(4)} = \frac{F_{n-2}^{(4)}}{n(n-1)} + \frac{4\cdot z\cdot F_{n-3}^{(3)}}{n(n-1)(n-2)} + \frac{6\cdot F_{n-4}^{(3)}}{n(n-1)(n-2)(n-3)}$$

Differentiate, about, z

1]. $F_{n-2}^{(4)}$

$$\frac{\partial}{\partial z}F_{n-2}^{(4)} = \frac{\partial}{\partial z}\left[\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}C_{n-2}(q)\right]$$

$$= C_{n-2}(q)\frac{\partial}{\partial z}\left(z^2+R^2\right)^{-\frac{(n-5)}{2}} + \left(z^2+R^2\right)^{-\frac{(n-5)}{2}}\frac{\partial C_{n-2}(q)}{\partial z}$$

$$\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}}{\partial z} = -\frac{(n-5)}{2}\left(z^2+R^2\right)^{-\frac{(n-5)}{2}-1}\cdot 2z$$

$$= -(n-5)zr^{-n/2+3/2} = -(n-5)zr^{-(n-3)/2}$$

$$= -(n-5)zr^{-(n-3)/2}\cdot C_{n-2} + r^{-(n-5)/2}C'_{n-2}\left(r^{-1/2} - z^2r^{-3/2}\right)$$

$$= r^{-(n-4)/2}\left[-(n-5)zr^{-1/2}\cdot C_{n-2} + C'_{n-2}\left(1-z^2r^{-1}\right)\right]$$

$$= r^{-(n-4)/2}\left[-(n-5)q\cdot C_{n-2} + C'_{n-2}\left(1-q^2\right)\right]$$

1. The, first, method ,

For, the, first, time, convert, to Legendre, function,

Legendre $P_m(q)$

$$C_{n-2} = \left(P_{n-4} - P_{n-2}\right)/(2n-5)$$

$$C'_{n-2} = -P_{n-3}(q)$$

$$= r^{-(n-4)/2}\left[-(n-5)q\cdot\frac{\left(P_{n-4}-P_{n-2}\right)}{(2n-5)} - \left(1-q^2\right)P_{n-3}\right]$$

to,[ ] apply, the Beltrami formula

$$\left(1-q^2\right)P'_n = \frac{n(n+1)}{(2n+1)}\left(P_{n-1}-P_{n+1}\right)$$

$$\left(1-q^2\right)P'_{n-3} = \frac{(n-3)(n-2)}{(2n-5)}\left(P_{n-4}-P_{n-2}\right)$$

$$[\ ] = \left[-(n-5)\cdot q\frac{\left(1-q^2\right)P'_{n-3}}{(n-3)(n-2)} - \left(1-q^2\right)P_{n-3}\right]$$

$$= -\left(1-q^2\right)\left[\frac{(n-5)\cdot q\cdot P'_{n-3}}{(n-3)(n-2)} + P_{n-3}\right]$$

By, the, Legendre, recurrent, formula $P_{n-3} \to P_{n-2}$

$$P'_{n+1} = qP'_n + (n+1)P_n$$

$$q\cdot P'_{n-3} = P'_{n-2} - (n-2)P_{n-3}$$

$$= -\left(1-q^2\right)\left[\frac{(n-5)\cdot\left(P'_{n-2}-(n-2)P_{n-3}\right)}{(n-3)(n-2)} + P_{n-3}\right]$$

$$= -\left(1-q^2\right)\left[\frac{(n-5)}{(n-3)(n-2)}P'_{n-2} - \frac{(n-5)}{(n-3)}P_{n-3} + P_{n-3}\right]$$

$$= -\left(1-q^2\right)\left[\frac{(n-5)}{(n-3)(n-2)}P'_{n-2} + \frac{2}{(n-3)}P_{n-3}\right]$$

In, addition,

$$\left(1-q^2\right)P'_n = (n+1)\cdot n\cdot C_{n+1}$$

Hene, we, have

$$\left(1-q^2\right)P'_{n-2} = (n-2)\cdot(n-1)\cdot C_{n-1}$$

$$= -\left[\frac{(n-5)(n-2)(n-1)}{(n-3)(n-2)}C_n + \frac{2\cdot\left(1-q^2\right)}{(n-3)}P_{n-3}\right]$$

$$= -\left[\frac{(n-5)(n-1)}{(n-3)}C_{n-1} + \frac{2\cdot\left(1-q^2\right)}{(n-3)}P_{n-3}\right] \to C_n \to C_{n-1}$$

Hence,

$$\frac{\partial}{\partial z}F^{(4)}_{n-2} = -r^{-\frac{(n-4)}{2}}\left[\frac{(n-5)(n-1)}{(n-3)}C_{n-1} + 2\cdot\frac{\left(1-q^2\right)}{(n-3)}P_{n-3}\right]$$

2.The, second, method, using the recurrent formu $C_n(q)$

$$-(n-1)\cdot q\cdot C_n + \left(1-q^2\right)C'_n = -(n+1)C_{n+1}$$

$$-(n-3)\cdot q\cdot C_{n-2} + \left(1-q^2\right)C'_{n-2} = -(n-1)C_{n-1}$$

Therefore

$$-(n-5)qC_{n-2} + \left(1-q^2\right)C'_{n-2} = -(n-1)C_{n-1} + 2qC_{n-2}$$

Thus,

$$\frac{\partial}{\partial z}F^{(4)}_{n-2} = -r^{-\frac{(n-4)}{2}}\left[-(n-1)C_{n-1} + 2qC_{n-2}\right]$$

