

Method for Macroscopic Motion of Non Spherical Bio Molecular Particles.

Prolate and Oval Particles.

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Mathematical and computational method were introduced for analyzing the creeping flow around non spherical bio molecular particle having arbitrary axisymmetrical axis near the cellular surface. This method was originally proposed by Wangyi1984. The Sampson's spherical infinite series expansion was selected as the fundamental singularities. The solutions for Stokes flow and pressure are consisted of linear combinations of Legendre function and Gegenbauer functions. For the line distribution of continuous singularities, several modified method including analytical integrations were utilized. The introduced method was verified to have a moderate approximations for creeping motions of the bio molecular particles.

Non spherical particle. Creeping flow. Legendre function, Gegenbauer polynomials. Prolate.

楕円形生体粒子の有機生体膜近傍での挙動解析

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任意の軸対称性を有する非球状生体分子の運動を近似的解析する方法を紹介する。本方法はWangによって1984年に提唱された遅いながれにおける非球粒子の挙動解析法である。 Sampsonの無限級数展開法をもちいて、流速、圧力をレジェンドレ関数および、ゲゲンバウエル関数らの線形結合で展開できた。またこれらを有限積分することで連続的 singularitiesが在る場合でも流速、圧力を計算できた。本方法は生体分子とくに生体膜近傍での非球状を呈する一般的な分子形状の粒子の遅い運動に応用することが可能である。

非球状生体分子. Legendre function, Gegenbauer polynomials. 楕円球. ストークス流れ

1. Introduction.

Cellular reaction to bio molecular transmitter starts by binding the transmitter to the target cellular surface. Most of this process has been simply described by the simplest creeping flow and spherical particles that approximate the geometric features of the bio molecular particles. In our previous works, we introduced extensive theoretical investigations of this field. All the biological particles were approximated by the spheres. This is because the mathematical treatment can be available for such spherical molecules where using Legendre functions enabled us to calculate the viscous drag and flow velocities.

In actual biological molecules, however, the geometry of the bio molecules are not necessarily sphere but has a variety of configurations such as prolate, oval, helical etc. Moreover, most of protein particles have definite electrical charges whereby the bio molecular particle interact when they approach each at their vicinity. Hence, there is a reasonable necessity to investigate the creeping flow near around the non spherical bio molecular particles. In the present work, we introduce a method proposed by Wangy to simulate the flow, pressure and drag around the prolate.

2. The Line distribution method of discrete singularities.

A uniform flow with instantaneous velocity U slowly passes an arbitrary prolate axisymmetrical body. The cylindrical coordinate R, z, θ is given in Fig 1. Z coincided with axisymmetrical axis of the body. The normalized stream function of the Stokes flow satisfies

$$\nabla^2 (\nabla^2 \psi) = 0 \quad (1)$$

The dimensionless velocity components v_z, v_R and pressure relate with ψ by

$$v_z = \frac{1}{R} \frac{\partial \psi}{\partial R}, \quad v_R = -\frac{1}{R} \frac{\partial \psi}{\partial z}, \quad (2.2)$$

$$\frac{\partial p}{\partial R} = -\frac{1}{R} \frac{\partial}{\partial z} (D^2 \psi), \quad \frac{\partial p}{\partial z} = \frac{1}{R} \frac{\partial}{\partial R} (D^2 \psi). \quad (2.3)$$

Sampson's⁽¹⁾ proposal, the disturbance produced by sphere can be following infinite series.

$$\begin{aligned} v_z &= \sum_{n=2}^{\infty} [c_{nj} F_n^{(1)}(R, z - z_j) + D_{nj} F_n^{(2)}(R, z - z_j)], \\ v_R &= \sum_{n=2}^{\infty} [c_{nj} F_n^{(3)}(R, z - z_j) + D_{nj} F_n^{(6)}(R, z - z_j)], \\ \psi &= \sum_{n=2}^{\infty} [c_{nj} F_n^{(3)}(R, z - z_j) + D_{nj} F_n^{(6)}(R, z - z_j)], \\ p &= p_{\infty} + \sum_{n=2}^{\infty} D_{nj} \frac{4n-6}{n} F_n^{(1)}(R, z - z_j), \end{aligned} \quad (2.4)$$

where $(0, z_j)$ is the coordinate of the sphere center. C_{nj}, D_{nj} are unknown coefficients, p_{∞} is the pressure at infinite. The function $F_{n-1}^{(k)}$ is

$$\begin{aligned} F_n^{(1)}(R, z) &= (R^2 + z^2)^{-\frac{n+1}{2}} P_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right], \\ F_n^{(2)}(R, z) &= (R^2 + z^2)^{-\frac{n-1}{2}} \left\{ P_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right] + 2J_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right] \right\}, \\ F_n^{(3)}(R, z) &= (R^2 + z^2)^{-\frac{n-1}{2}} J_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right], \end{aligned} \quad (2.5)$$

$$\begin{aligned} F_n^{(4)}(R, z) &= (R^2 + z^2)^{-\frac{n-1}{2}} J_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right], \\ F_n^{(5)}(R, z) &= (n+1)(R^2 + z^2)^{-n/2} \frac{1}{R} J_{n+1} \left[\frac{z}{\sqrt{R^2 + z^2}} \right], \\ F_n^{(6)}(R, z) &= (n+1)(R^2 + z^2)^{-\frac{n-2}{2}} \frac{1}{R} J_{n+1} \left[\frac{z}{\sqrt{R^2 + z^2}} \right] \\ &\quad - 2z(R^2 + z^2)^{-\frac{n-1}{2}} \frac{1}{R} J_n \left[\frac{z}{\sqrt{R^2 + z^2}} \right], \end{aligned}$$

where P_n and J_n are Legendre polynomials and Gegenbauer polynomials of the order n . We call the singularities positioned at $(0, z)$ and expressed by (2,4) the Sampson spherical singularities. A segment AB is taken on the axis of symmetry inside the body. If the nose and tail of the body are rounded, then, their centers of curvature can be chosen as A and B. Based on these consideration, the M different points at which the Sampson spherical singularities are placed, are prescribed on the segment AB with the coordinates $(0, Z_j)$ where $j=1$ to M . According to (2,4) the disturbance evoked by these M singularities in addition to the uniform flow have

$$v_z = 1 + \sum_{j=1}^M \sum_{n=2}^{\infty} [c_{nj} F_n^{(1)}(R, z - z_j) + D_{nj} F_n^{(2)}(R, z - z_j)] \quad (2.6a)$$

$$v_R = \sum_{j=1}^M \sum_{n=2}^{\infty} [c_{nj} F_n^{(3)}(R, z - z_j) + D_{nj} F_n^{(6)}(R, z - z_j)], \quad (2.6b)$$

$$\psi = \frac{1}{2} R^2 + \sum_{j=1}^M \sum_{n=2}^{\infty} [c_{nj} F_n^{(3)}(R, z - z_j) + D_{nj} F_n^{(6)}(R, z - z_j)], \quad (2.6c)$$

$$p = p_{\infty} + \sum_{j=1}^M \sum_{n=2}^{\infty} D_{nj} \frac{4n-6}{n} F_n^{(1)}(R, z - z_j). \quad (2.6d)$$

The non slip condition requires

$$\sum_{j=1}^M \sum_{n=2}^{\infty} [c_{nj} F_n^{(1)}(\bar{R}, \bar{z} - z_j) + D_{nj} F_n^{(2)}(\bar{R}, \bar{z} - z_j)] = -1, \quad (2.7a)$$

$$\sum_{j=1}^M \sum_{n=2}^{\infty} [c_{nj} F_n^{(3)}(\bar{R}, \bar{z} - z_j) + D_{nj} F_n^{(6)}(\bar{R}, \bar{z} - z_j)] = 0 \quad (2.7b)$$

where R, z are the coordinates of the points on the surface. C_{nj} and D_{nj} are unknown. Truncate the infinite series in (2,6) and the boundary conditions (2,7) are satisfied at properly selected $M \times N$ discrete points. Then, the $2(M \times N)$ simultaneous linear algebraic equations are obtained to determine the $2MN$ unknown C_{nj} and D_{nj} where $n=1$ to $N, J=1$ to M .

