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Macroscopic Motion Analysis of Two Interacting Bio Molecular Particles.

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A mathematical method was proposed for interacting two spherical bio molecular particles in bio chemical reactions which was introduced by Wakiya (1967). The bi-spherical coordinates for slow motion of a viscous fluid was extended to the general flow with plane symmetry around two interacting sphere obstacles. The creeping flow (stokes) equations with the equation of continuity was solved by setting adequate boundary conditions. Rigorous mathematical treatment utilized recurrent formulas of Legendre bi functions and series expansions of the products of hyperbolic function and Legendre functions. The present work originated by Wakiya is available for analyzing interacting bio molecular particles in creeping flow.

Two interaction bio molecular particles. Stokes flow. Legendre function. bi spherical coordinates.

相互に干渉し合う生体分子の挙動解析

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生体の分子レベルでの反応における生体分子の球状粒子が互いに相互干渉しあう場合の数学的解析方法を紹介する。原方法は脇屋（１９６７）によって提唱されている。２球座標系を遅い粘性流体の流れに適用した。　ストークス式を連続式とを適切に境界条件を設定することで解析的に解くことが可能であった。厳密な数学的扱により解はレジェンドレの陪関数と双曲線関数との積の級数で表現できた。係数は境界条件とレジェンドレ関数の直交性から決定できた。　脇屋によって流体力学分野に紹介された研究も遅い流れにおける生体分子の相互干渉運動を分析するうえではきわめて重要な知研である。

相互干渉生体分子.　２球座標系.　ストークス式.　連続式.　レジェンドレ陪関数

1. Introduction

Bio molecular interaction plays the essential part of biological reaction on molecular level. The present paper introduces a method for interaction analysis on a pair particles in the same phase space in a limited laminar shear flow that has been proposed by Wakiya.

2. Wakiya's Mathematical method.

1.The fluid motion equations

$$1/\mu \ \partial p/\partial r = (\nabla^2 - 1/r^2)\, v_r - 2/r^2 \ \partial v_\phi/\partial \phi$$

$$1/\mu \ \partial p/(r\,\partial \phi) = (\nabla^2 - 1/r^2)\, v_\phi + 2/r^2\, \partial v_r/\partial \phi$$

$$1/\mu \ \partial p/\partial z = \nabla^2 v_z \quad \text{and}$$

$$(\partial/\partial r + 1/r)\, v_r + \partial v_\phi/(r\,\partial \phi) + \partial v_z/\partial z = 0 \quad \text{----(1)}$$

$$\nabla^2 = \partial^2/\partial r^2 + 1/r\ \partial/\partial r + 1/r^2\ \partial^2/\partial \phi^2 + \partial^2/\partial z^2$$

where (vr, vϕ, vz) are fluid velocity. The solutions are

$$p = \mu/c\ \Sigma\, Q_m \cos(m\phi), \quad v_\phi = 1/2\,\Sigma(u_m - v_m)\sin(m\phi)$$

$$v_r = 1/2\,\Sigma(r/c\, Q_m + u_m + v_m)\cos(m\phi)$$

$$v_z = 1/2\,\Sigma(z/c\, Q_m + 2w_m)\cos(m\phi) \quad \text{--------------(2)}$$

where c is constant length and

$$L^2{}_m Q_m = L^2{}_m w_m = L^2{}_{m-1} v_m = L^2{}_{m+1} u_m = 0 \quad \text{----(3)}$$

$$L_1{}^2 = \partial^2/\partial r^2 + 1/r\ \partial/\partial r - l^2/r^2 + \partial^2/\partial z^2 \quad \text{---(4)}$$

(l = m-1, m+1 and m).The equation of continuity is

$$(3 + r\,\partial/\partial r + z\,\partial/\partial z)\, Q_m + c\{(\partial/\partial r + (m+1)/r)\, u_m + (\partial/\partial r - (m-1)/r)\, v_m + 2\,\partial w_m/\partial z\} = 0 \quad \text{-----(5)}$$

For two spheres which centers are on the z- axis at z = da and z = - d_b. We set bi-spherical coordinates (ξ, η, ϕ).

$$r = c\sin(\eta)/(\cosh\xi - \cos\eta) \quad \text{and}$$

$$z = c\sinh\xi/(\cosh\xi - \cos\eta) \quad \text{----------------(6)}$$

The sphere is $\xi = \alpha$, $\xi = -\beta$. Their radii are

$$a = c\,\mathrm{cosech}\ \alpha \quad b = c\,\mathrm{cosech}\ \beta \quad \text{---- ------(7)}$$

$$d_a = c\coth\alpha \quad d_b = c\coth\beta \quad \text{-----------(8)}$$

Equation (3) are satisfied by

$$w_m = (\cosh\xi - t)^{1/2}\ \Sigma\{A_n{}^m \sinh((n+1/2)\xi) + B_n{}^m \cosh((n+1/2)\xi)\}\, P_n{}^m(t)$$

$$Q_m = (\cosh\xi - t)^{1/2}\ \Sigma\{C_n{}^m \cosh((n+1/2)\xi) + D_n{}^m \sinh((n+1/2)\xi)\}\, P_n{}^m(t)$$

$$v_m = (\cosh\xi - t)^{1/2}\ \Sigma\{E_n{}^m \cosh((n+1/2)\xi) + F_n{}^m \sinh((n+1/2)\xi)\}\, P_n{}^{m-1}(t)$$

$$u_m = (\cosh\xi - t)^{1/2}\ \Sigma\{G_n{}^m \cosh((n+1/2)\xi) + H_n{}^m \sinh((n+1/2)\xi)\}\, P_n{}^{m+1}(t) \quad \text{---------(9)}$$

where $t = \cos\eta$. The conservation law (5) requires

$$5\, C_n{}^m - (n-m)\, C_{n-1}{}^m + (n+m+1)\, C_{n+1}{}^m + 2\, E_n{}^m - E_{n-1}{}^m - E_{n+1}{}^m - 2(n-m)(n+m+1)\, G_n{}^m + (n-m-1)(n-m)\, G_{n-1}{}^m + (n+m+1)(n+m+2)\, G_{n+1}{}^m + 2(2n+1)\, A_n{}^m - 2(n-m)\, A_{n-1}{}^m - 2(n+m+1)\, A_{n+1}{}^m = 0 \quad \text{----(10)}$$

2. The forces and Couple coefficients.

Cartesian components of the force exerted by the fluid on the sphere α is

$$D_z = -\sqrt{2}\,\mu\,\pi c\ \Sigma\{2(A_n{}^0 + B_n{}^0) + (2n+1)(C_n{}^0 + D_n{}^0)\}$$

$$D_x = -\sqrt{2}\,\mu\,\pi c\ \Sigma\{n(n+1)(C_n{}^1 + D_n{}^1) + (E_n{}^1 + F_n{}^1)\}$$

$$C_y = \sqrt{2}\,\mu\,\pi c^2\ \Sigma\{2n(n+1)(A_n{}^1 + B_n{}^1) - (2n+1)(E_n{}^1 + F_n{}^1)\} - D_x d_a \quad \text{-----(11)}$$

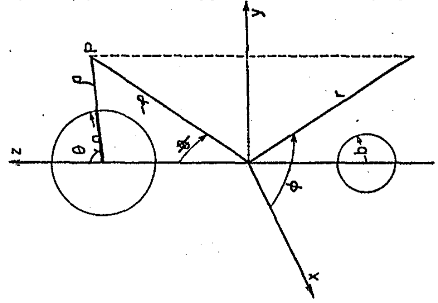


Fig. 1. Coordinate systems employed.