II] $z\cdot F^{(3)}_{n-3}$

$$\frac{\partial}{\partial z}\left(z\cdot F^{(3)}_{n-3}(z)\right) = F^{(3)}_{n-3}(z) + z\cdot\frac{\partial F^{(3)}_{n-3}}{\partial z}$$

$$\frac{\partial F^{(3)}_{n-3}(z)}{\partial z} = \frac{\partial}{\partial z}\left[\left(z^2+R^2\right)^{-\frac{(n-4)}{2}}C_{n-3}(q)\right]$$

$$= C_{n-3}(q)\frac{\partial}{\partial z}\left(z^2+R^2\right)^{-\frac{(n-4)}{2}} + \left(z^2+R^2\right)^{-\frac{(n-4)}{2}}\frac{\partial}{\partial z}C_{n-3}(q)$$

$$\frac{\partial\left(z^2+R^2\right)^{-\frac{(n-4)}{2}}}{\partial z} = -\frac{(n-4)}{2}\left(z^2+R^2\right)^{-\frac{(n-4)}{2}-1}2z$$

$$= -(n-4)\cdot z\cdot\left(z^2+R^2\right)^{-n/2+1}$$

$$= -(n-4)\cdot z\cdot\left(z^2+R^2\right)^{-(n-2)/2}$$

$$= -(n-4)\cdot z\cdot r^{-(n-2)/2}\cdot C_{n-3} + r^{-\frac{(n-4)}{2}}C'_{n-3}\cdot\left(r^{-1/2} - z^2r^{-3/2}\right)$$

$$= r^{-\frac{(n-3)}{2}}\cdot\left[-(n-4)\cdot z\cdot r^{-1/2}\cdot C_{n-3} + C'_{n-3}\cdot\left(1-z^2r^{-1}\right)\right]$$

$$= r^{-\frac{(n-3)}{2}}\cdot\left[-(n-4)\cdot q\cdot C_{n-3} + C'_{n-3}\left(1-q^2\right)\right]$$

1].The.first.method, by, converting C_m to *Legendre*

$$C_{n-3} = \left(P_{n-5} - P_{n-3}\right)/(2n-7)$$

$$C'_{n-3} = -P_{n-4}$$

$$= r^{-\frac{(n-3)}{2}}\cdot\left[-(n-4)\cdot q\cdot\frac{\left(P_{n-5}-P_{n-3}\right)}{(2n-7)} - \left(1-q^2\right)\cdot P_{n-4}\right]$$

From, the, Beltrami formula

$$\left(1-q^2\right)P'_n = \frac{n(n+1)}{(2n+1)}\left(P_{n-1}-P_{n+1}\right)$$

$$\left(1-q^2\right)P'_{n-4} = (n-4)(n-3)\frac{\left(P_{n-5}-P_{n-3}\right)}{(2n-7)}$$

Thus

$$= r^{-\frac{(n-3)}{2}}\left[-(n-4)\cdot q\cdot\frac{\left(1-q^2\right)\cdot P'_{n-4}}{(n-4)(n-3)} - \left(1-q^2\right)\cdot P_{n-4}\right]$$

$$= -r^{-\frac{(n-3)}{2}}\cdot\left(1-q^2\right)\left[-\frac{q\cdot P'_{n-4}}{(n-3)} - P_{n-4}\right]$$

Elevate, theorder, of P_{n-4}

$$P'_{n+1} = qP'_n + (n+1)P_n$$

$$P'_{n-3} = qP'_{n-4} + (n-3)P_{n-4}$$

$$= -r^{-\frac{(n-3)}{2}}\cdot\left(1-q^2\right)\left[\frac{P'_{n-3} - (n-3)P_{n-4}}{(n-3)} + P_{n-4}\right]$$

$$= -r^{-\frac{(n-3)}{2}}\frac{\left(1-q^2\right)\cdot P'_{n-3}}{(n-3)}$$

while

$$= r^{-\frac{(n-3)}{2}}\cdot(n-2)C_{n-2} \qquad \left(1-q^2\right)\cdot P'_{n-3} = (n-2)(n-3)C_{n-2}$$

2].The.second.method. for. solution.

Using, the, recurrent. formula. of C_m

$$-(n-1)qC_n + \left(1-q^2\right)C'_n = -(n+1)C_{n+1}$$

Puttig. $n \to n-3$

$$-(n-3-1)qC_{n-3} + \left(1-q^2\right)C'_{n-3} = -(n-2)C_{n-2}$$

From, this, we have, directly

$$\frac{\partial}{\partial z}F^{(3)}_{n-3} = r^{-\frac{(n-3)}{2}}\left[-(n-2)\cdot C_{n-2}\right]$$

Therefore

$$\frac{\partial}{\partial z}\left(z\cdot F_{n-3}^{(3)}\right)=F_{n-3}^{(3)}+z\left(-r^{-\frac{(n-3)}{2}}\cdot(n-2)\cdot C_{n-2}\right)$$

$$=r^{-\frac{(n-4)}{2}}C_{n-3}-z\cdot r^{-\frac{(n-3)}{2}}\cdot(n-2)\cdot C_{n-2}$$

$$=r^{-\frac{(n-4)}{2}}\left[C_{n-3}-z\cdot r^{-1/2}\cdot(n-2)\cdot C_{n-2}\right]$$

$$=r^{-\frac{(n-4)}{2}}\left[C_{n-3}-q\cdot(n-2)\cdot C_{n-2}\right]$$

III] $F_{n-4}^{(3)}$

$$\frac{\partial}{\partial z}F_{n-4}^{(3)}=\frac{\partial}{\partial z}\left[\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}C_{n-4}(q)\right]$$

$$=C_{n-4}(q)\frac{\partial}{\partial z}\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}+\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}\frac{\partial C_{n-4}(q)}{\partial z}$$

$$\frac{\partial}{\partial z}\left(z^2+R^2\right)^{-\frac{(n-5)}{2}}=-\frac{(n-5)}{2}\cdot\left(z^2+R^2\right)^{-\frac{(n-5)}{2}-1}2z$$

$$=-(n-5)z\cdot\left(z^2+R^2\right)^{-n/2+3/2}$$

$$=(n-5)z\cdot r^{-(n-3)/2}C_{n-4}+r^{-\frac{(n-5)}{2}}C_{n-4}'\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)$$

$$=r^{-(n-4)/2}\left[(n+5)z\cdot r^{-1/2}\cdot C_{n-4}+C_{n-4}'\cdot\left(1-z^2r^{-1}\right)\right]$$

$$=r^{-(n-4)/2}\left[(n+5)\cdot q\cdot C_{n-4}+\left(1-q^2\right)C_{n-4}'\right]$$