3. The Line distribution method of Continuous singularities.

A segment AB $(-d, d)$ is taken on the axis of symmetry inside the prolate body. $2d$ is the length of the segment. Disturbing the Sampson spherical singularities continuously over AB we get the following relations for the field.

$$\begin{aligned} v_z &= 1 + \sum_{n=2}^{\infty} \left[\int_{-d}^d c_n(\xi) F_n^{(1)}(R, z - \xi) d\xi \right. \\ &\quad \left. + \int_{-d}^d D_n(\xi) F_n^{(2)}(R, z - \xi) d\xi \right], \end{aligned} \quad (3.1a)$$

$$v_R = \sum_{n=2}^{\infty} \left[\int_{-d}^d c_n(\xi) F_n^{(0)}(R, z - \xi) d\xi + \int_{-d}^d D_n(\xi) F_n^{(0)}(R, z - \xi) d\xi \right], \quad (3.1b)$$

$$\phi = \frac{1}{2} R^2 + \sum_{n=2}^{\infty} \left[\int_{-d}^d c_n(\xi) F_n^{(0)}(R, z - \xi) d\xi + \int_{-d}^d D_n(\xi) F_n^{(0)}(R, z - \xi) d\xi \right],$$

$$p = p_{\infty} + \sum_{n=2}^{\infty} \int_{-d}^d D_n(\xi) \frac{4n-6}{n} F_{n-1}^{(1)}(R, z - \xi) d\xi, \quad (3.1d)$$

where $C_n(\xi)$ and $D_n(\xi)$ are unknown density distribution function. p_{∞} is the pressure at infinite. Divide AB into M segments. The density distributions on each segment are substituted by constant. After truncating the infinite series at N terms and performing the integral,

$$\begin{aligned} v_z &= 1 + \sum_{n=2}^{N+1} \sum_{j=1}^M \{C_{nj}[G_n^{(1)}(R, z - d_{j1}) - G_n^{(1)}(R, z - d_{j2})] \\ &\quad + D_{nj}[G_n^{(2)}(R, z - d_{j1}) - G_n^{(2)}(R, z - d_{j2})]\}, \\ v_R &= \sum_{n=2}^{N+1} \sum_{j=1}^M \{C_{nj}[G_n^{(3)}(R, z - d_{j1}) - G_n^{(3)}(R, z - d_{j2})] \\ &\quad + D_{nj}[G_n^{(4)}(R, z - d_{j1}) - G_n^{(4)}(R, z - d_{j2})]\}, \\ \phi &= \frac{R^2}{2} + \sum_{n=2}^{N+1} \sum_{j=1}^M \{C_{nj}[G_n^{(3)}(R, z - d_{j1}) - G_n^{(3)}(R, z - d_{j2})] \\ &\quad + D_{nj}[G_n^{(4)}(R, z - d_{j1}) - G_n^{(4)}(R, z - d_{j2})]\}, \\ p &= p_{\infty} + \sum_{n=2}^{N+1} \sum_{j=1}^M D_{nj} \frac{4n-6}{n} [G_{n-1}^{(1)}(R, z - d_{j1}) \\ &\quad - G_{n-1}^{(1)}(R, z - d_{j2})], \end{aligned}$$

where

$$\begin{aligned} G_n^{(k)}(R, z) &= -\frac{1}{n} F_{n-1}^{(k)}(R, z), \quad k = 1, 3, 5 \\ G_n^{(2)}(R, z) &= \begin{cases} -\frac{1}{n} F_{n-1}^{(2)}(R, z) - \frac{2}{n(n-1)} z F_{n-2}^{(1)}(R, z) \\ -\frac{2}{n(n-1)(n-2)} F_{n-3}^{(1)}(R, z) \end{cases} \quad n \geq 2 \\ &\quad \left\{ -\frac{1}{2} \frac{z}{(R^2 + z^2)^{3/2}} + \sin^{-1} \frac{z}{R}, \quad n = 2 \right. \\ G_n^{(4)}(R, z) &= \begin{cases} -\frac{1}{n} F_{n-1}^{(4)}(R, z) - \frac{2}{n(n-1)} z F_{n-2}^{(3)}(R, z) \\ -\frac{2}{n(n-1)(n-2)} F_{n-3}^{(3)}(R, z) \end{cases} \quad n \geq 2 \\ &\quad \left\{ \frac{R^2}{2} \sin^{-1} \frac{z}{R}, \quad n = 2 \right. \\ G_n^{(6)}(R, z) &= -\frac{1}{n} F_{n-1}^{(6)}(R, z) - \frac{2}{n(n-1)} z F_{n-2}^{(5)}(R, z), \end{aligned}$$

where d_{j2} and d_{j1} are coordinates of the end points for each segment. By satisfying the non slip boundary condition on the surface of the body. we arrive the linear algebraic equations of MN order for C_{nj} and D_{nj} as

$$\begin{aligned} \sum_{j=1}^M \sum_{n=2}^N \{C_{nj}[G_n^{(1)}(\bar{R}, \bar{z} - d_{j1}) - G_n^{(1)}(\bar{R}, \bar{z} - d_{j2})] \\ + D_{nj}[G_n^{(2)}(\bar{R}, \bar{z} - d_{j1}) - G_n^{(2)}(\bar{R}, \bar{z} - d_{j2})]\} &= -1, \\ \sum_{j=1}^M \sum_{n=2}^N \{C_{nj}[G_n^{(3)}(\bar{R}, \bar{z} - d_{j1}) - G_n^{(3)}(\bar{R}, \bar{z} - d_{j2})] \\ + D_{nj}[G_n^{(4)}(\bar{R}, \bar{z} - d_{j1}) - G_n^{(4)}(\bar{R}, \bar{z} - d_{j2})]\} &= 0, \end{aligned}$$

where R and z are coordinates of the surface points.

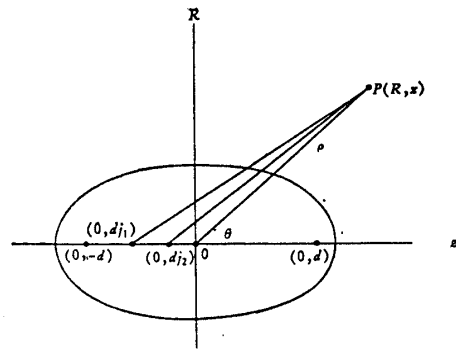


Fig 1

Pressure distribution

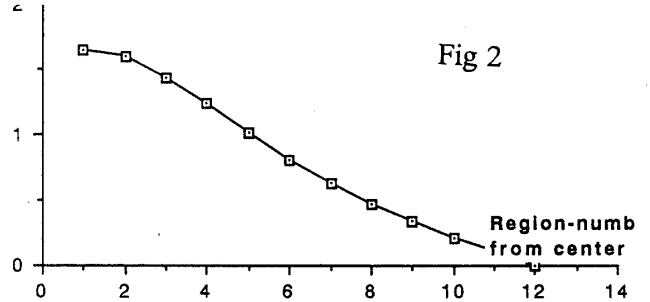


Fig 2

Pressure on the oval surface

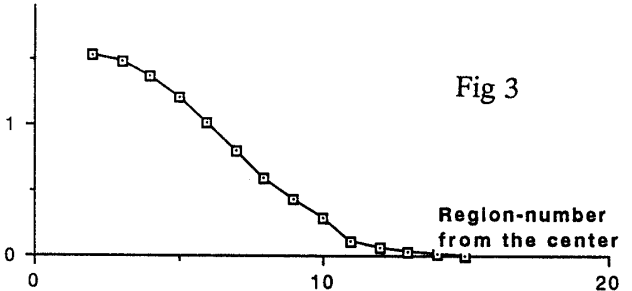


Fig 3

Results.