we have

$$\Sigma[2A_n - (2n+1)C_n] = \Sigma[2B_n - (2n+1)D_n] = 0$$

$$\Sigma[n(n+1)C_n - E_n] = \Sigma[n(n+1)D_n - F_n] = 0$$

$$\Sigma[2n(n+1)A_n + (2n+1)E_n] = \Sigma[2n(n+1)B_n + (2n+1)F_n] = 0 \quad \text{------------}$$

§ 5. Fixed Spheres in Laminar Shear Flow

As another problem, two equal-sized spheres in a laminar shear flow are considered. The spheres are supposed to be fixed in space. If they have translational or angular velocities, the solutions for those cases, such as mentioned in § 4, can be simply superposed. The original flow is assumed as

$$v_{x'} = Gz' \quad (G\text{: constant}), \quad v_y = v_{z'} = 0, \quad (25)$$

referred to Cartesian coordinates (x', y', z'), the origin and the y-axis being in common with the coordinates (x, y, z). Let χ be the angle between the z'- and z-axis, then the components of $v_{x'}$ in the cylindrical coordinates are

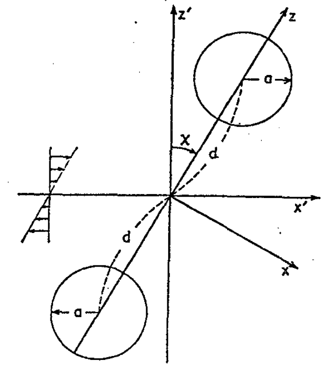


Fig. 2. Equal-sized spheres in laminar shear flow.

$$\left.\begin{aligned} v_r' &= Gz'\cos\chi\cos\phi, & v_\phi' &= -Gz'\cos\chi\sin\phi, \\ v_z' &= Gz'\sin\chi; & z' &= z\cos\chi - r\sin\chi\cos\phi. \end{aligned}\right\}$$

Hence, the boundary conditions give

$$\left.\begin{aligned} Q_0 &= -cG\sin 2\chi - 2\frac{c}{z}w_0, & \frac{1}{2}(u_0 + v_0) &= \frac{3}{4}G\sin 2\chi + \frac{r}{z}w_0, \\ Q_1 &= 2(cG\sin^2\chi)\frac{r}{z} - 2\frac{c}{z}w_1, & u_1 &= -(G\sin^2\chi)\frac{r^2}{z} + \frac{r}{z}w_1, \\ v_1 &= -(G\sin^2\chi)\frac{r^2}{z} - 2(G\cos^2\chi)z + \frac{r}{z}w_1, & v_2 &= \frac{1}{2}(G\sin 2\chi)r + \frac{r}{z}w_2 \end{aligned}\right\}$$

$$Q2 = -2\,c\,w2/z, \quad v2 = \gamma\, w2/z$$

In the general solution (2), terms for $m \geq 3$ can be out of consideration.

The relations (27) yield

$$C_n{}^0 = -cG\sin 2\chi\cdot\lambda_n\,\mathrm{sech}\left(n+\frac{1}{2}\right)\alpha - 2h_n A_n{}^0 + 2(h_n - 1)n\frac{A^0_{n-1}}{2n-1} + 2(h_n + 1)(n+1)\frac{A^0_{n+1}}{2n+3} \quad (n \geq 0)$$

$$\frac{1}{2}\left(G_n{}^0 - \frac{E_n{}^0}{n(n+1)}\right) = \frac{3}{2}cG\sin 2\chi\cdot\lambda_n\,\mathrm{sech}\left(n+\frac{1}{2}\right)\alpha + (h_n - 1)\frac{A^0_{n-1}}{2n-1} - (h_n + 1)\frac{A^0_{n+1}}{2n+3} \quad (n \geq 1).$$

$$E_n{}^2 = cG\sin 2\chi\cdot\lambda_n\,\mathrm{sech}\left(n+\frac{1}{2}\right)\alpha \quad (n \geq 1), \quad (+(A^2_{n-1},\ A^2_{n+1})$$

$$B_n{}^0 = D_n{}^0 = F_n{}^0 = H_n{}^0 = B_n{}^2 = D_n{}^2 = F_n{}^2 = H_n{}^2 = 0,$$

h_n being defined in (21), and

$$D_n{}^1 = \frac{2cG}{\sinh\alpha}\sin^2\chi\left(\frac{\lambda_{n-1}}{2n-1} - \frac{\lambda_{n+1}}{2n+3}\right)\mathrm{cosech}\left(n+\frac{1}{2}\right)\alpha - 2k_n B_n{}^1 + 2(k_n - 1)(n-1)\frac{B^1_{n-1}}{2n-1} + 2(k_n + 1)(n+2)\frac{B^1_{n+1}}{2n+3} \quad (n \geq 1), \quad (31)$$

$$F_n{}^1 = \frac{2cG}{\sinh\alpha}\left[\sin^2\chi\left\{(n-1)n\frac{\lambda_{n-1}}{2n-1} - (n+1)(n+2)\frac{\lambda_{n+1}}{2n+3}\right\} - \cos^2\chi\left(n+\frac{1}{2}\right)(\lambda_{n-1} - \lambda_{n+1})\right]\mathrm{cosech}\left(n+\frac{1}{2}\right)\alpha - (k_n - 1)(n-1)n\frac{B^1_{n-1}}{2n-1} + (k_n + 1)(n+1)(n+2)\frac{B^1_{n+1}}{2n+3} \quad (n \geq 0), \quad (32)$$

$$H_n{}^1 = -\frac{2cG}{\sinh\alpha}\sin^2\chi\left(\frac{\lambda_{n-1}}{2n-1} - \frac{\lambda_{n+1}}{2n+3}\right)\mathrm{cosech}\left(n+\frac{1}{2}\right)\alpha + (k_n - 1)\frac{B^1_{n-1}}{2n-1} - (k_n + 1)\cdot\frac{B^1_{n+1}}{2n+3} \quad (n \geq 2). \quad (33)$$

$$k_n = \coth\left(n+\frac{1}{2}\right)\alpha\coth\alpha.$$

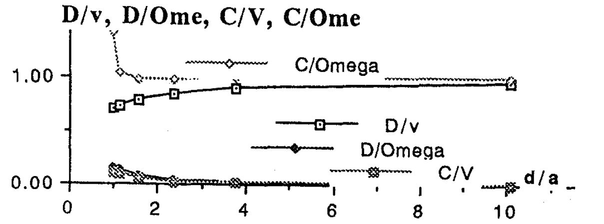
The equations for $A_n{}^0$ can be found to be

$$\begin{aligned} &\{(2n-1)(h_{n-1} - 1) - (2n-3)(h_n - 1)\}\, n\left(\frac{A^0_{n-1}}{2n-1} - \frac{A_n{}^0}{2n+1}\right) \\ &- \{(2n+5)(h_n + 1) - (2n+3)(h_{n+1} + 1)\}(n+1)\left(\frac{A_n{}^0}{2n+1} - \frac{A^0_{n+1}}{2n+3}\right) \\ &= \frac{1}{2}cG\sin 2\chi\left[n(3n-2)\left\{\lambda_n\,\mathrm{sech}\left(n+\frac{1}{2}\right)\alpha - \lambda_{n-1}\,\mathrm{sech}\left(n-\frac{1}{2}\right)\alpha\right\}\right. \\ &\left.+ (n+1)(3n+5)\left\{\lambda_n\,\mathrm{sech}\left(n+\frac{1}{2}\right)\alpha - \lambda_{n+1}\,\mathrm{sech}\left(n+\frac{3}{2}\right)\alpha\right\}\right] \quad (n \geq 0). \quad (35) \end{aligned}$$