1].*The.first.method.for.solution,by Beltrami formula*

$$\left(1-q^2\right)P_{n-5}'=\frac{(n-4)(n-5)}{(2n-9)}\left(P_{n-6}-P_{n-4}\right)$$

Convert,the,Legendre,function,to, C_m

$$C_{n-4}=\left(P_{n-6}-P_{n-4}\right)/(2n-9)$$

$$C_{n-4}'=-P_{n-5}$$

$$=-r^{-(n-4)/2}\left[(n-5)\cdot q\cdot\frac{\left(P_{n-6}-P_{n-4}\right)}{(2n-9)}+\left(1-q^2\right)P_{n-5}\right]$$

To,above,applying, Beltrami formula

$$\frac{\left(P_{n-6}-P_{n-4}\right)}{(2n-9)}=\left(1-q^2\right)\frac{P_{n-5}'}{(n-4)(n-5)}$$

$$=r^{-(n-4)/2}\cdot\left[-(n-5)q\frac{\left(1-q^2\right)P_{n-5}'}{(n-4)(n-5)}-\left(1-q^2\right)P_{n-5}\right]$$

$$=\frac{-r^{-(n-4)/2}\cdot\left(1-q^2\right)}{(n-4)}\left[q\cdot P_{n-5}'+(n-4)P_{n-5}\right]$$

From,the,recurrent,formula

$$P_{n+1}'=qP_n'+(n+1)P_n$$

$$P_{n-4}'=qP_{n-5}'+(n-4)P_{n-5}$$

$$=\frac{-r^{-(n-4)/2}\cdot\left(1-q^2\right)P_{n-4}'}{(n-4)}$$

where

$$\left(1-q^2\right)P_{n-4}'=(n-3)(n-4)C_{n-3}$$

$$=-r^{-(n-4)/2}\cdot(n-3)\cdot C_{n-3}$$

2].The second. method.by, using,*the recurrent* C_m

$$-(n-1)q\cdot C_n+\left(1-q^2\right)C_n'=-(n+1)C_{n+1}$$

Putting $n\rightarrow n-4$

$$-(n-5)q\cdot C_{n-4}+\left(1-q^2\right)C_{n-4}'=-(n-3)C_{n-3}$$

Associate, these

$$\frac{1}{n(n-1)}\frac{\partial}{\partial z}\left[F_{n-2}^{(4)}+(4)\frac{z\cdot F_{n-3}^{(3)}}{(n-2)}+\frac{(6)F_{n-4}^{(3)}}{(n-2)(n-3)}\right]$$

$$=\frac{r^{-\frac{(n-4)}{2}}}{n(n-1)}\cdot\left[-\frac{(n-5)(n-1)}{(n-3)}C_{n-1}-\frac{2\left(1-q^2\right)}{(n-3)}P_{n-3}\right.$$

$$+\frac{(4)2}{(n-2)}\left(C_{n-3}-q(n-2)C_{n-2}\right)$$

$$\left.+\frac{(6)2}{(n-2)(n-3)}\left(-(n-3)\cdot C_{n-3}\right)\right]$$

$$=\frac{r^{-\frac{(n-4)}{2}}}{n(n-1)}\left[-\frac{(n-5)(n-1)}{(n-3)}C_{n-1}-\frac{2\cdot\left(1-q^2\right)}{(n-3)}P_{n-3}-2qC_{n-2}\right]$$

Associate, the, second, and, the third terms

$$\frac{-2}{(n-3)}\left[\left(1-q^2\right)P_{n-3}+q\cdot(n-3)\cdot C_{n-2}\right]$$

$$C_{n-2}=\left(P_{n-4}-P_{n-2}\right)/(2n-5)$$

From,the Beltrami

$$\left(1-q^2\right)P_n'=\frac{n(n+1)}{(2n+1)}\left(P_{n-1}-P_{n+1}\right)$$

$$\left(1-q^2\right)P_{n-3}'=\frac{(n-3)(n-2)\left(P_{n-4}-P_{n-2}\right)}{(2n-5)}$$

Thus,

$$C_{n-2}=\frac{\left(1-q^2\right)P_{n-3}'}{(n-3)(n-2)}$$

Hence

$$-\frac{2}{(n-3)}\left[\left(1-q^2\right)P_{n-3}+q\cdot(n-3)\frac{\left(1-q^2\right)P_{n-3}'}{(n-3)(n-2)}\right]$$

$$=-\frac{2\left(1-q^2\right)}{(n-3)(n-2)}\left[(n-2)P_{n-3}+q\cdot P_{n-3}'\right]$$

From,the,recurrent,formula

$$P_{n+1}'=qP_n'+(n+1)P_n$$

$$P_{n-2}'=qP_{n-3}'+(n-2)P_{n-3}$$

$$=-\frac{2\left(1-q^2\right)}{(n-3)(n-2)}P_{n-2}'(q)$$

because,$(1-q\text{^}2)P(n-2)'=(n-1)(n-2)C(n-1)$

$=-2(n-1)C(n-1)/(n-3)$

Sum, with, the, first, term,

$$\left[-\frac{(n-5)(n-1)}{(n-3)}C_{n-1}-\frac{2(n-1)}{(n-3)}C_{n-1}\right]$$

$$=-(n-1)\cdot C_{n-1}$$

Hence,

$$\frac{\partial}{\partial z}H_n^{(4)}=\frac{-(n-1)\cdot C_{n-1}\cdot r^{-(n-4)/2}}{n(n-1)}$$

$$=-\frac{r^{-(n-4)/2}}{n}C_{n-1}=-\frac{1}{n}F_{n-1}^{(4)}$$

$$H_n^{(5)}=\frac{1}{n(n-1)}F_{n-2}^{(5)}$$

$$=\frac{1}{n(n-1)}(n-2+1)\left(z^2+R^2\right)^{-\frac{(n-2)}{2}}C_{n-1}(q)$$

Differentiate, about z

$$\frac{\partial}{\partial z}\left\{\left(z^2+R^2\right)^{-\frac{(n-2)}{2}}C_{n-1}(q)\right\}$$

$$= C_{n-1}(q)\frac{\partial}{\partial z}(z^2+R^2)^{-\frac{(n-2)}{2}} + (z^2+R^2)^{-\frac{(n-2)}{2}}\frac{\partial C_{n-1}(q)}{\partial z}$$