Fig 2 and Fig 3 shows the computed pressure distribution on the surfaces of prolate (Fig 2) and oval (at $c=0.8$ Fig 3). The present method will be available for evaluating non spherical bio molecular particles on the cellular surface.

Reference

1. Wu.Wangyi. Scientia Sinica. vol 27. pp 731-. 1984.

APPENDIX

Integration of (3.2)

We certify this process by differentiation.

$$F_{n-1}^{(1)}(R, z) = (R^2 + z^2)^{-\frac{n}{2}} P_{n-1} \left[\frac{z}{\sqrt{R^2 + z^2}} \right]$$

Differentiate with respect to z

$$\begin{aligned} F_{n-1}^{(1)'} &= \left(-\frac{n}{2} \right) (R^2 + z^2)^{-\frac{n}{2}-1} \cdot 2z \cdot P_{n-1} \left(\frac{z}{\sqrt{R^2 + z^2}} \right) \\ &\quad + (R^2 + z^2)^{-\frac{n}{2}} \frac{\partial}{\partial z} P_{n-1} \left(\frac{z}{\sqrt{R^2 + z^2}} \right) \\ &\quad x = \frac{z}{\sqrt{R^2 + z^2}} \\ &= -n \cdot z \cdot (R^2 + z^2)^{-\frac{(n+2)}{2}} \cdot P_{n-1} \left(\frac{z}{\sqrt{R^2 + z^2}} \right) \\ &\quad + (R^2 + z^2)^{-\frac{n}{2}} \frac{\partial P_{n-1}(x)}{\partial x} \frac{\partial x}{\partial z} \end{aligned}$$

where

$$\begin{aligned}\frac{\partial x}{\partial z} &= \frac{\partial}{\partial z} \left(\frac{z}{\sqrt{R^2 + z^2}} \right) = \frac{(R^2 + z^2)^{-1/2} - \frac{1}{2}(R^2 + z^2)^{-3/2} \cdot 2z^2}{(R^2 + z^2)} \\ &= (R^2 + z^2)^{-1/2} - z^2 \cdot (R^2 + z^2)^{-3/2} \\ &= -n \cdot z \cdot (R^2 + z^2)^{\frac{(n+2)}{2}} \cdot P_{n-1}(x) \\ &\quad + (R^2 + z^2)^{n/2} \left((R^2 + z^2)^{-1/2} - z^2 \cdot (R^2 + z^2)^{-3/2} \right) \cdot \frac{\partial P_{n-1}}{\partial x} \\ &= -n \cdot z \cdot r^{-\frac{(n+2)}{2}} \cdot P_{n-1}(x) + \left(r^{-\frac{(n+1)}{2}} - z^2 \cdot r^{-\frac{(n+3)}{2}} \right) \cdot \frac{\partial P_{n-1}}{\partial x} \\ &= r^{-\frac{(n+1)}{2}} \left(-n \cdot z \cdot r^{-1/2} \cdot P_{n-1} + (1 - z^2 \cdot r^{-1/2}) \cdot P_{n-1}' \right)\end{aligned}$$

On the other hand

$$\begin{aligned}z \cdot r^{-1/2} &= \frac{z}{\sqrt{z^2 + R^2}} = x \\ z^2 r^{-1} &= x^2 \\ &= r^{-\frac{(n+1)}{2}} \left[-n \cdot x \cdot P_{n-1} + (1 - x^2) P_{n-1}' \right]\end{aligned}$$

The Legendre recurrent formula

$$\begin{aligned}P_{n+1}'(x) - x \cdot P_n'(x) &= (n+1)P_n(x) \\ \text{Putting } n &\rightarrow (n-1) \\ P_n'(x) - x \cdot P_{n-1}'(x) &= n \cdot P_{n-1}(x) \\ \text{Multiply } x &\text{ on both sides} \\ x \cdot P_n'(x) - x^2 \cdot P_{n-1}'(x) &= n \cdot x \cdot P_{n-1}(x)\end{aligned}$$

Then, we have

$$\begin{aligned}-nxP_{n-1} - x^2P_{n-1}' &= -xP_n' \\ -nxP_{n-1} + P_{n-1}' - x^2P_{n-1}' &= -xP_n' + P_{n-1}'\end{aligned}$$

therefore

$$\begin{aligned}&= r^{-\frac{(n+1)}{2}} (-x \cdot P_n' + P_{n-1}') \\ \text{Applying } zP_n' - P_{n-1}' &= nP_n(x) \\ &= -r^{-\frac{(n+1)}{2}} \cdot n \cdot P_n(x)\end{aligned}$$

From the Beltrami's formula, we have

$$(x^2 - 1)P_n'(x) = -(n+1)(xP_n(x) - P_{n+1}(x))$$

Putting $n \rightarrow (n-1)$

$$\begin{aligned}(1 - x^2)P_{n-1}' &= +n(xP_{n-1} - P_n) \\ \therefore (1 - x^2)P_{n-1}' - nxP_{n-1} &= -nP_n\end{aligned}$$

Hence,

$$r^{-\frac{(n+1)}{2}} \cdot (-n) \cdot P_n(x)$$

つまり

$$\begin{aligned}\frac{\partial}{\partial z} F_{n-1}^{(1)}(z, R) &= -r^{-\frac{(n+1)}{2}} \cdot n \cdot P_n(x) \\ &= -n \cdot F_n^{(1)}\end{aligned}$$

$$\therefore \int F_n^{(1)} dz = -\frac{1}{n} F_{n-1}^{(1)}(z, R)$$

$$F_{n-1}^{(3)} = (z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

Differentiate with respect to z

$$\frac{\partial F_{n-1}^{(3)}}{\partial z} = -\frac{(n-2)}{2} (z^2 + R^2)^{\frac{(n-2)}{2}-1} \cdot 2z \cdot C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$\begin{aligned}&+ (z^2 + R^2)^{\frac{(n-2)}{2}} \frac{\partial}{\partial z} \cdot C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \\ r &= (z^2 + R^2) \quad x = \frac{z}{\sqrt{z^2 + R^2}} \\ &= -(n-2) \cdot z \cdot r^{-1/2} \cdot C_{n-1}(x) \\ &\quad + r^{-\frac{(n-2)}{2}} \frac{\partial}{\partial x} C_{n-1}(x) \cdot [r^{-1/2} - z^2 r^{-3/2}] \\ &= -(n-2) \cdot z \cdot r^{-1/2} C_{n-1} + (r^{-1/2} - z^2 r^{-3/2}) C_{n-1}'(x) \\ &= r^{-\frac{(n-1)}{2}} \left[-(n-2) \cdot z r^{-1/2} \cdot C_{n-1} + (1 - z^2 r^{-1}) C_{n-1}' \right] \\ &\quad z \cdot r^{-1/2} = z / \sqrt{z^2 + R^2} = x \\ &\quad z^2 r^{-1} = x^2 \\ &= r^{-\frac{(n-1)}{2}} \left[-(n-2) \cdot x \cdot C_{n-1} + (1 - x^2) C_{n-1}' \right]\end{aligned}$$