The equations for $B_n{}^1$ is

$$(2n-1)(k_{n-1} - 1) - (2n-3)(k_n - 1)\}\left\{(n-1)\frac{B^1_{n-1}}{2n-1} - n\frac{B_n{}^1}{2n+1}\right\}$$

$$-\{(2n+5)(k_n+1)-(2n+3)(k_{n+1}+1)\}\left\{(n+1)\frac{B_n'}{2n+1}-(n+2)\frac{B_{n+1}'}{2n+3}\right\}$$
$$=\frac{cG}{\sinh\alpha}\left\langle \sin^2\chi\left[\left\{(n-1)\lambda_{n-2}-((n-1)^2+n(n+1))\frac{\lambda_n}{2n+1}\right\}\operatorname{cosech}\left(n-\frac{1}{2}\right)\alpha\right.\right.$$
$$-\left\{(4n^2+1)\frac{\lambda_{n-1}}{2n-1}-(4(n+1)^2+1)\frac{\lambda_{n+1}}{2n+3}\right\}\operatorname{cosech}\left(n+\frac{1}{2}\right)\alpha$$
$$\left.+\left\{((n+2)^2+n(n+1))\frac{\lambda_n}{2n+1}-(n+2)\lambda_{n+2}\right\}\operatorname{cosech}\left(n+\frac{3}{2}\right)\alpha\right]$$
$$-\cos^2\chi\left[\frac{1}{2}(2n-1)(\lambda_{n-2}-\lambda_n)\operatorname{cosech}\left(n-\frac{1}{2}\right)\alpha-(2n+1)(\lambda_{n-1}-\lambda_{n+1})\operatorname{cosech}\left(n+\frac{1}{2}\right)\alpha\right.$$
$$\left.\left.+\frac{1}{2}(2n+3)(\lambda_n-\lambda_{n+2})\operatorname{cosech}\left(n+\frac{3}{2}\right)\alpha\right]\right\rangle \quad (n\geq 1). \tag{36}$$



Results

Fig.3 shows the Force (D) and Couple (C) as functions of d/a: D/V and C/V when couple =0 and D/omeg, C/Omeg when V=0.

[Reference]

1. Wakiya S. J. Phy. Society. Japan. vol 22. No 4. p1101-1109. 1967.

APPENDIX.

0. Induction of equation (33).

For the left side, putting m=1 in (9),

$$u1 = f^{1/2} \sum_{n=2} Hn' \sinh(n+1/2)\alpha \;\; Pn^2$$

The right side is

$$- G \sin^2\chi \;\; c^2 \sin^2\eta \;\; (\cosh\alpha - \cos\eta) \,/\, [(\cosh\alpha - \cos\eta)^2 \; c \sin\alpha\,]$$
$$+ c \sin\eta\,(\cosh\alpha - \cos\eta)\, f^{1/2} \sum_{n=2} Bn' \cosh(n+1/2)\alpha \;\; Pn' \,/[(\cosh\alpha - \cos\eta)\;\; c \sin\alpha\,]$$

Therefore

$$\sum_{n=2} Hn' \sinh(n+1/2)\alpha \;\; Pn^2$$
$$= - G \sin^2\chi \;\; c \sin^2\eta \;\; /\,[f^{3/2} \sin\alpha\,]$$
$$+ \sin\eta / \sinh\alpha \;\; \sum_{n=2} Bn' \cosh(n+1/2)\alpha \;\; Pn'$$

We associate the right side by $Pn^2(\cos\eta)$

1. In the first term of the right side

$$(\cosh\alpha - \cos\eta)^{-1/2} = \sum \lambda n \;\; Pn^o(\cos\eta)$$

Differentiate with respect to η

$$(-1/2)(\cosh\alpha - \cos\eta)^{-1/2} \sin\eta = \sum \lambda n \;\; \partial Pn^o(t)/\partial t$$
$$* \partial t/\partial\eta = \sum \lambda \;\; \partial Pn^o(t)/\partial t\,(-\sin\eta)$$

Therefore

$$f^{-3/2} = 2 \sum \lambda n \;\; \partial Pn^o(t)/\partial t$$

Hence

$$\sin^2\eta \,/\, f^{3/2} = 2\sum \lambda n \;\; \sin^2\eta \;\; \partial Pn^o(t)/\partial t$$
$$= 2 \sum \lambda n\,(1 - t^2)\; \partial Pn^o(t)/\partial t$$

In the recurrent formula of Legendre function

$$(1 - t^2)\;\partial Pn^o(t)/\partial t = (1 - t^2)^{1/2} Pn^1 - 0\, t\, Pn(t)^0$$
$$= (1 - t^2)^{1/2} Pn^1$$
$$P_{n+1}{}^2 - P_{n-1}{}^2 = (2n+1)(1 - t^2)^{1/2} \;\; Pn(t)^1$$

hence

$$(1 - t^2)\;\partial Pn^o(t)/\partial t = (P_{n+1}{}^2 - P_{n-1}{}^2)/(2n+1)$$

Hence

$$\sin^2\eta \,/\, f^{3/2} = 2\sum \lambda n\,(1 - t^2)\;\partial Pn^o(t)/\partial t$$
$$= 2 \sum \lambda n\,(P_{n+1}{}^2 - P_{n-1}{}^2)/(2n+1)$$
$$= 2\,[\,\lambda_1 (P_2{}^2 - P_0{}^2)/3 + \lambda_2 (P_3{}^2 - P_1{}^2)/5 +$$
$$+ \lambda_3 (P_4{}^2 - P_2{}^2)/7 + \lambda_4 (P_5{}^2 - P_3{}^2)/9 +$$
$$+ \lambda_5 (P_6{}^2 - P_4{}^2)/11 + \lambda_6 (P_7{}^2 - P_5{}^2)/13 +$$

$$= P_2{}^2\,(\lambda_1/3 - \lambda_3/7) + P_3{}^2\,(\lambda_2/5 - \lambda_4/9) +$$
$$+ P_4{}^2\,(\lambda_3/7 - \lambda_5/11) + P_5{}^2\,(\lambda_4/9 - \lambda_6/13) +$$
$$+ P_6{}^2\,(\lambda_5/11 - \lambda_7/15) + P_7{}^2\,(\lambda_6/13 - \lambda_8/17) +$$
$$= \sum Pn^2\,(\lambda_{n-1}/(2n-1) - \lambda_{n+1}/(2n+3))$$