$$\frac{\partial}{\partial z}(z^2+R^2)^{-\frac{(n-2)}{2}} = -\frac{(n-2)}{2}(z^2+R^2)^{-\frac{(n-2)}{2}-1}2z$$

$$= -(n-2)zr^{-n/2}$$

$$\frac{\partial C_{n-1}(q)}{\partial z} = \left(r^{-1/2} - z^2 r^{-3/2}\right)C'_{n-1}$$

$$= -(n-2)zr^{-n/2}\cdot C_{n-1} + r^{-\frac{(n-2)}{2}}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)C'_{n-1}$$

$$= r^{-(n-1)/2}\left[-(n-2)zr^{-1/2}\cdot C_{n-1} + (1-z^2r^{-1})C'_{n-1}\right]$$

$$= r^{-(n-1)/2}\left[-(n-2)q\cdot C_{n-1} + (1-q^2)C'_{n-1}\right]$$

1].The.first. method. for. solution

Using,the.recurrent.form,of $C_m^{-1/2}$

$$-(n-1)qC_n + (1-q^2)C'_n = -(n+1)C_{n+1}$$

$$n \to n-1$$

$$-(n-2)qC_{n-1} + (1-q^2)C'_{n-1} = -nC_n$$

$$= -nr^{-(n-1)/2}C_n$$

2].*The.second.method.by.converting the* $C_m^{-1/2}$ to *Legendre*

$$C_{n-1} = (P_{n-3}-P_{n-1})/(2n-3)$$

$$C'_{n-1} = -P_{n-2}$$

$$[\ ] = -(n-2)\cdot q\cdot\frac{(P_{n-3}-P_{n-1})}{(2n-3)} + (1-q^2)(-P_{n-2})$$

Apply, the, first, term, on *Beltrami formula*

$$(1-q^2)P'_n = \frac{n(n+1)}{(2n+1)}(P_{n-1}-P_{n+1})$$

$$(1-q^2)P'_{n-2} = \frac{(n-2)(n-1)}{(2n-3)}(P_{n-3}-P_{n-1})$$

$$= -(n-2)\cdot q\cdot\frac{(1-q^2)P'_{n-2}}{(n-2)(n-1)} - (1-q^2)P_{n-2}$$

$$= -(1-q^2)\left[\frac{q\cdot P'_{n-2}}{(n-1)} + P_{n-2}\right] = -\frac{(1-q^2)}{(n-1)}\left[q\cdot P'_{n-2} + (n-1)P_{n-2}\right]$$

Re*current, forluma of .Legendre*

$$P'_{n+1} - qP'_n = (n+1)P_n$$

$$P'_{n-1} - qP'_{n-2} = (n-1)P_{n-2}$$

$$= -\frac{(1-q^2)}{(n-1)}\cdot P'_{n-1}$$

$$(1-q^2)\cdot P'_{n-1} = n(n-1)C_n$$

$= -n\cdot C_n$.*Thus*

$$\frac{\partial}{\partial z}H_n^{(5)} = r^{-(n-1)/2}\frac{(-n)C_n}{R}\frac{1}{n} = -\frac{F_{n-1}^{(5)}}{n}$$

$$H_n^{(6)}(R,z) = \frac{F_{n-2}^{(6)}}{n(n-1)} + 4\frac{zF_{n-3}^{(5)}}{n(n-1)(n-2)} + 2\frac{F_{n-4}^{(5)}}{n(n-1)(n-2)(n-3)}$$

Differentiate, about z

1]. $$\frac{\partial F_{n-2}^{(6)}(R,z)}{\partial z} = \frac{\partial}{\partial z}\left[(n-1)(z^2+R^2)^{-(n-4)/2}\cdot\frac{C_{n-1}(q)}{R} - 2\cdot z\cdot(z^2+R^2)^{-(n-3)/2}\cdot C_{n-2}(q)\Big/R\right]$$

a) $$\frac{\partial}{\partial z}\left[(z^2+R^2)^{-(n-4)/2}\cdot C_{n-1}(q)\right]$$

$$= \frac{C_{n-1}(q)\partial}{\partial z}\left[(z^2+R^2)^{-(n-4)/2}\right] + (z^2+R^2)^{-(n-4)/2}\frac{\partial C_{n-1}(q)}{\partial z}$$

here

$$\frac{\partial}{\partial z}\left[(z^2+R^2)^{-(n-4)/2}\right] = -(n-4)/2\,(z^2+R^2)^{-n/2+2-1}\cdot 2\cdot z$$

$$= -(n-4)\cdot z\cdot(z^2+R^2)^{-n/2+1}$$

$$\frac{\partial C_{n-1}(q)}{\partial z} = \frac{\partial C_{n-1}(q)}{\partial q}\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right) = \frac{(z^2+R^2)^{-1/2} - 1/2(z^2+R^2)^{-1/2}\cdot 2z^2}{(z^2+R^2)}$$

$$= (z^2+R^2)^{-1/2} - z^2\cdot(z^2+R^2)^{-3/2}$$

$$= r^{-1/2} - z^2\cdot r^{-3/2}$$

$$= C_{n-1}(q)\left[-(n-4)\cdot z\cdot r^{-n/2+1}\right] + r^{-n/2+2}\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)C'_{n-1}(q)$$

$$= r^{-n/2+3/2}\left[-(n-4)\cdot C_{n-1}(q)\cdot z\cdot r^{-1/2} + r^{1/2}\left(r^{-1/2}-z^2\cdot r^{-3/2}\right)C'_{n-1}(q)\right]$$

$$= r^{-(n-3)/2}\cdot\left[-(n-4)\cdot C_{n-1}(q)\cdot z\cdot r^{-1/2} + (1-z^2\cdot r^{-1})C'_{n-1}(q)\right]$$

while $\dfrac{z}{r^{1/2}} = \dfrac{z}{(z^2+R^2)^{1/2}} = q$

$$= r^{-(n-3)/2}\cdot\left[-(n-4)\cdot q\cdot C_{n-1}(q) + (1-q^2)C'_{n-1}(q)\right]$$