Recurent formula for Cm

$$\begin{aligned}(1-n) \cdot s \cdot C_n + (1-s^2) C_n' &= -(n+1) C_{n+1} \\ (1 - (1-n)s) C_n + (1-s^2) C_n' &= -(n) C_n\end{aligned}$$

$$\begin{aligned}&= r^{-\frac{(n-1)}{2}} (-nC_n) \\ &= -n \cdot F_n^{(3)}\end{aligned}$$

[Another process]

Convert all $C_n(x)$ to Legendre function by

$$C_n = (P_{n-2} - P_{n-1}) / (2n-1)$$

$$C_n' = -P_{n-1}$$

Hence

$$[] = \frac{-(n-2)(P_{n-3} - P_{n-1})}{(2n-3)} + (1-x^2)(-P_{n-2})$$

Apply the Beltrami's formula for the first term on the Right side

$$\begin{aligned}(1-x^2)P_n' &= \frac{n(n+1)}{(2n+1)}(P_{n-1} - P_{n+1}) \\ \text{putting } n &\rightarrow (n-2) \\ (1-x^2)P_{n-2}' &= \frac{(n-2)(n-1)}{(2n-3)}(P_{n-3} - P_{n-1})\end{aligned}$$

Hence

$$\begin{aligned}[] &= \frac{-(n-2) \cdot x \cdot (1-x^2)P_{n-2}'}{(n-2)(n-1)} - (1-x^2)P_{n-2} \\ &= -(1-x^2) \left[\frac{x \cdot P_{n-2}'}{(n-1)} + P_{n-2} \right] \\ &= -(1-x^2) \left[\frac{x \cdot P_{n-2}' + (n-1)P_{n-2}}{(n-1)} \right]\end{aligned}$$

By the recurrent formula of legendre function

$$P_{n+1}' - x \cdot P_n' = (n+1)P_n$$

Putting $n \rightarrow n-2$

$$\begin{aligned}P_{n-1}' &= xP_{n-2}' + (n-1)P_{n-2} \\ &= \frac{-(1-x^2)P_{n-1}'}{(n-1)}\end{aligned}$$

The relation between Legendre and C_n

$$\begin{aligned}(1-s^2)P_{n-1}' &= n(n-1)C_n^{-1/2} \\ &= -n \cdot C_n^{-1/2}\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial F_{n-1}^{(3)}}{\partial z} &= -nr^{-\frac{(n-1)}{2}} \cdot C_n = -n \cdot F_n^{(3)} \\ \therefore \int F_n^{(3)} dz &= -\frac{1}{n} F_{n-1}^{(3)}\end{aligned}$$

$$F_{n-1}^{(s)} = n \cdot (z^2 + R^2)^{\frac{(n-1)}{2}} \cdot \frac{1}{R} \cdot C_n \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

Factor out n/R and differentiate about z

$$\begin{aligned} \frac{\partial F_{n-1}^{(s)}}{\partial z} &= -\frac{(n-1)}{2} (z^2 + R^2)^{\frac{(n-1)}{2}-1} \cdot 2z \cdot C_n \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \\ &\quad + (z^2 + R^2)^{\frac{(n-1)}{2}} \frac{\partial}{\partial z} C_n \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \\ r &= (z^2 + R^2) \quad x = \frac{z}{\sqrt{z^2 + R^2}} \\ &= -(n-1) \cdot z \cdot r^{-\frac{1}{2}} \cdot C_n(x) \\ &\quad + r^{-\frac{1}{2}} \cdot C_n'(x) \cdot \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) \\ &= r^{-\frac{1}{2}} \left[-(n-1) \cdot z \cdot r^{-\frac{1}{2}} \cdot C_n + (1 - z^2 r^{-1}) C_n' \right] \\ z \cdot r^{-\frac{1}{2}} &= x \\ &= r^{-\frac{1}{2}} \left[-(n-1) \cdot x \cdot C_n + (1 - x^2) C_n' \right] \\ \text{Recurrent formula of } C_n & \\ (1-n)C_n + (1-x^2)C_n' &= -(n+1)C_{n+1} \\ &= -r^{-\frac{1}{2}} \cdot (n+1) \cdot C_{n+1} \end{aligned}$$

Thus

$$\frac{\partial F_{n-1}^{(s)}}{\partial z} = -n F_n^{(s)}$$

[Another process]

$$C_n = (P_{n-2} - P_n) / (2n-1)$$

$$C_n' = -P_{n-1}$$

Hence

$$[] = -(n-1) \cdot \frac{(P_{n-2} - P_n)}{(2n-1)} + (1 - x^2) \cdot (-P_{n-1})$$

Beltram formula

$$\begin{aligned} (1 - z^2) P_n' &= \frac{n(n+1)}{(2n+1)} (P_{n-1} - P_{n+1}) \\ (1 - z^2) P_{n-1}' &= \frac{(n-1)n}{(2n-1)} (P_{n-2} - P_n) \\ &= -(n-1)x \frac{(1-x^2) P_{n-1}'}{(n-1)n} - (1-x^2) P_{n-1} \\ &= (1-x^2) \left[\frac{x \cdot P_{n-1}'}{n} + P_{n-1} \right] \end{aligned}$$

$$\text{Legendre formulae } P_{n+1}' - x \cdot P_n' = (n+1)P_n$$

$$x P_{n+1}' + n P_{n-1} = P_{n+1}'$$

$$= -(1-x^2) \frac{P_{n+1}'}{n}$$

$$(1-x^2) P_n' = (n+1)P_{n+1}$$

$$= -(n+1)C_{n+1}$$

$$G_n^{(2)} = -\frac{1}{n} F_{n-1}^{(2)} - \frac{2}{n(n-1)} z \cdot F_{n-2}^{(1)} - \frac{2}{n(n-1)(n-2)} F_{n-3}^{(0)}$$

Differentiate each term with respect to z

I) The first term

$$\frac{\partial}{\partial z} F_{n-1}^{(2)} = \frac{\partial}{\partial z} \left\{ (z^2 + R^2)^{\frac{(n-2)}{2}} \left[P_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) + 2C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right] \right\}$$

Differentiate P_{n-1}, C_{n-1} independently

1-a ① $P_{n-1}(x)$

$$\begin{aligned} \frac{\partial}{\partial z} \left\{ (z^2 + R^2)^{\frac{(n-2)}{2}} \cdot P_{n-1}(x) \right\} \\ &= -\frac{(n-2)}{2} \cdot (z^2 + R^2)^{\frac{(n-2)}{2}-1} \cdot 2z \cdot P_{n-1}(x) \\ &\quad + (z^2 + R^2)^{\frac{(n-2)}{2}} \frac{\partial P_{n-1}(x)}{\partial x} \cdot \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) \end{aligned}$$

$$\begin{aligned} &= -(n-2) \cdot z \cdot r^{-\frac{1}{2}} \cdot P_{n-1} + r^{-\frac{1}{2}} \cdot \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) P_{n-1}' \\ &= r^{-\frac{1}{2}} \cdot \frac{1}{2} \left[-(n-2) \cdot z \cdot r^{-\frac{1}{2}} \cdot P_{n-1} + (1 - z^2 r^{-1}) P_{n-1}' \right] \\ &= r^{-\frac{1}{2}} \cdot \frac{1}{2} \left[-(n-2) \cdot x \cdot P_{n-1} + (1 - x^2) P_{n-1}' \right] \end{aligned}$$