2. In the second term of the right side

$$\sin\eta/\sinh\alpha \;\; \sum_{n=2} Bn' \cosh(n+1/2)\alpha \;\; Pn'$$
$$= 1/\sinh\alpha \;\; \sum_{n=2} Bn' \cosh(n+1/2)\alpha \sin\eta \;\; Pn'$$

since

$$P_{n+1}{}^{\mu} - P_{n-1}{}^{\mu} = (2n+1)(1 - \cos\eta^{\,2})^{1/2} \;\; Pn^{\mu-1}$$
$$P_{n+1}{}^2 - P_{n-1}{}^2 = (2n+1)(\sin\eta)\;\; Pn^1$$
$$= 1/\sinh\alpha \;\; \sum_{n=2} Bn' \cosh(n+1/2)\alpha \;\; (P_{n+1}{}^2 - P_{n-1}{}^2)/(2n+1)$$

$$= 1/\sinh\alpha\,(\,B1' \cosh3/2\alpha \;\; (P_2{}^2 - P_0{}^2)/3 +$$
$$+ B2' \cosh5/2\alpha \;\; (P_3{}^2 - P_1{}^2)/5 + B_3'\cosh7/2\alpha \;\; (P_4{}^2 - P_2{}^2)/7$$
$$+ B_4'\cosh9/2\alpha \;\; (P_5{}^2 - P_3{}^2)/5 + B_5'\cosh11/2\alpha \;\; (P_6{}^2 - P_4{}^2)/7$$
$$+ B_6'\cosh13/2\alpha \;\; (P_7{}^2 - P_5{}^2)/5 + B_7'\cosh15/2\alpha \;\; (P_8{}^2 - P_6{}^2)/7$$
$$= 1/\sinh\alpha\,[\,P_2{}^2\,(B1'/3 \cosh(3/2\alpha) - B3'/7 \cosh(7/2\alpha))$$
$$+ P_3{}^2\,(B2'/5 \cosh(5/2\alpha) - B4'/9 \cosh(9/2\alpha))$$
$$+ P_4{}^2\,(B3'/7 \cosh(7/2\alpha) - B5'/11 \cosh(11/2\alpha))$$
$$+ P_5{}^2\,(B4'/9 \cosh(9/2\alpha) - B6'/13 \cosh(13/2\alpha))\,]$$
$$= 1/\sinh\alpha\,[\,\sum Pn^2\,(B_{n-1}'/(2n-1)\cosh(2n-1)\alpha/2 - B_{n+1}'/(2n+3)\;\cosh(2n+3)\alpha/2\,]$$
$$= 1/\sinh\alpha \;\; \sum Pn^2\,[\,B_{n-1}'/(2n-1)\,(\cosh(n/2+1)\alpha * \cosh\alpha - \sinh(n/2+1)\alpha * \sinh\alpha)$$
$$- B_{n+1}'/(2n+3)(\cosh(n/2+1)\alpha * \cosh\alpha + \sinh(n/2+1)\alpha * \sinh\alpha)]$$

$$= \sum Pn^2\,[\,B_{n-1}'/(2n-1)\,(\cosh(n/2+1)\alpha * \coth\alpha - \sinh(n/2+1)\alpha)$$
$$- B_{n+1}'/(2n+3)(\cosh(n/2+1)\alpha * \coth\alpha + \sinh(n/2+1)\alpha)]$$
$$= \sum Pn^2 \sinh(n/2+1)\alpha\;[\,B_{n-1}'/(2n-1)\,(\cosh(n/2+1)\alpha * \coth\alpha \,/\, \sinh(n/2+1)\alpha - 1)$$
$$- B_{n+1}'/(2n+3)(\cosh(n/2+1)\alpha * \coth\alpha / \sinh(n/2+1)\alpha + 1)]$$

Putting

$$kn = \cosh(n/2+1)\alpha * \coth\alpha \,/\, \sinh(n/2+1)\alpha$$
$$= \coth(n/2+1)\alpha * \coth\alpha$$
$$= \sum Pn^2 \sinh(n/2+1)\alpha\;[\,B_{n-1}'/(2n-1)(kn - 1) - B_{n+1}'/(2n+3)(kn+1)\,]$$

Hence, totally, we have

$$\sum_{n=2} Hn' \sinh(n+1/2)\alpha \;\; Pn^2$$
$$= - 2\, c\, G\sin^2\chi \;\; /\sinh\alpha \;\; \sum Pn^2\,[\,\lambda_{n-1}/(2n-1) - \lambda_{n+1}/(2n+3)\,]$$
$$+ \sum Pn^2 \sinh(n/2+1)\alpha\;[\,B_{n-1}'/(2n-1)(kn - 1) - B_{n+1}'/(2n+3)(kn+1)\,]$$

Therefore,

$$Hn' = - 2\, c\, G\sin^2\chi \;\; /\sinh\alpha \;\; [\,\lambda_{n-1}/(2n-1) - \lambda_{n+1}/(2n+3)\,]\operatorname{cosech}(n+1/2)\alpha$$
$$+ [\,B_{n-1}'/(2n-1)(kn - 1) - B_{n+1}'/(2n+3)(kn+1)\,]$$

1. Induction of Equatrion (31).

From the boundary condition of

$$Q_1 = 2cG\sin^2 x\frac{r}{z} - 2\frac{C}{z}w_1$$

In the general form of the solution

Qm = (cosh ζ - t)1/2 Σ{ Cnm cosh(n+1/2) ζ
+ Dnm sinh(n+1/2) ζ } Pnm(t) -----(9-b)

Putting $m = 1$, and setting
$f = \cosh\alpha - \cos\eta$, we have

$$Q_1 = f^{\frac{1}{2}}\sum_{n=1} D_n^1 \sinh\left(n+\frac{1}{2}\right)\alpha\, P_n^1$$

The right side is

$$2cG\sin^2\frac{c\sin\eta}{\cosh\xi - \cos\eta}\;\frac{\cosh\xi\cos\eta}{c\sinh\xi}$$

$$-2\frac{c(\cosh\xi - \cos\eta)}{c\sinh\xi}f^{\frac{1}{2}}\sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\, P_n^1$$

Therefore

$$f^{\frac{1}{2}}\sum_{n=1} D_n^1\sinh\left(n+\frac{1}{2}\right)\alpha\, P_n^1$$

$$= 2cG\sin^2 x\frac{\sin\eta}{\sinh\alpha} - 2\frac{f\cdot f^{\frac{1}{2}}}{\sinh\alpha}\sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\, P_n^1$$

Eliminate $f^{\frac{1}{2}}$ from the both sides

$$\sum_{n=1} D_n^1\sinh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

$$= 2cG\sin^2 x\frac{\sin\eta}{\sinh\alpha}f^{-\frac{1}{2}}$$

$$-2\frac{(\cosh\alpha - \cos\eta)}{\sinh\alpha}\sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\, P_n^1$$

while,w e have

$$f^{-\frac{1}{2}} = \sum_{n=0}\lambda_n\cdot P_n^0(t)$$

Thus, the right side of above equation is

$$= 2cG\sin^2 x\frac{\sin\eta}{\sinh\alpha}\sum_{n=0}\lambda_n\cdot P_n^0(t)$$

$$-2\coth\alpha\sum_{n=1} B_n^1\coth\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

$$+2\frac{\cos\eta}{\sinh\alpha}\sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

① The third term in the above equation is modifeid by using the recurrent formula of Legendre equation

$$(n-\mu+1)P_{n+1}^{\mu} - (2n+1)xP_n^{\mu}$$
$$+(n+\mu)P_{n-1}^{\mu} = 0$$

Putting

$\mu = 0$ and $x = \cos\eta$, above formula is

$$\cos\eta\cdot P_n^1 = \frac{n\cdot P_{n+1}^1 + (n+1)P_{n-1}^1}{(2n+1)}$$

Applying this

$$\cos\eta\sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

$$= \sum_{n=1} B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\left[\frac{nP_{n+1}^1 + (n+1)P_{n-1}^1}{2n+1}\right]$$