Apply, the, recurrent, formula .of $C_n^{-1/2}(q)$ to, the, second, term

$$(1-q^2)C_n^{-1/2}(q)' = -(n+1)C_{n+1}^{-1/2}(q) + (n-1)qC_n^{-1/2}$$

putting $n \to n-1$

$$(1-q^2)C_{n-1}^{-1/2}(q)' = -n\cdot C_n^{-1/2}(q) + (n-2)\cdot q\cdot C_{n-1}^{-1/2}$$

$$= r^{-(n-3)/2}\cdot\left[-(n-4)\cdot q\cdot C_{n-1} - nC_n + (n-2)\cdot q\cdot C_{n-1}\right]$$

$$= r^{-(n-3)/2}\cdot\left[-nC_n + 2qC_{n-1}\right]$$

b) $$\frac{\partial}{\partial z}\left[z\cdot(z^2+R^2)^{-(n-3)/2}\cdot C_{n-2}(q)\right]$$

$$= (z^2+R^2)^{-(n-3)/2}\cdot C_{n-2}(q) + z\cdot\frac{\partial}{\partial z}\left[(z^2+R^2)^{-(n-3)/2}\cdot C_{n-2}(q)\right]$$

$$= r^{-(n-3)/2}\cdot C_{n-2}(q) + z\cdot\left[C_{n-2}(q)\cdot\frac{\partial(z^2+R^2)^{-(n-3)/2}}{\partial z} + (z^2+R^2)^{-(n-3)/2}\frac{\partial C_{n-2}}{\partial z}\right]$$

$$\frac{\partial(z^2+R^2)^{-(n-3)/2}}{\partial z} = -\frac{(n-3)}{2}(z^2+R^2)^{-n/2+3/2-1}\cdot 2z$$

$$= -(n-3)\cdot z\cdot(z^2+R^2)^{-n/2+1/2}$$

$$\frac{\partial C_{n-2}(q)}{\partial z} = \frac{\partial C_{n-2}}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right) = C'_{n-2}(q)\cdot\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right)$$

$$= C'_{n-2}(q)\cdot\left(r^{-1/2} - z^2r^{-3/2}\right)$$

Therefore

$$= r^{-(n-3)/2}C_{n-2}(q) + z\left[-(n-3)\cdot z\cdot r^{-n/2+1/2}\cdot C_{n-2} + r^{-n/2+3/2}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)C'_{n-2}\right]$$

$$= r^{-(n-3)/2}C_{n-2}(q) + z\cdot r^{-n/2+1}\left[-(n-3)\cdot z\cdot r^{-1/2}\cdot C_{n-2} + r^{1/2}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)C'_{n-2}\right]$$

$$= r^{-(n-3)/2}C_{n-2}(q) + z\cdot r^{-(n-2)/2}\left[-(n-3)\cdot z\cdot r^{-1/2}\cdot C_{n-2} + (1-z^2r^{-1})C'_{n-2}\right]$$

$$q = \frac{z}{(z^2+R^2)^{1/2}} = \frac{z}{r^{1/2}}$$

$$= r^{-(n-3)/2}C_{n-2}(q) + z\cdot r^{-(n-2)/2}\left[-(n-3)\cdot q\cdot C_{n-2} + (1-q^2)C'_{n-2}\right]$$

To, the , third, term, apply, the, recurrent

$$\left(1-q^2\right)C_n'(q) = -(n+1)C_{n+1}(q) + (n-1)qC_n(q)$$

changing $n \to (n-2)$

$$\left(1-q^2\right)C_{n-2}'(q) = -(n-1)C_{n-1}(q) + (n-3)qC_{n-2}(q)$$

$$= r^{-(n-3)/2}C_{n-2}(q) + z\cdot r^{-(n-2)/2}\left[-(n-3)\cdot q\cdot C_{n-2} - (n-1)C_{n-1}(q) + (n-3)qC_{n-2}(q)\right]$$

$$= r^{-(n-3)/2}C_{n-2}(q) \quad -z\cdot(n-1)\cdot r^{-(n-2)/2}\cdot C_{n-1}$$

$$= r^{-(n-3)/2}\cdot\left[C_{n-2}(q) - z\cdot(n-1)\cdot r^{-1/2}\cdot C_{n-1}(q)\right]$$

$$= r^{-(n-3)/2}\cdot\left[C_{n-2} - (n-1)\cdot q\cdot C_{n-1}\right]$$

Hence, the, (1) is

$$\frac{\partial F_{n-2}^{(6)}}{\partial z} = \frac{(n-1)}{R}r^{-(n-3)/2}\cdot\left[-nC_n + 2\cdot q\cdot C_{n-1}\right] - \frac{2}{R}\cdot r^{-(n-3)/2}\cdot\left[C_{n-2} - (n-1)\cdot q\cdot C_{n-1}\right]$$

$$= \frac{r^{-(n-3)/2}}{R}\left[-(n-1)\cdot nC_n + 2\cdot q\cdot(n-1)\cdot C_{n-1} - 2C_{n-2} + 2(n-1)\cdot q\cdot C_{n-1}\right]$$

2) $\dfrac{\partial}{\partial z}\left(z\cdot F_{n-3}^{(5)}(q)\right) = F_{n-3}^{(5)}(q) + z\cdot\dfrac{\partial F_{n-3}^{(5)}}{\partial z}(q)$

$$\frac{\partial F_{n-3}^{(5)}}{\partial z}(q) = \frac{\partial}{\partial z}\left[(n-2)\left(z^2+R^2\right)^{-(n-3)/2}\frac{C_{n-2}(q)}{R}\right]$$

$$= \frac{(n-2)}{R}\left[C_{n-2}(q)\frac{\partial\left(z^2+R^2\right)^{-(n-3)/2}}{\partial z} + \left(z^2+R^2\right)^{-(n-3)/2}\frac{\partial C_{n-2}(q)}{\partial z}\right]$$

$$\frac{\partial\left(z^2+R^2\right)^{-(n-3)/2}}{\partial z} = -\frac{(n-3)}{2}\left(z^2+R^2\right)^{-n/2+3/2-1}2\cdot z$$