By the recurrent formula for Legendre

$$\begin{aligned} (1-x^2) P_n' &= (n+1)(x P_n - P_{n+1}) \\ \therefore (1-x^2) P_{n-1}' &= n x P_{n-1} - P_n \\ &= r^{-\frac{1}{2}} \cdot \frac{1}{2} \cdot [2x P_{n-1} - n P_n] \end{aligned}$$

1-b Differentiation of $C_{n-1}(x)$ is equivalent

$$2 \cdot F_{n-1}^{(3)}(x) \quad \text{Thus} \quad C_{n-1}(x) = -2n F_n^{(3)} = -2nr^{\frac{(n-1)}{2}} C_n(x)$$

Hence

$$\frac{\partial}{\partial z} (F_{n-1}^{(2)}) = r^{-\frac{(n-1)}{2}} [2x \cdot P_{n-1} - n P_n - 2n C_n]$$

$$\text{II) } \frac{\partial}{\partial z} (z \cdot F_{n-2}^{(1)}) = \frac{\partial}{\partial z} \left[z \cdot (z^2 + R^2)^{\frac{(n-1)}{2}} P_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right]$$

Firstly

$$\begin{aligned} &\frac{\partial}{\partial z} \left[(z^2 + R^2)^{\frac{(n-1)}{2}} P_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right] \\ &= -\frac{(n-1)}{2} (z^2 + R^2)^{\frac{(n-1)}{2}-1} \cdot 2z P_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \\ &\quad + (z^2 + R^2)^{\frac{(n-1)}{2}} \frac{\partial}{\partial z} \left(P_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right) \\ &= -(n-1) \cdot z r^{-\frac{1}{2}} \cdot P_{n-2}(x) + r^{-\frac{(n-1)}{2}} \cdot \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) P_{n-2}' \\ &= r^{-\frac{1}{2}} \left[-(n-1) z r^{-\frac{1}{2}} P_{n-2}(x) + (1 - z^2 r^{-1}) P_{n-2}'(x) \right] \\ &= r^{-\frac{1}{2}} \left[-(n-1) x P_{n-2} + (1 - x^2) P_{n-2}' \right] \end{aligned}$$

BY the recurrent formula of Legendre function

$$\begin{aligned} (1-x^2) P_n' &= (n+1)(x P_n - P_{n+1}) \\ (1-x^2) P_{n-2}' &= (n-1)(x P_{n-2} - P_{n-1}) \\ &= r^{-\frac{1}{2}} \left[-(n-1) \cdot P_{n-1} \right] \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial z} (z \cdot F_{n-2}^{(1)}) &= F_{n-2}^{(1)} + z \cdot r^{-\frac{1}{2}} \cdot (-(n-1) \cdot P_{n-1}) \\ &= r^{-\frac{(n-1)}{2}} \cdot P_{n-2} - z \cdot r^{-\frac{1}{2}} \cdot (n-1) \cdot P_{n-1} \\ &= r^{-\frac{(n-1)}{2}} \cdot \left(P_{n-2} - z \cdot r^{-\frac{1}{2}} \cdot (n-1) \cdot P_{n-1} \right) \\ &= r^{-\frac{(n-1)}{2}} \cdot (P_{n-2} - (n-1) \cdot x \cdot P_{n-1}) \end{aligned}$$

$$\text{III) } \frac{\partial}{\partial z} F_{n-3}^{(0)} = \frac{\partial}{\partial z} \left[(z^2 + R^2)^{\frac{(n-2)}{2}} P_{n-3} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right]$$

$$\begin{aligned} &= -\frac{(n-2)}{2} (z^2 + R^2)^{\frac{(n-2)}{2}-1} 2z P_{n-3}(x) \\ &\quad + (z^2 + R^2)^{\frac{(n-2)}{2}} \frac{\partial}{\partial z} \left(P_{n-3} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right) \\ &= -(n-2) \cdot z \cdot r^{-\frac{1}{2}} \cdot P_{n-3} + r^{-\frac{1}{2}} \cdot P_{n-3}' \cdot \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) \\ &= r^{-\frac{1}{2}} \cdot \frac{1}{2} \left[-(n-2) \cdot z \cdot r^{-\frac{1}{2}} \cdot P_{n-3} + (1 - z^2 r^{-1}) P_{n-3}' \right] \\ &= r^{-\frac{(n-1)}{2}} \left[-(n-2) \cdot x \cdot P_{n-3} + (1 - x^2) P_{n-3}' \right] \end{aligned}$$

By the recurrent formula of Legendre

$$-r^{-(n-1)/2} \left[-(n-2)P_{n-2} \right] \quad (1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$$

$$(1-x^2)P'_{n-3} = (n-2)(xP_{n-3} - P_{n-2})$$

Associating those three terms

$$-\frac{\partial}{\partial z} \left[\frac{1}{n} \cdot F_{n-1}^{(2)} + \frac{2}{n(n-1)} \cdot z \cdot F_{n-2}^{(1)} + \frac{2}{n(n-1)(n-2)} \cdot F_{n-3}^{(1)} \right]$$

$$= -r^{-(n-1)/2} \left[\frac{2x \cdot P_{n-1}}{n} - P_n - 2C_n + \frac{2}{n(n-1)} (P_{n-2} - (n-1)xP_{n-1}) \right.$$

$$\left. - \frac{2}{n(n-1)(n-2)} \cdot (n-2) \cdot P_{n-2} \right]$$

$$= -r^{-(n-1)/2} \left[\frac{2xP_{n-1}}{n} - P_n - 2C_n + \frac{2 \cdot P_{n-2}}{n(n-1)} - \frac{2xP_{n-1}}{n} \right.$$

$$\left. - \frac{2}{n(n-1)} P_{n-2} \right] = r^{-(n-1)/2} (P_n + 2C_n) = F_n^{(2)}$$

$$G_n^{(4)} = -\frac{1}{n} F_{n-1}^{(4)} - \frac{2}{n(n-1)} \cdot z \cdot F_{n-2}^{(3)} - \frac{2}{n(n-1)(n-2)} F_{n-3}^{(3)}$$

Differentiate about z

$$I) \frac{\partial}{\partial z} F_{n-1}^{(4)} = \frac{\partial}{\partial z} \left[(z^2 + R^2)^{\frac{(n-4)}{2}} C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right]$$

$$= -\frac{(n-4)}{2} (z^2 + R^2)^{\frac{(n-4)}{2}-1} \cdot 2z \cdot C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$+ (z^2 + R^2)^{\frac{(n-4)}{2}} \frac{\partial}{\partial z} C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$= -(n-4) \cdot z \cdot r^{-\frac{1}{2}+1} \cdot C_{n-1}(x) + r^{-\frac{(n-4)}{2}} \left(r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}} \right) \cdot C'_{n-1}(x)$$

$$= r^{-\frac{(n-3)}{2}} \left[-(n-4) \cdot z \cdot r^{-\frac{1}{2}} \cdot C_{n-1} + (1 - z^2 r^{-1}) C'_{n-1}(x) \right]$$

$$= r^{-\frac{(n-3)}{2}} \left[-(n-4) \cdot x \cdot C_{n-1}(x) + (1-x^2) C'_{n-1}(x) \right]$$