Expand this by the series as

$$= B_1^1\cosh\frac{3}{2}\alpha\cdot\left(\frac{1}{3}P_2^1 + \frac{2}{3}P_0^1\right)$$
$$+ B_2^1\cosh\frac{5}{2}\alpha\cdot\left(\frac{2}{5}P_3^1 + \frac{3}{5}P_1^1\right)$$
$$= B_3^1\cosh\frac{7}{2}\alpha\cdot\left(\frac{7}{3}P_4^1 + \frac{7}{4}P_2^1\right)$$
$$+ B_4^1\cosh\frac{9}{2}\alpha\cdot\left(\frac{4}{9}P_5^1 + \frac{5}{9}P_3^1\right)$$

Rearrange with respect to Pnm

$$= P_2^1\left(\frac{1}{3}B_1^1\cosh\frac{3}{2}\alpha + \frac{4}{7}B_3^1\cosh\frac{7}{2}\alpha\right)$$
$$+ P_3^1\left(\frac{2}{5}B_2^1\cosh\frac{5}{2}\alpha + \frac{5}{9}B_4^1\cosh\frac{9}{2}\alpha\right)$$
$$+ P_4^1\left(\frac{3}{7}B_3^1\cosh\frac{7}{2}\alpha + \frac{6}{11}B_5^1\cosh\frac{11}{2}\alpha\right)$$
$$= \sum P_n^1\left(\frac{(n-1)}{(2n-1)}B_{n-1}^1\cosh\frac{(2n-1)}{2}\alpha\right.$$
$$\left. + \frac{(n+2)}{(2n+3)}B_{n+1}^1\cosh\frac{(2n+3)}{2}\alpha\right)$$

We expand the hyperbolic functions as

$$\cosh\frac{(2n-1)}{2}\alpha = \cosh\left(n+\frac{1}{2}-1\right)\alpha$$
$$= \cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha - \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha$$
$$\cosh\frac{(2n+3)}{2}\alpha = \cosh\left(n+\frac{1}{2}+1\right)\alpha$$
$$= \cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha + \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha$$

Hence, the third term is

$$\frac{\cos\eta}{\sinh\alpha}\sum_{n=1} B_n^1\quad\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$
$$= P_n^1\left[\frac{(n-1)}{(2n-1)}B_{n-1}^1\left(\cosh\left(n+\frac{1}{2}\right)\alpha\coth\alpha - \sinh\left(n+\frac{1}{2}\right)\alpha\right)\right.$$
$$\left. + \frac{(n+2)}{(2n+3)}B_{n+1}^1\left(\cosh\left(n+\frac{1}{2}\right)\alpha\coth\alpha + \sinh\left(n+\frac{1}{2}\right)\alpha\right)\right]$$

Since the left side of the Q1 is a function of series of

$$\sinh\left(n+\frac{1}{2}\right)\alpha$$

Hence, we factor out

$$\sinh\left(n+\frac{1}{2}\right)\alpha$$

$$\sum P_n^1\sinh\left(n+\frac{1}{2}\right)\alpha\cdot\left[\frac{(n-1)}{(2n-1)}B_{n-1}^1\left(\frac{\cosh\left(n+\frac{1}{2}\right)\alpha}{\sinh\left(n+\frac{1}{2}\right)\alpha}\coth\alpha - 1\right)\right.$$
$$\left. + \frac{(n+2)}{(2n+3)}B_{n+1}^1\left(\frac{\cosh\left(n+\frac{1}{2}\right)\alpha}{\sinh\left(n+\frac{1}{2}\right)\alpha}\coth\alpha + 1\right)\right]$$

Putting

$$k_n = \coth\left(n+\frac{1}{2}\right)\alpha\cdot\coth\alpha$$

Then,

$$= \sum P_n^1\sinh\left(n+\frac{1}{2}\right)\alpha\left[\frac{(n-1)}{(2n-1)}B_{n-1}^1(k_n - 1)\right.$$
$$\left. + \frac{(n+2)}{(2n+3)}B_{n+1}^1(k_n + 1)\right]$$

②The second term in the right can be expressed by using

$$k_n = \coth\left(n+\frac{1}{2}\right)\alpha\cdot\coth\alpha$$

Hence

$$-2\sum k_n\cdot B_n^1\cdot P_n^1$$

③ The first term is given in terms of $P_n^0(t)$. We expand this by the series of $P_n^1(t)$. From the recurrent formula of Legends function, we have

$$P_{n+1}^{\mu} - P_{n-1}^{\mu} = (2n+1)\sqrt{1-x^2}\,P_n^{\mu-1}(x)$$

Putting

$x = \cos\eta$ and

$$\sqrt{1-x^2} = \sqrt{1-\cos^2\eta} = \sin\eta$$

Putting $\mu = 1$, we have

$$\sin\eta\cdot P_n^0(x) = \frac{P_{n+1}^1 - P_{n-1}^1}{2n+1}$$

Therefore

$$\sin\eta\cdot\sum_{n=0}\lambda_n\cdot P_n^0(t)\quad = \sum\lambda_n\cdot\frac{(P_{n+1}^1 - P_{n-1}^1)}{2n+1}$$
$$= \frac{\lambda_1}{3}(P_2^1 - P_0^1) + \frac{\lambda_2}{5}(P_3^1 - P_1^1) + \frac{\lambda_3}{7}(P_4^1 - P_2^1)$$
$$+ \frac{\lambda_4}{9}(P_5^1 - P_3^1) + \frac{\lambda_5}{11}(P_6^1 - P_4^1) + \frac{\lambda_6}{13}(P_7^1 - P_5^1)$$
$$= P_2^1\left(\frac{1}{3}\lambda_1 - \frac{1}{7}\lambda_3\right) + P_3^1\left(\frac{1}{5}\lambda_2 - \frac{1}{9}\lambda_4\right)$$
$$+ P_4^1\left(\frac{1}{7}\lambda_3 - \frac{1}{11}\lambda_5\right) + P_5^1\left(\frac{1}{9}\lambda_4 - \frac{1}{13}\lambda_6\right)$$
$$= \sum P_n^1\left(\frac{\lambda_{n-1}}{2n-1} - \frac{\lambda_{n+1}}{2n+3}\right)$$

As a result, the first term is obtained by eliminating

$$\sinh\left(n+\frac{1}{2}\right)\alpha$$

as

$$2cG\sin^2 x\frac{1}{\sinh\alpha}\sum P_n^1\left(\frac{\lambda_{n-1}}{2n-1} - \frac{\lambda_{n+1}}{2n+3}\right)\mathrm{cosinh}\left(n+\frac{1}{2}\right)\alpha$$

2. Induction of Equation (32)

From the boundary condition (27)

$$v_1 = -G\sin^2 x\frac{r^2}{z} - 2G\cos^2 x\cdot z + \frac{r}{z}w_1$$

In the general form of the solution

vm = (cosh ζ - t)1/2 Σ{ Enm cosh(n+1/2) ζ + Fnm sinh(n+1/2) ζ } Pn,m-1 (t) -----(9-c)

Setting m = 1 , ew have

$$v_1 = f^{\frac{1}{2}}\sum_{n=0} F_n^1\sinh\left(n+\frac{1}{2}\right)\alpha P_n^0$$