$$= -(n-3)\cdot z\cdot\left(z^2+R^2\right)^{-n/2+1/2}$$

$$\frac{\partial C_{n-2}(q)}{\partial z} = \frac{\partial C_{n-2}(q)}{\partial q}\frac{\partial q}{\partial z} = C_{n-2}'(q)\frac{\partial}{\partial z}\left(\frac{z}{\left(z^2+R^2\right)^{1/2}}\right)$$

$$= C_{n-2}'(q)\left[\left(z^2+R^2\right)^{-1/2} - \tfrac{1}{2}\left(z^2+R^2\right)^{-3/2}\cdot 2\cdot z^2\right]$$

$$= C_{n-2}'(q)\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)$$

$$= \frac{(n-2)}{R}\left[-(n-3)\cdot z\cdot r^{-n/2+1/2}\cdot C_{n-2}(q) + r^{-n/2+3/2}\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)C_{n-2}'(q)\right]$$

$$= \frac{(n-2)}{R}r^{-n/2+1}\left[-(n-3)\cdot z\cdot r^{-1/2}\cdot C_{n-2} + r^{1/2}\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)C_{n-2}'\right]$$

$$= \frac{(n-2)}{R}r^{-n/2+1}\left[-(n-3)\cdot q\cdot C_{n-2} + \left(1-q^2\right)C_{n-2}'\right]$$

To, this, apply, the, recurrent, formula

$$\left(1-q^2\right)C_n'(q) = -(n+1)C_{n+1}(q) + (n-1)qC_n(q)$$

converting $n \to (n-2)$

$$\left(1-q^2\right)C_{n-2}'(q) = -(n-1)C_{n-1}(q) + (n-3)qC_{n-2}(q)$$

$$= \frac{(n-2)}{R}r^{-n/2+1}\left[-(n-3)\cdot q\cdot C_{n-2}(q) - (n-1)C_{n-1}(q) + (n-3)\cdot q\cdot C_{n-2}(q)\right]$$

$$= \frac{(n-2)}{R}r^{-n/2+1}\left[-(n-1)C_{n-1}(q)\right]$$

Therefore

$$\frac{\partial\left(z\cdot F_{n-3}^{(5)}(q)\right)}{\partial z} = (n-2)\cdot r^{-n/2+3/2}\cdot\frac{C_{n-2}}{R} - z\cdot\frac{(n-2)}{R}\cdot r^{-n/2+1}\cdot(n-1)\cdot C_{n-1}(q)$$

$$= \frac{(n-2)}{R}\cdot r^{-n/2+3/2}\left[C_{n-2} - z\cdot(n-1)\cdot r^{-1/2}\cdot C_{n-1}\right]$$

$$= \frac{(n-2)}{R}\cdot r^{-n/2+3/2}\left[C_{n-2} - (n-1)\cdot q\cdot C_{n-1}\right]$$

3]. $\dfrac{\partial}{\partial z}F_{n-4}^{(5)}$

$$= \frac{(n-3)}{R}\cdot\frac{\partial}{\partial z}\left[\left(z^2+R^2\right)^{-(n-4)/2}\cdot C_{n-3}(q)\right]$$

$$= \frac{(n-3)}{R}\cdot\left[C_{n-3}(q)\cdot\frac{\partial\left(z^2+R^2\right)^{-n/2+2}}{\partial z} + \left(z^2+R^2\right)^{-n/2+2}\cdot\frac{\partial C_{n-3}(q)}{\partial z}\right]$$

$$\frac{\partial\left(z^2+R^2\right)^{-n/2+2}}{\partial z} = \left(-\frac{n}{2}+2\right)\left(z^2+R^2\right)^{-n/2+1}\cdot 2\cdot z$$

$$= -(n-4)\cdot z\cdot r^{-n/2+1}$$

$$\frac{\partial C_{n-3}(q)}{\partial z} = \frac{\partial C_{n-3}(q)}{\partial q}\cdot\frac{\partial q}{\partial z} = C_{n-3}'\cdot\frac{\partial}{\partial z}\left(\frac{z}{\left(z^2+R^2\right)^{1/2}}\right)$$

$$= C_{n-3}'\cdot\left(r^{-1/2} - z\cdot 2\cdot z\cdot r^{-3/2}/2\right)$$

$$= C_{n-3}'\cdot\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)$$

$$= \frac{(n-3)}{R}\cdot\left[-(n-4)\cdot z\cdot r^{-n/2+1}\cdot C_{n-3} + r^{-n/2+2}C_{n-3}'\cdot\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)\right]$$

$$= \frac{(n-3)}{R}\cdot r^{-n/2+3/2}\left[-(n-4)\cdot z\cdot r^{-1/2}\cdot C_{n-3} + r^{1/2}C_{n-3}'\cdot\left(r^{-1/2} - z^2\cdot r^{-3/2}\right)\right]$$

$$q = z\cdot r^{-1/2}$$

$$= \frac{(n-3)}{R}\cdot r^{-n/2+3/2}\left[-(n-4)\cdot q\cdot C_{n-3} + C_{n-3}'\cdot\left(1-q^2\right)\right]$$

To, the, second, term, apply, the, recurrent

$$\left(1-q^2\right)C_n' = -(n+1)C_{n+1} + (n-1)\cdot q\cdot C_n$$

$$\left(1-q^2\right)C_{n-3}' = -(n-2)C_{n-2} + (n-4)\cdot q\cdot C_{n-3}$$

$$= \frac{(n-3)}{R}\cdot r^{-n/2+3/2}\left[-(n-4)\cdot q\cdot C_{n-3} - (n-2)C_{n-2} + (n-4)\cdot q\cdot C_{n-3}\right]$$

$$= \frac{(n-3)}{R}\cdot r^{-n/2+3/2}\cdot(n-2)C_{n-2}(q)$$

Associating

$$\frac{\partial H_n^{(6)}}{\partial z} = \frac{r^{-n/2+3/2}}{n(n-1)R}\left[-(n-1)\cdot n\cdot C_n + 4q\cdot(n-1)\cdot C_{n-1} - 2\cdot C_{n-2}\right.$$

$$\left. + \frac{4(n-2)}{(n-2)}\left(C_{n-2} - (n-1)q\cdot C_{n-1}\right) + \frac{2\left(-(n-3)(n-2)\right)C_{n-2}}{(n-2)(n-3)}\right]$$

$$= \frac{r^{-(n-3)/2}}{R}\frac{\left(-(n-1)\cdot n\cdot C_n(q)\right)}{n(n-1)} = F_n^{(4)}$$

$$H_n^{(2)} = \frac{F_{n-2}^{(2)}}{n(n-1)} + 4\cdot\frac{z\cdot F_{n-3}^{(1)}}{n(n-1)(n-2)} + 6\cdot\frac{F_{n-4}^{(1)}}{n(n-1)(n-2)(n-3)}$$