C_n Recurrent formula of $C_n(x)$

$$(1-n)x C_n + (1-x^2) C'_n = -(n+1) C_{n+1}$$

$$(1-(n-1))x C_{n-1} + (1-x^2) C'_{n-1} = -n C_n$$

$$= r^{-\frac{(n-3)}{2}} \left[-(n-4)x C_{n-1} - n C_n - (2-n)x C_{n-1} \right]$$

$$= r^{-\frac{(n-3)}{2}} \left[-n C_n + 2x C_{n-1} \right]$$

[Another method]

 C_n Expand C_n by Legendre function

$$C_n = (P_{n-2} - P_n) / (2n-1)$$

$$C'_n = -P_{n-1}$$

Thus

$$[] = \frac{-(n-4)x \cdot (P_{n-3} - P_{n-1}) + (1-x^2)(-P_{n-2})}{(2n-3)}$$

For the first term

$$(1-x^2)P'_n = \frac{n(n+1)}{2n+1} (P_{n-1} - P_{n+1})$$

$$(1-x^2)P'_{n-2} = \frac{(n-2)(n-1)}{2n-3} (P_{n-3} - P_{n+1})$$

$$= -(n-4)x \frac{(1-x^2)P'_{n-2}}{(n-2)(n-1)} - (1-x^2)P_{n-2}$$

$$= -(1-x^2) \left[\frac{(n-4)xP'_{n-2}}{(n-2)(n-1)} + P_{n-2} \right]$$

Applying the following recurrent formula of Legendre Function for the content of []

$$P'_{n+1} - xP'_n = (n+1)P_n$$

$$n \rightarrow (n-2)$$

$$xP'_{n-2} = -(n-1)P_{n-2} + P'_{n-1}$$

$$= -(1-x^2) \left[\frac{(n-4)x \cdot (P_{n-2} + P'_{n-1})}{(n-2)(n-1)} + P_{n-2} \right]$$

$$= -(1-x^2) \left[-\frac{(n-4)}{(n-2)} P_{n-2} + \frac{(n-4)}{(n-2)(n-1)} P'_{n-1} + P_{n-2} \right]$$

$$= -(1-x^2) \left[\frac{2}{(n-2)} P_{n-2} + \frac{(n-4)}{(n-2)(n-1)} P'_{n-1} \right]$$

$$(1-x^2)P'_{n-1} = n(n-1)C_n$$

$$= -\frac{2 \cdot (1-x^2)}{(n-2)} P_{n-2} - \frac{(n-4)n(n-1)C_n}{(n-2)(n-1)}$$

$$= -\frac{2 \cdot (1-x^2)}{(n-2)} P_{n-2} - \frac{(n-4)nC_n}{(n-2)}$$

$$II) \frac{\partial}{\partial z} \left[z \cdot F_{n-2}^{(3)} \right]$$

$$\frac{\partial}{\partial z} F_{n-2}^{(3)}(z) = \frac{\partial}{\partial z} \left[(z^2 + R^2)^{\frac{(n-3)}{2}} C_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right]$$

$$= -\frac{(n-3)}{2} (z^2 + R^2)^{\frac{(n-3)}{2}-1} \cdot 2z \cdot C_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$+ (z^2 + R^2)^{\frac{(n-3)}{2}} \frac{\partial}{\partial z} C_{n-2} \left(\frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$= -(n-3)z \cdot r^{-\frac{1}{2}+1/2} C_{n-2} + r^{-\frac{1}{2}+3/2} C'_{n-2} \cdot (r^{-\frac{1}{2}} - z^2 r^{-\frac{3}{2}})$$

$$= r^{-\frac{1}{2}+1} \left[-(n-3)z \cdot r^{-\frac{1}{2}} C_{n-2} + C'_{n-2} (1 - z^2 r^{-1}) \right]$$

$$= r^{-\frac{(n-2)}{2}} \left[-(n-3)x C_{n-2} + C'_{n-2} (1-x^2) \right]$$

Recurrent formula for $C_n(x)$

$$-(n-1)x C_n + (1-x^2) C'_n = -(n+1) C_{n+1}$$

$$-(n-3)x C_{n-2} + (1-x^2) C'_{n-2} = -(n-1) C_{n-1}$$

$$= -r^{-\frac{(n-2)}{2}} \cdot (n-1) \cdot C_{n-1}$$

$$\text{Since } C_{n-2} = \frac{(P_{n-4} - P_{n-2})}{(2n-5)}$$

$$C_{n-2} = -P_{n-3}$$

$$[] = \frac{-(n-3)x(P_{n-4} - P_{n-2}) - (1-x^2)P_{n-3}}{(2n-5)}$$

Applying the Beltrami's formula

$$(1-x^2)P'_n = \frac{n(n+1)}{(2n+1)} (P_{n-1} - P_{n+1})$$

$$(1-x^2)P'_{n-3} = \frac{(n-3)(n-2)}{(2n-5)} (P_{n-4} - P_{n-2})$$

$$= -\frac{(n-3)x \cdot (1-x^2)P'_{n-3}}{(n-3)(n-2)} - (1-x^2)P_{n-3}$$

$$= -(1-x^2) \left[\frac{xP'_{n-3}}{(n-2)} + P_{n-3} \right]$$

From the recurrent formula of Legendre function

$$P'_{n+1} - x \cdot P'_n = (n+1)P_n$$

$$P'_{n-2} - x \cdot P'_{n-3} = (n-2)P_{n-3}$$

$$= -\frac{(1-x^2)P'_{n-2}}{(n-2)}$$

$$(1-x^2)P'_{n-2} = (n-1)(n-2)C_{n-1}$$

$$= -(n-1) \cdot C_{n-1}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial z} (z \cdot F_{n-2}^{(3)}) &= F_{n-2}^{(3)} + z \frac{\partial F_{n-2}^{(3)}}{\partial z} \\ &= r^{-\frac{(n-3)}{2}} C_{n-2} - z \cdot r^{-\frac{(n-2)}{2}} \cdot (n-1) \cdot C_{n-1} \\ &= r^{-\frac{(n-3)}{2}} [C_{n-2} - z r^{-1} \cdot (n-1) \cdot C_{n-1}] \\ &= r^{-\frac{(n-3)}{2}} [C_{n-2} - x \cdot (n-1) \cdot C_{n-1}] \end{aligned}$$

$$\text{III) } \frac{\partial}{\partial z} (F_{n-3}^{(3)})$$

$$\begin{aligned} &= \frac{\partial}{\partial z} \left[(z^2 + R^2)^{\frac{(n-4)}{2}} C_{n-3} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right] \\ &= -\frac{(n-4) \cdot (z^2 + R^2)^{\frac{(n-4)}{2}-1} \cdot 2z \cdot C_{n-3}(x)}{2} \\ &\quad + r^{-\frac{(n-4)}{2}} C'_{n-3}(x) \cdot (r^{-1/2} - z^2 r^{-3/2}) \\ &= -(n-4) r^{-\frac{1}{2}+1} \cdot C_{n-3} + r^{-\frac{1}{2}+2} (r^{-1/2} - z^2 r^{-3/2}) C'_{n-3} \\ &= r^{-\frac{1}{2}+3/2} \left\{ -(n-4) r^{-1/2} \cdot C_{n-3} + (1 - z^2 r^{-1}) C'_{n-3} \right\} \\ &= r^{-\frac{(n-3)}{2}} \left[-(n-4) \cdot x \cdot C_{n-3} + (1 - x^2) C'_{n-3} \right] \end{aligned}$$