The right side can be

$$-G\sin^2 x\frac{(c\sin\eta)^2}{f^2}\frac{f}{c\sinh\alpha} - 2G\cos^2 x\frac{c\sinh\alpha}{f}$$

$$+\frac{\sin\eta}{\sinh\alpha}f^{\frac{1}{2}}\sum B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

Hence, we eliminate $f^{\frac{1}{2}}$ from the both sides and associate the right side by $P_n^0(t)$

The right side is

$$= -cG\frac{\sin^2 x\cdot\sin^2\eta}{\sinh\alpha}\frac{1}{f^{\frac{3}{2}}} - 2cG\cos^2 x\frac{\sinh\alpha}{f^{\frac{3}{2}}}$$

$$+\frac{\sin\eta}{\sinh\alpha}\sum B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

We convert the third term of the right by $P_n^1 \rightarrow P_n^0$

Putting

$\cos\eta = t$ and

$$\sin\eta P_n^1(\cos\eta) = \sqrt{1-t^2}\,P_n^1(t)$$

From the current formula of legends function putting $\mu = 0$

$$n\cos\eta P_n^0(t) - nP_{n-1}^0(t) = -\sqrt{1-t^2}\cdot P_n^1(t) \quad \text{— (e)}$$

More over

$$(n+1)P_{n+1}^0 - (n+1)\cos\eta P_n^0 = -\sqrt{1-t^2}P_n^1(t) \quad \text{\_\_(a)}$$

Putting right sides of these formulae, we have

$$n\cos\eta P_n^0 - nP_{n-1}^0 = (n+1)P_{n+1}^0 - (n+1)\cos\eta P_n^0$$

$$(2n+1)\cos\eta P_n^0 = (n+1)P_{n+1}^0 + nP_{n-1}^0$$

Hence,

$$\cos\eta P_n^0 = \frac{(n+1)P_{n+1}^0 + nP_{n-1}^0}{(2n+1)}$$

Substituting this to (e)

$$-\sin\eta P_n^1 = n\cos\eta P_n^0 - nP_{n-1}^0$$

$$= \frac{n(n+1)P_{n+1}^0 - n(n+1)P_{n-1}^0}{2n+1}$$

Hence,

$$\sum B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot\frac{[n(n+1)P_{n-1}^0 - n(n+1)P_{n+1}^0]}{2n+1}$$

$$= B_1^1\cosh\frac{3}{2}\alpha\cdot\left[\frac{1\cdot 2\cdot P_0^0}{3} - \frac{1\cdot 2\cdot P_2^0}{3}\right]$$

$$+ B_2^1\cosh\frac{5}{2}\alpha\cdot\left[\frac{2\cdot 3\cdot P_1^0}{5} - \frac{2\cdot 3\cdot P_3^0}{5}\right]$$

$$= P_2^0\left[\frac{3\cdot 4}{7}B_3^1\cosh\frac{7}{2}\alpha - \frac{1\cdot 2}{3}B_1^1\cosh\frac{3}{2}\alpha\right]$$

$$+ P_3^0\left[\frac{4\cdot 5}{9}B_4^1\cosh\frac{9}{2}\alpha - \frac{2\cdot 3}{5}B_2^1\cosh\frac{5}{2}\alpha\right]$$

$$= P_n^0\left[\frac{(n+1)(n+2)}{(2n+3)}B_{n+1}^1\cosh\frac{(2n+3)}{2}\alpha - \frac{(n-1)n}{(2n-1)}B_{n-1}^1\cosh\frac{(2n-1)}{2}\alpha\right]$$

The hyperbolic function can be decomposed to

$$\cosh\frac{(2n+3)}{2}\alpha = \cosh\left(n+\frac{1}{2}+1\right)\alpha$$

$$= \cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha + \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha$$

$$\cosh\frac{(2n-1)}{2}\alpha$$

$$= \cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha - \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha$$

Since, the left side is a series function of $\sinh\left(n+\frac{1}{2}\right)\alpha$ we have

$$\frac{\sin\eta}{\sinh\alpha}\sum B_n^1\cosh\left(n+\frac{1}{2}\right)\alpha\cdot P_n^1$$

$$= \frac{1}{\sinh\alpha}\sum P_n^0\left[\frac{(n+1)(n+2)}{(2n+3)}B_{n+1}^1\right.$$

$$\cdot\left(\cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha + \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha\right)$$

$$-\frac{(n-1)n}{(2n-1)}B_{n-1}^1$$

$$\left.\cdot\left(\cosh\left(n+\frac{1}{2}\right)\alpha\cosh\alpha - \sinh\left(n+\frac{1}{2}\right)\alpha\sinh\alpha\right)\right]$$

$$= \sum P_n^0\cdot\sinh\left(n+\frac{1}{2}\right)\alpha$$

$$\left[\frac{(n+1)(n+2)}{(2n+3)}B_{n+1}^1\left(\frac{\cosh\left(n+\frac{1}{2}\right)\alpha}{\sinh\left(n+\frac{1}{2}\right)\alpha}\cdot\frac{\cosh\alpha}{\sinh\alpha}+1\right)\right.$$

$$\left.-\frac{(n-1)n}{(2n-1)}B_{n-1}^1\left(\coth\left(n+\frac{1}{2}\right)\alpha\cdot\coth\alpha - 1\right)\right]$$

$$= \sum P_n^0\sinh\left(n+\frac{1}{2}\right)\alpha\left[\frac{(n+1)(n+2)}{(2n+3)}B_{n+1}^1(k_n+1)\right.$$

$$\left.-\frac{(n-1)n}{(2n-1)}B_{n-1}^1(k_n-1)\right]$$

About the first term

$$\frac{\sin^2\eta}{\sinh\alpha\cdot f^{\frac{3}{2}}}$$

From the series expansion of

$$(\cosh\alpha - \cos\eta)^{-\frac{1}{2}} = \sum\lambda_n\cdot P_n^0(\cos\eta)$$

$$\lambda n = 2^{1/2}\exp(-(n+1/2)\alpha)$$

Differntiate both sides by η

$$\left(-\frac{1}{2}\right)(\cosh\alpha - \cos\eta)^{-\frac{3}{2}}\sin\eta = \sum\lambda_n\cdot\frac{\partial P_n^0}{\partial\eta}(\cos\eta)$$

$$= \sum\lambda_n\cdot\frac{\partial P_n^0(\cos\eta)}{\partial(\cos\eta)}\frac{\partial(\cos\eta)}{\partial\eta} = -\sin\eta\sum\lambda_n\cdot\frac{\partial P_n^0(t)}{\partial t}$$

Therefore

$$\frac{\sin\eta}{f^{\frac{3}{2}}} = 2\sin\eta\sum\lambda_n\cdot\frac{\partial P_n^0}{\partial t}$$

Hence

$$\frac{1}{\sinh\alpha}\frac{\sin^2\eta}{f^{\frac{3}{2}}} = \frac{1}{\sinh\alpha}2\sin^2\eta\sum\lambda_n\cdot\frac{\partial P_n^0}{\partial t}$$

About the term of $\partial P_n^0(t)/\partial t$ From the recurrent formula of Legendre function

$$(1-x^2)\frac{\partial P_n^\mu}{\partial x} = \sqrt{1-x^2}\cdot P_n^{\mu+1} - \mu\cdot x\cdot P_n^\mu$$