Differentiate, about, z

1]. $\dfrac{\partial F_{n-2}^{(2)}(R,z)}{\partial z} = \dfrac{\partial}{\partial z}\left\{\left(z^2+R^2\right)^{-(n-3)/2}\left[P_{n-2}(q) + C_{n-2}^{1/2}(q)\right]\right\}$

The, 1st, term $= \left(z^2+R^2\right)^{-(n-3)/2}\cdot P_{n-2}'(q) + P_{n-2}(q)\dfrac{\partial\left(z^2+R^2\right)^{-(n-3)/2}}{\partial z}$

$$\frac{\partial\left(z^2+R^2\right)^{-(n-3)/2}}{\partial z} = -\frac{(n-3)}{2}\cdot\left(z^2+R^2\right)^{-n/2+3/2-1}\cdot 2\cdot z$$

$$= -(n-3)\cdot z\cdot\left(z^2+R^2\right)^{-n/2+1/2}$$

$$\frac{\partial P_{n-2}(q)}{\partial z} = \frac{\partial P_{n-2}(q)}{\partial q}\cdot\frac{\partial q}{\partial z} = P_{n-2}'(q)\frac{\partial}{\partial z}\cdot\left(\frac{z}{\left(z^2+R^2\right)^{1/2}}\right)$$

$$= P_{n-2}'(q)\left(r^{-1/2} - z^2r^{-3/2}\right)$$

$$q = z\cdot r^{-1/2}$$

Therefore

the.1st.term $= r^{-n/2+3/2}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)\cdot P'_{n-2}-(n-3)\cdot z\cdot r^{-n/2+1/2}\cdot P_{n-2}$

$$= r^{-n/2+1}\left[r^{1/2}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)\cdot P'_{n-2}-(n-3)\cdot z\cdot r^{-1/2}\cdot P_{n-2}\right]$$

$$= r^{-n/2+1}\left[(1-q^2)\cdot P'_{n-2}-(n-3)\cdot q\cdot P_{n-2}\right]$$

(1 - q^ 2)P' (n - 2) = (n -1)(q * P(n - 2) - P(n -1))

$$= r^{-n/2+1}\left[(n-1)qP_{n-2}-(n-1)P_{n-1}-(n-3)\cdot q\cdot P_{n-2}\right]$$

$$= r^{-n/2+1}\left[2qP_{n-2}-(n-1)P_{n-1}\right]$$

.indicates,differentiation.about.z

The,second,term $= r^{-n/2+3/2}\cdot C'_{n-2}(q)+C_{n-2}(q)\left[-(n-3)\cdot z\cdot r^{-n/2+1/2}\right]$

$$= r^{-n/2+1}\left[r^{1/2}\cdot C'_{n-2}(q)-C_{n-2}(q)\cdot(n-3)\cdot z\cdot r^{-1/2}\right]$$

$$C'_{n-2}(q)=\frac{\partial C_{n-2}(q)}{\partial z}=\frac{\partial C_{n-2}(q)}{\partial q}\cdot\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right)$$

$$= C'_{n-2}(q)\left(r^{-1/2}-z^2r^{-3/2}\right)$$

$$= r^{-n/2+1}\left[r^{1/2}\cdot\left(r^{-1/2}-z^2r^{-3/2}\right)\cdot C'_{n-2}-(n-3)\cdot z\cdot r^{-1/2}\cdot C_{n-2}\right]$$

$$= r^{-n/2+1}\left[(1-q^2)C'_{n-2}-(n-3)q\cdot C_{n-2}\right]$$

To, the first, factor, apply, the, recurrent.form

$$(1-q^2)C'_{n-2}=-(n-1)C_{n-1}+(n-3)q\cdot C_{n-2}$$

$$= r^{-n/2+1}\cdot(n-1)C_{n-1}$$

Hence,

$$\frac{\partial F^{(2)}_{n-2}}{\partial z}= r^{-n/2+1}\cdot\left[2\cdot q\cdot P_{n-2}-(n-1)\cdot P_{n-1}-2(n-1)C_{n-1}\right]$$

(2) $\frac{\partial}{\partial z}\left[z\cdot F^{(1)}_{n-3}\right]=\frac{\partial}{\partial z}\left[z\cdot(z^2+R^2)^{-(n-2)/2}\cdot P_{n-3}(q)\right]$

$$= r^{-n/2+1}\cdot P_{n-3}(q)+z\cdot\frac{\partial}{\partial z}\left\{(z^2+R^2)^{-(n-2)/2}\cdot P_{n-3}(q)\right\}$$

$$= r^{-n/2+1}\cdot P_{n-3}(q)+z\left[P_{n-3}(q)\frac{\partial(z^2+R^2)^{-(n-2)/2}}{\partial z}+r^{-n/2+1}\cdot\frac{\partial P_{n-3}(q)}{\partial z}\right]$$

$$\frac{\partial(z^2+R^2)^{-n/2+1}}{\partial z}=-\frac{(n-2)}{2}(z^2+R^2)^{-n/2+1-1}\cdot 2\cdot z=-(n-2)\cdot z\cdot r^{-n/2}$$

$$\frac{\partial P_{n-3}(q)}{\partial z}=\frac{\partial P_{n-3}(q)}{\partial q}\cdot\frac{\partial q}{\partial z}=P'_{n-3}(q)\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right)$$

$$= P'_{n-3}(q)\left(r^{-1/2}-z^2r^{-3/2}\right)$$

$$= r^{-n/2+1}\cdot P_{n-3}(q)+z\cdot\left[P_{n-3}(-(n-2)\cdot z)\cdot r^{-n/2}+r^{-n/2+1}\cdot P'_{n-3}\left(r^{-1/2}-z^2r^{-3/2}\right)\right]$$

$$= r^{-n/2+1}\cdot P_{n-3}(q)+z\cdot r^{-n/2+1/2}\left[-(n-2)\cdot z\cdot P_{n-3}\cdot r^{-1/2}+r^{1/2}\cdot P'_{n-3}\left(r^{-1/2}-z^2r^{-3/2}\right)\right]$$

$$= r^{-n/2+1}\cdot P_{n-3}(q)+z\cdot r^{-n/2+1/2}\left[-(n-2)\cdot q\cdot P_{n-3}+P'_{n-3}(1-q^2)\right]$$