C_n Recurrent formulae

$$\begin{aligned} -(n-1)xC_n + (1-x^2)C'_n &= -(n+1)C_{n+1} \\ -(n-4)xC_{n-3} + (1-x^2)C'_{n-3} &= -(n-2)C_{n-2} \end{aligned}$$

$$= -r^{-\frac{(n-3)}{2}} \cdot (n-2) \cdot C_{n-2}$$

$$\text{Since } C_{n-3} = (P_{n-5} - P_{n-3})/(2n-7)$$

$$C'_{n-3} = -P_{n-4}$$

$$[] = \frac{-(n-4)x(P_{n-5} - P_{n-3})}{(2n-7)} - (1-x^2)P_{n-4}$$

B f

$$(1-x^2)P'_{n-4} = \frac{(n-4)(n-3)}{(2n-7)}(P_{n-5} - P_{n-3})$$

$$[] = -\frac{(n-4)x \cdot (1-x^2)P'_{n-4}}{(n-4)(n-3)} - (1-x^2)P_{n-4}$$

$$= -(1-x^2) \left[\frac{x \cdot P'_{n-4} + (n-3)P_{n-4}}{(n-3)} \right]$$

Recurrent formula of Legendre function

$$P'_{n+1} - xP'_n = (n+1)P_n$$

$$P'_{n-3} - xP'_{n-4} = (n-3)P_{n-4}$$

$$= -\frac{(1-x^2)P'_{n-3}}{(n-3)}$$

$$(1-x^2)C'_{n-3} = (n-2)(n-3)C_{n-2}$$

$$= -(n-2) \cdot C_{n-2}$$

Associating those

$$\begin{aligned} &= \frac{\partial}{\partial z} \left[\frac{1}{n} F_{n-1}^{(4)} + \frac{2}{n(n-1)} z \cdot F_{n-2}^{(3)} + \frac{2}{n(n-1)(n-2)} F_{n-3}^{(2)} \right] \\ &= \frac{(n-4)}{(n-2)} C_n + \frac{2(1-x^2)}{n(n-2)} P_{n-2} - \frac{2}{n(n-1)} [C_{n-2} - x(n-1)C_{n-1}] \\ &\quad + \frac{2}{n(n-1)(n-2)} (n-2)C_{n-2} \\ &= \frac{(n-4)}{(n-2)} C_n + \frac{2(1-x^2)}{n(n-2)} P_{n-2} + \frac{2x}{n} C_{n-1} \end{aligned}$$

The second and the third terms are

$$\begin{aligned} &\frac{2}{n} \left[\frac{(1-x^2)}{(n-2)} P_{n-2} + x C_{n-1} \right] \\ &\quad C_{n-1} = (P_{n-3} - P_{n-1})/(2n-3) \\ &= \frac{2}{n} \left[\frac{(1-x^2)}{(n-2)} P_{n-2} + x \frac{(P_{n-3} - P_{n-1})}{(2n-3)} \right] \end{aligned}$$

Applying the Beltrami's formula

$$(1-x^2)P'_n = \frac{n(n+1)}{(2n+1)}(P_{n-1} - P_{n+1})$$

$$(1-x^2)P'_{n-2} = \frac{(n-2)(n-1)}{(2n-3)}(P_{n-3} - P_{n-1})$$

$$\begin{aligned} &= \frac{2}{n} \left[\frac{(1-x^2)}{(n-2)} P_{n-2} + x \frac{(1-x^2)P'_{n-2}}{(n-2)(n-1)} \right] \\ &= \frac{2}{n} \frac{(1-x^2)}{(n-2)} \frac{[(n-1)P_{n-2} + xP'_{n-2}]}{(n-1)} \end{aligned}$$

$$\text{since } P'_{n+1} - xP'_n = (n+1)P_n$$

$$P'_{n-1} - xP'_{n-2} = (n-1)P_{n-2}$$

$$= \frac{2(1-x^2) \cdot P'_{n-1}}{n(n-1)(n-2)}$$

$$\text{since } (1-s^2)P'_{n-1} = n(n-1)C_n$$

$$= \frac{2C_n}{(n-2)} \text{ Thus}$$

$$\frac{(n-4)}{(n-2)} C_n + \frac{2C_n}{(n-2)} = C_n \rightarrow F_n^{(4)}$$

$$G_n^{(6)} = -\frac{1}{n} F_{n-1}^{(6)} - \frac{2}{n(n-1)} z \cdot F_{n-2}^{(5)}$$

$$\text{I) } \frac{\partial}{\partial z} F_{n-1}^{(6)} = \frac{\partial}{\partial z} \left\{ n(z^2 + R^2)^{\frac{(n-3)}{2}} C_n - 2z(z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \right\}$$

I-a The first term

$$\begin{aligned} &\frac{n}{R} \frac{\partial}{\partial z} \left\{ (z^2 + R^2)^{\frac{(n-3)}{2}} C_n \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right\} \\ &= \frac{n}{R} \left\{ -\frac{(n-3)}{2} (z^2 + R^2)^{\frac{(n-3)}{2}-1} \cdot 2z \cdot C_n(x) \right. \\ &\quad \left. + r^{-\frac{(n-3)}{2}} C'_n \cdot (r^{-1/2} - z^2 r^{-3/2}) \right\} \\ &= \frac{n \left\{ -(n-3) r^{-1/2+1/2} C_n + r^{-1/2+3/2} C'_n \cdot (r^{-1/2} - z^2 r^{-3/2}) \right\}}{R} \\ &= \frac{n}{R} r^{-1/2+1} \left\{ -(n-3) r^{-1/2} C_n + r^{1/2} C'_n \cdot (r^{-1/2} - z^2 r^{-3/2}) \right\} \\ &= \frac{n}{R} r^{-\frac{(n-2)}{2}} \left\{ (n-3)xC_n + (1-z^2r^{-1})C'_n \right\} \\ &= \frac{n}{R} r^{-\frac{(n-2)}{2}} \left\{ (n-3)xC_n + (1-x^2)C'_n \right\} \\ &\quad (1-x^2)C'_n = -(n+1)C_{n+1} + (n-1)xC_n \\ &= \frac{n}{R} r^{-\frac{(n-2)}{2}} [2xC_n - (n+1)C_{n+1}] \end{aligned}$$

{ Another approach } since

$$C_n = (P_{n-2} - P_n)/(2n-1)$$

$$C'_n = -P_{n-1}$$

$$[] = -(n-3)x \frac{(P_{n-2} - P_n)}{(2n-1)} - (1-x^2)P_{n-1}$$

$$(1-x^2)P'_{n-1} = \frac{(n-1)n}{(2n-1)}(P_{n-2} - P_n)$$

$$\begin{aligned}
 &= -\frac{(n-3)x \cdot (1-x^2)P'_{n-1}}{(n-1)n} - (1-x^2)P_{n-1} \\
 &= -(1-x^2) \left[\frac{(n-3)x \cdot P'_{n-1}}{(n-1)n} + P_{n-1} \right] \\
 &\quad P'_{n+1} - xP'_n = (n+1)P_n \\
 &\quad P'_n - xP'_{n-1} = n \cdot P_{n-1} \\
 &= -(1-x^2) \left[\frac{(n-3)(P_n' - nP_{n-1}) + n(n-1)P_{n-1}}{(n-1)n} \right] \\
 &= -(1-x^2) \left[\frac{(n-3)}{(n-1)n} P'_n + \frac{2}{(n-1)} P_{n-1} \right] \\
 &\quad \text{Since } (1-x^2)P_n' = (n+1)n C_{n+1} \\
 &= -(n-3)(n+1)n / ((n-1)n) C_{n+1} - 2(1-x^2)/(n-1) P_{n-1}
 \end{aligned}$$