Putting $\mu = 0$

$$(1-x^2)\frac{\partial P_n^0}{\partial x} = \sqrt{1-x^2}\cdot P_n^1$$

Set $\cos\eta = x = t$ then

$$\sin^2\eta\frac{\partial P_n^0}{\partial x} = \sqrt{1-x^2}\cdot P_n^1 = \sin\eta\cdot P_n^1$$

Therefore

$$\frac{1}{\sinh\alpha}\frac{\sin^2\eta}{f^{\frac{3}{2}}} = \frac{2}{\sinh\alpha}\sum\lambda_n\cdot\sin\eta\cdot P_n^1$$

Using the relation as before

$$-\sin\eta\cdot P_n^1 = \frac{n(n+1)P_{n+1}^0 - n(n+1)P_{n-1}^0}{2n+1}$$

Hence

$$\frac{1}{\sinh\alpha}\frac{\sin^2\eta}{f^{\frac{3}{2}}}$$

$$= \frac{2}{\sinh\alpha}\sum\lambda_n\cdot\frac{[n(n+1)P_{n-1}^0 - n(n+1)P_{n+1}^0]}{2n+1}$$

This can be modified as we have induced the third term by setting $B_n \rightarrow \lambda_n$

$$= \frac{2}{\sinh\alpha}\sum P_n^0\left[\frac{(n-1)n}{2n-1}\lambda_{n-1} - \frac{(n+1)(n+2)}{(2n+3)}\lambda_{n+1}\right]$$

About the second term

$$-2cG\cos^2 x\cdot\frac{1}{\sinh\alpha}\,\frac{\sinh^2\alpha}{f^{\frac{3}{2}}}$$

From the series expansion of

$$(\cosh\alpha-\cos\eta)^{-\frac{1}{2}}=\sum\lambda_n P_n^0(\cos\eta)$$

$$=\sum\sqrt{2}\,e^{-(n+\frac{1}{2})\alpha}\cdot P_n^0(\cos\eta)$$

Differentiate both sides by α

$$(-\tfrac{1}{2})(\cosh\alpha-\cos\eta)^{-\frac{3}{2}}\cdot\sinh\alpha$$

$$=\sum\sqrt{2}(-n-\tfrac{1}{2})e^{-(n+\frac{1}{2})\alpha}\cdot P_n^0(\cos\eta)$$

$$\frac{\sinh^2\alpha}{f^{\frac{3}{2}}}=\sum(2n+1)\lambda_n\cdot P_n^0\sinh\alpha$$

$$=\sqrt{2}\sum(2n+1)e^{-(n+\frac{1}{2})\alpha}\sinh\alpha\cdot P_n$$

$$=\sqrt{2}\sum(2n+1)e^{-(n+\frac{1}{2})\alpha}\frac{(e^{\alpha}-e^{-\alpha})}{2}\cdot P_n$$

$$=\sqrt{2}\sum(n+\tfrac{1}{2})(e^{-(n-\frac{1}{2})\alpha}-e^{-(n+\frac{3}{2})\alpha})\cdot P_n$$

$$\lambda_n=\sqrt{2}\,e^{-(n+\frac{1}{2})\alpha}\qquad\lambda_{n-1}=\sqrt{2}\,e^{-(n-\frac{1}{2})\alpha}$$

$$\lambda_{n+1}=\sqrt{2}\,e^{-(n+\frac{3}{2})\alpha}$$

$$=\sum(n+\tfrac{1}{2})(\lambda_{n-1}-\lambda_{n+1})\cdot P_n$$

3. Induction of Equation (35)

From the equation (29)

$$G_n^0=\frac{E_n^0}{n(n+1)}+3cG\cdot\sin 2x\cdot\lambda_n\cdot\mathrm{sech}(n+\tfrac{1}{2})\alpha$$

$$+(h_n-1)\frac{2}{(2n-1)}A_{n-1}^0-(h_n+1)\frac{2}{(2n+3)}A_{n+1}^0$$

Setting n-1 instead of n

$$G_{n-1}^0=\frac{E_{n-1}^0}{(n-1)n}+3cG\cdot\sin 2x\cdot\lambda_{n-1}\cdot\mathrm{sech}(n-\tfrac{1}{2})\alpha$$

$$+(h_{n-1}-1)\frac{2}{(2n-3)}A_{n-2}^0-(h_{n-1}+1)\frac{2}{(2n+1)}A_n^0$$

Setting n+1 instead of n

$$G_{n+1}^0=\frac{E_{n+1}^0}{(n+1)(n+2)}+3cG\cdot\sin 2x\cdot\lambda_{n+1}\cdot\mathrm{sech}(n+\tfrac{3}{2})\alpha$$

$$+(h_{n+1}-1)\frac{2}{(2n+1)}A_n^0-(h_{n+1}+1)\frac{2}{(2n+5)}A_{n+2}^0$$

From the equation of continuity

5 Cnm - (n-m) Cn-1,m + (n+m+1) Cn+1,m
+ 2 Enm - En-1 m - En+1 m - 2 (n-m)(n+m+1) Gnm
+(n-m-1)(n-m) Gn-1,m + (n+m+1)(n+m+2)Gn+1,m
+2(2n+1)Anm -2(n-m) An-1,m -2(n+m+1)An+1,m =0

Setting m = 0

$$5C_n^0-nC_{n-1}^0+(n+1)C_{n+1}^0+2E_n^0-E_{n-1}^0-E_{n+1}^0$$

$$-2n(n+1)G_n^0+(n-1)nG_{n-1}^0+(n+1)(n+2)G_{n+1}^0$$

$$+2(2n+1)A_n^0-2nA_{n-1}^0-2(n+1)A_{n+1}^0=0$$

Firstyl, we substitute G_n G_{n-1}, G_{n+1}

$$-2\Big[E_n^0+3cG\cdot\sin 2x\cdot\lambda_n\cdot\mathrm{sech}(n+\tfrac{1}{2})\alpha\cdot n(n+1)$$

$$+\frac{(h_n-1)2\cdot n(n+1)}{(2n-1)}A_{n-1}^0$$

$$-\frac{(h_n+1)2\cdot(n+1)n}{(2n+3)}A_{n+1}^0\Big]$$

$$+\Big[E_{n-1}^0+3cG\cdot\sin 2x\cdot\lambda_{n-1}\cdot\mathrm{sech}(n-\tfrac{1}{2})\alpha\cdot(n-1)n$$

$$+\frac{(h_{n-1}-1)2\cdot(n-1)n}{(2n-3)}A_{n-2}^0$$

$$-\frac{(h_{n-1}+1)2\cdot(n-1)n}{(2n+1)}A_n^0\Big]$$

$$+\Big[E_{n+1}^0+3cG\cdot\sin 2x\cdot\lambda_{n+1}\cdot\mathrm{sech}(n+\tfrac{3}{2})\alpha\cdot(n+1)(n+2)$$

$$+\frac{(h_{n+1}-1)\cdot 2(n+1)(n+2)}{(2n+1)}A_n^0$$

$$-\frac{(h_{n+1}+1)\cdot 2(n+1)(n+2)}{(2n+5)}A_{n+2}^0\Big]$$

About An, from equation (28), we have

$$-5cG\cdot\sin 2x\cdot\lambda_n\cdot\mathrm{sech}(n+\tfrac{1}{2})\alpha-10h_nA_n^0$$