To,the,3rd,term.apply,the,recurrent,formula

$$(1-q^2)\cdot P'_n(q)=(n+1)(q\cdot P_n(q)-P_{n+1}(q))$$

changing $n\to(n-3)$

$$(1-q^2)\cdot P'_{n-3}(q)=(n-2)(q\cdot P_{n-3}(q)-P_{n-2}(q))$$

$$= r^{-n/2+1}\cdot P_{n-3}+z\cdot r^{-n/2+1/2}(-(n-2)\cdot q\cdot P_{n-3}+(n-2)q\cdot P_{n-3}-(n-2)P_{n-2})$$

$$= r^{-n/2+1}\cdot P_{n-3}-(n-2)\cdot z\cdot r^{-n/2+1/2}\cdot P_{n-2}$$

$$= r^{-n/2+1}\cdot\left(P_{n-3}-(n-2)\cdot z\cdot r^{-1/2}\cdot P_{n-2}\right)$$

$$= r^{-n/2+1}\cdot(P_{n-3}-(n-2)\cdot q\cdot P_{n-2})$$

3). $\frac{\partial F^{(1)}_{n-4}}{\partial z}=\frac{\partial}{\partial z}\left[(z^2+R^2)^{-(n-3)/2}\cdot P_{n-4}(q)\right]$

$$= P_{n-4}(q)\frac{\partial(z^2+R^2)^{-(n-3)/2}}{\partial z}+(z^2+R^2)^{-(n-3)/2}\frac{\partial P_{n-4}(q)}{\partial z}$$

$$\frac{\partial(z^2+R^2)^{-(n-3)/2}}{\partial z}=-\frac{(n-3)}{2}(z^2+R^2)^{-n/2+3/2-1}\cdot 2\cdot z$$

$$=-(n-3)r^{-n/2+1/2}\cdot z$$

$$\frac{\partial P_{n-4}(q)}{\partial z}=\frac{\partial P_{n-4}(q)}{\partial q}\cdot\frac{\partial q}{\partial z}=P'_{n-4}(q)\frac{\partial}{\partial z}\left(\frac{z}{(z^2+R^2)^{1/2}}\right)$$

$$=P'_{n-4}(q)\left(r^{-1/2}-z^2r^{-3/2}\right)$$

$$=-(n-3)r^{-n/2+1/2}\cdot z\cdot P_{n-4}(q)+r^{-n/2+3/2}\cdot P'_{n-4}\left(r^{-1/2}-z^2r^{-3/2}\right)$$

$$= r^{-n/2+1}\left[-(n-3)\cdot z\cdot r^{-1/2}\cdot P_{n-4}(q)+r^{1/2}\cdot P'_{n-4}\left(r^{-1/2}-z^2r^{-3/2}\right)\right]$$

$$= r^{-n/2+1}\left[-(n-3)\cdot q\cdot P_{n-4}(q)+P'_{n-4}(1-q^2)\right]$$

From,the,recurrent,formula.of,Legendre

$$(1-q^2)P'_n(q)=(n+1)(q\cdot P_n(q)-P_{n+1}(q))$$

$$(1-q^2)P'_{n-4}(q)=(n-3)(q\cdot P_{n-4}-P_{n-3})$$

$$= r^{-n/2+1}\left[-(n-3)\cdot q\cdot P_{n-4}(q)+(n-3)(q\cdot P_{n-4}-P_{n-3})\right]$$

$$=-r^{-n/2+1}(n-3)\cdot P_{n-3}(q)$$

Associating, these

$$\frac{\partial H^{(2)}_n}{\partial z}=\frac{1}{n(n-1)}\frac{\partial}{\partial z}\left[F^{(2)}_{n-2}(q)+\frac{2\cdot z\cdot F^{(1)}_{n-3}}{(n-2)}+\frac{2\cdot F^{(1)}_{n-4}}{(n-2)(n-3)}\right]$$

$$=\frac{r^{-n/2+1}}{n(n-1)}\cdot\left[2\cdot q\cdot P_{n-2}-(n-1)\cdot P_{n-1}-2(n-1)\cdot C_{n-1}\right.$$

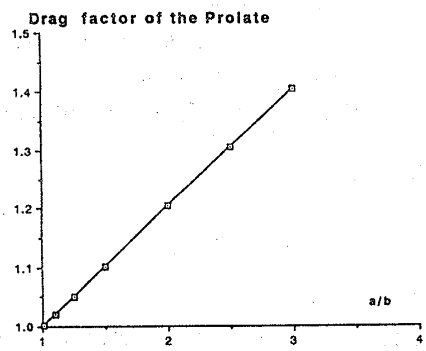
$$\left.+\frac{2}{(n-2)}\left[P_{n-3}-(n-2)qP_{n-2}\right]+\frac{2(-(n-3))\cdot P_{n-3}}{(n-2)(n-3)}\right]$$

$$=\frac{r^{-n/2+1}}{n(n-1)}\left[2\cdot q\cdot P_{n-2}-(n-1)\cdot P_{n-1}-2(n-1)\cdot C_{n-1}-2qP_{n-2}\right]$$

$$=-r^{-n/2+1}\frac{1}{n}\left[P_{n-1}(q)+2C_{n-1}(q)\right]$$

3. Result

Fig 2 shows the integrated drag factor. suffered from the fluid as a function of ratio between long and short axis. As the geometry of the non spherical particle changes, the magnitude of the drag changes significantly.



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