Hence

$$\begin{aligned}
 \frac{\partial}{\partial z} F_{n-1}^{(6)} &= -\frac{n}{R} r^{-\frac{(n-2)}{2}} \left[\frac{(n-3)(n+1)}{(n-1)} C_{n+1} + \frac{2(1-x^2)}{(n-1)} P_{n-1} \right] \\
 &= -\frac{n}{R} r^{-\frac{(n-2)}{2}} (2xC_n - (n+1)C_{n+1})
 \end{aligned}$$

2) For the second term, factor out $-2/R$

$$\frac{\partial}{\partial z} \left\{ z \cdot (z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \right\}$$

Firstly

$$\begin{aligned}
 &\frac{\partial}{\partial z} \left\{ (z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right\} \\
 &= -\frac{(n-2)}{2} (z^2 + R^2)^{\frac{(n-2)}{2}-1} \cdot 2z \cdot C_{n-1}(x) \\
 &\quad + r^{-\frac{(n-2)}{2}} C_{n-1}' \left(r^{-\frac{(n-2)}{2}} - zr^{-\frac{(n-2)}{2}} \right) \\
 &= -(n-2)zr^{-\frac{(n-2)}{2}} C_{n-1} + r^{-\frac{(n-2)}{2}} \cdot 2(1-x^2) C_{n-1}' \\
 &= r^{-\frac{(n-1)}{2}} \left(-(n-2)zr^{-\frac{(n-2)}{2}} C_{n-1} + (1-zr^{-1}) C_{n-1}' \right) \\
 &= r^{-\frac{(n-1)}{2}} \left(-(n-2)xC_{n-1} + (1-x^2) C_{n-1}' \right) \\
 &\quad \text{since } -(n-2)xC_{n-1} + (1-x^2) C_{n-1}' = -n \\
 &= -r^{-\frac{(n-1)}{2}} \cdot n \cdot C_n
 \end{aligned}$$

[Another approach]

$$\begin{aligned}
 C_{n-1} &= (P_{n-3} - P_{n-1}) / (2n-3) \\
 C'_{n-1} &= -P_{n-2} \\
 [] &= \frac{-(n-2)x \cdot (P_{n-3} - P_{n-1})}{(2n-3)} + (1-x^2)(-P_{n-2}) \\
 &\quad (1-x^2)P'_n = \frac{n(n+1)}{(2n+1)}(P_{n-1} - P_{n+1}) \\
 &\quad (1-x^2)P'_{n-2} = \frac{(n-2)(n-1)}{(2n-3)}(P_{n-3} - P_{n-1})
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{(n-2)x(1-x^2)P'_{n-2}}{(n-2)(n-1)} - (1-x^2)P_{n-2} \\
 &= -(1-x^2) \left[\frac{x \cdot P'_{n-2}}{(n-1)} + P_{n-2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &\quad P'_{n+1} - xP'_n = (n+1)P_n \\
 &\quad P'_n - xP'_{n-2} = (n-1)P_{n-2} \\
 &= -\frac{(1-x^2)P'_{n-1}}{(n-1)} = -nC_n(x) \\
 &\quad \text{since } (1-x^2)P'_{n-1} = n(n-1)C_n
 \end{aligned}$$

Thus

$$\begin{aligned}
 &= -\frac{2}{R} \frac{\partial}{\partial z} \left\{ z(z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \right\} \\
 &= -\frac{2}{R} \left\{ r^{-\frac{(n-2)}{2}} C_{n-1} + z \left(-r^{-\frac{(n-2)}{2}} n C_n \right) \right\} \\
 &= -\frac{2r^{-\frac{(n-2)}{2}}}{R} [C_{n-1} - zr^{-\frac{1}{2}} n C_n] = -\frac{2r^{-\frac{(n-2)}{2}}}{R} (C_{n-1} - x \cdot n C_n)
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\frac{\partial}{\partial z} \left(zr^{-\frac{(n-2)}{2}} C_{n-1} \right) \\
 &= r^{-\frac{(n-2)}{2}} C_{n-1} - zr^{-\frac{(n-2)}{2}} n \cdot C_n \\
 &= r^{-\frac{(n-2)}{2}} [C_{n-1} - zr^{-\frac{1}{2}} n C_n] \\
 &= r^{-\frac{(n-2)}{2}} [C_{n-1} - xn C_n]
 \end{aligned}$$

Hence, the differentiated second term is

$$-\frac{2}{R} r^{-\frac{(n-2)}{2}} (C_{n-1} - x \cdot n \cdot C_n)$$

$$\text{II) } \frac{\partial}{\partial z} (z F_{n-2}^{(5)}(z))$$

$$\frac{\partial}{\partial z} F_{n-2}^{(5)}(z) = \frac{(n-1)}{R} \frac{\partial}{\partial z} \left\{ (z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1} \left(\frac{z}{\sqrt{z^2 + R^2}} \right) \right\}$$

This is equivalent to the differentiation of $F_{n-1}^{(3)}(z)$

$$= \frac{(n-1)}{R} \left(-r^{-\frac{(n-1)}{2}} n \cdot C_n \right) \text{ Hence}$$

$$\begin{aligned}
 \frac{\partial}{\partial z} (z \cdot F_{n-2}^{(5)}) &= \frac{(n-1)}{R} \frac{\partial}{\partial z} \left[z(z^2 + R^2)^{\frac{(n-2)}{2}} C_{n-1}(x) \right] \\
 &= \frac{(n-1)}{R} \left[r^{-\frac{(n-2)}{2}} C_{n-1} - z \cdot n \cdot r^{-\frac{(n-1)}{2}} C_n \right] \\
 &= \frac{(n-1)}{R} r^{-\frac{(n-2)}{2}} [C_{n-1} - zr^{-\frac{1}{2}} \cdot n \cdot C_n] \\
 &= \frac{(n-1)}{R} r^{-\frac{(n-2)}{2}} [C_{n-1} - x \cdot n \cdot C_n]
 \end{aligned}$$

Associate the I), II)

$$\begin{aligned}
 &= -\frac{\partial}{\partial z} \left[\frac{1}{n} F_{n-1}^{(6)} + \frac{2}{n(n-1)} z \cdot F_{n-2}^{(5)} \right] \\
 &= -\frac{r^{-\frac{(n-2)}{2}}}{R} \left[\frac{(n-3)(n+1)}{(n-1)} C_{n+1} + \frac{2(1-x^2)}{(n-1)} P_{n-1} \right] \\
 &\quad + \frac{2(C_{n-1}/n - x C_n)}{n(n-1)} - \frac{2(n-1)}{n(n-1)} (C_{n-1} - xn C_n)
 \end{aligned}$$

Hence, the residual terms are

$$[(n-3)(n+1)C_{n+1} + 2(1-x^2)P_{n-1}]/(n-1)$$

which is the first term of the $\partial F_{n-1}^{(6)} / \partial z$
 since $(1-x^2)P_{n-1} = -(1-x^2)C_n'$, this is equal to

$$= [(n-3)(n+1)C_{n+1} - 2(1-x^2)C_n']/(n-1)$$

Moreover

$$\begin{aligned}
 &-(n-1)s C_n + (1-s^2)C_n' = -(n+1)C_{n+1} \\
 &-2(1-s^2)C_n' = -2(n-1)s C_n + 2(n+1)C_{n+1}
 \end{aligned}$$

$$\begin{aligned}
 &= [(n-3)(n+1)C_{n+1} - 2(n-1)x C_n + 2(n+1)C_{n+1}]/(n-1) \\
 &= [(n-3+2)(n+1)C_{n+1} - 2(n-1)x C_n]/(n-1) \\
 &= (n+1)C_{n+1} - 2x C_n ==> F_n^{(6)}
 \end{aligned}$$