$$+10(h_n-1)n\frac{A_{n-1}^0}{(2n-1)}+\frac{10(h_n+1)(n+1)A_{n+1}^0}{(2n+3)}$$

$$-n\Big[-cG\cdot\sin 2x\cdot\lambda_{n-1}\cdot\mathrm{sech}(n-\tfrac{1}{2})\alpha-2h_{n-1}A_{n-1}^0$$

$$+2(h_{n-1}-1)(n-1)\frac{A_{n-2}}{2n-3}+2(h_{n-1}+1)n\frac{A_n}{2n+1}$$

$$+(n+1)\Big[-cG\cdot\sin 2x\cdot\lambda_{n+1}\cdot\mathrm{sech}(n+\tfrac{3}{2})\alpha$$

$$-2h_{n+1}A_{n+1}$$

$$+2(h_{n+1}-1)(n+1)\frac{A_n}{(2n+1)}$$

$$+2(h_{n+1}+1)(n+2)\frac{A_{n+2}}{(2n+5)}\Big]$$

Rearrange those with respect to A_{n+m}

① A_{n-2}

$$-n\cdot 2(h_{n-1}-1)(n-1)\frac{1}{(2n-3)}$$

$$+(h_{n-1}-1)\cdot 2(n-1)n\frac{1}{(2n-3)}=0$$

② A_{n+2}

$$(n+1)\cdot 2\cdot(h_{n+1}+1)(n+2)\frac{1}{(2n+5)}$$

$$-(h_{n+1}+1)\cdot 2\cdot(n+1)(n+2)\frac{1}{(2n+5)}=0$$

③ A_{n-1}

$$10(h_n-1)n\frac{1}{(2n-1)}+n\cdot 2\cdot h_{n-1}$$

$$-2\frac{(h_n-1)\cdot 2\cdot n(n+1)}{(2n-1)}-2n$$

$$=\frac{n\cdot 2}{(2n-1)}\Big[(2n-1)(h_{n-1}-1)-(h_n-1)(2n-3)\Big]$$

④ A_{n+1}

$$10(h_n+1)(n+1)\frac{1}{(2n+3)}+2\frac{(h_n+1)\cdot 2n(n+1)}{(2n+3)}$$

$$-(n+1)\cdot 2\cdot h_{n+1}-2(n+1)$$

$$=\frac{(n+1)}{(2n+3)}\Big[(h_n+1)\cdot 2\cdot(2n+5)$$

$$-2(2n+3)(h_{n+1}+1)\Big]$$

⑤ A_n

$$-10h_nA_n^0-n\cdot 2(h_{n-1}+1)n\frac{1}{(2n+1)}$$

$$+(n+1)\cdot 2\cdot(h_{n+1}-1)(n+1)\frac{1}{(2n+1)}$$

$$-(h_{n-1}+1)\cdot 2\cdot(n-1)n\frac{1}{(2n+1)}$$

$$+(h_{n+1}-1)\cdot 2\cdot(n+1)(n+2)\frac{1}{(2n+1)}$$

$$+2\cdot(2n+1)$$

$$=\frac{2n}{(2n+1)}\Big[-n(h_{n-1}+1)-(h_{n-1}+1)(n-1)\Big]$$

$$+\frac{2(n+1)}{(2n+1)}\Big[(n+1)(h_{n+1}-1)+(n+2)(h_{n+1}-1)\Big]$$

$$=\frac{2n}{(2n+1)}(-1)(2n-1)(h_{n-1}+1)$$

$$+\frac{2(n+1)}{(2n+1)}(h_{n+1}-1)(2n+3)$$

$$-10h_n+2(2n+1)$$

$$=\frac{2}{(2n+1)}\Big[-n(2n-1)(h_{n-1}-1)-2n(2n-1)$$

$$+(n+1)(h_{n+1}+1)(2n+3)-2(n+1)(2n+3)$$

$$-(2n+5)(n+1)(h_n+1)+(2n-3)n(h_n-1)$$

$$+(2n+5)(n+1)+(2n-3)n+(2n+1)^2\Big]$$

$$=\frac{2n}{(2n+1)}\Big[-(2n-1)(h_{n-1}-1)+(2n-3)(h_n-1)\Big]$$

$+\frac{2(n+1)}{(2n+1)}\Big[(h_{n+1}+1)(2n+3)-(2n+5)(h_n+1)\Big]$

$+\frac{2}{(2n+1)}\Big[-2n(2n-1)-2(n+1)(2n+3)$

$+(2n+5)(n+1)+(2n-3)n+2\cdot(2n+1)^2\Big]$

Thus, the coefficient of A_n^1 are

$\frac{2n}{(2n+1)}\Big[-(2n-1)(h_{n-1}-1)+(2n-3)(h_n-1)\Big]$

$+\frac{2(n+1)}{(2n+1)}\Big[(2n+3)(h_{n+1}+1)-(2n+5)(h_n+1)\Big]$

$-cG\sin 2x\ \Big\{5\lambda_n\,\mathrm{sech}(n+\frac{1}{2})\alpha-n\lambda_{n-1}\,\mathrm{sech}(n-\frac{1}{2})\alpha$

$+(n+1)\lambda_{n+1}\,\mathrm{sech}(n+\frac{3}{2})\alpha$

$+6\lambda_n\,\mathrm{sech}(n+\frac{1}{2})\alpha\cdot n(n+1)$

$-3\lambda_{n-1}\,\mathrm{sech}(n-\frac{1}{2})\alpha\cdot(n-1)n$

$-3\lambda_{n+1}\,\mathrm{sech}(n+\frac{3}{2})\alpha\cdot(n+1)(n+2)\Big\}$

$=-cG\sin 2x$

$\Big\{\lambda_n\,\mathrm{sech}(n+\frac{1}{2})\alpha\cdot(5+6n(n+1))$

$-\lambda_{n-1}\,\mathrm{sech}(n-\frac{1}{2})\alpha\cdot(n+3\cdot(n-1)n)$

$-\lambda_{n+1}\,\mathrm{sech}(n+\frac{3}{2})\alpha\cdot(-(n+1)+3\cdot(n+1)(n+2))\Big\}$

$=-cG\sin 2x\ \Big\{(3n-2)n(\lambda_n\,\mathrm{sech}(n+\frac{1}{2})\alpha$

$-\lambda_{n-1}\cdot\mathrm{sech}(n-\frac{1}{2})\alpha)$

$+(3n+5)(n+1)(\lambda_n\,\mathrm{sech}(n+\frac{1}{2})\alpha$

$-\lambda_{n+1}\cdot\mathrm{sech}(n+\frac{3}{2})\alpha)\Big\}$

4. Induction of equation (36)

From the continuity equation

$5C_{nm}-(n-m)C_{n-1,m}+(n+m+1)C_{n+1,m}$
$+2E_{nm}-E_{n-1\,m}-E_{n+1\,m}-2(n-m)(n+m+1)G_{nm}$
$+(n-m-1)(n-m)G_{n-1,m}+(n+m+1)(n+m+2)G_{n+1,m}$
$+2(2n+1)A_{nm}-2(n-m)A_{n-1,m}-2(n+m+1)A_{n+1,m}=0$

Putting $m=1$

$5D_n^1-(n-1)D_{n-1}^1+(n+2)D_{n+1}^1$

$+2F_n^1-F_{n-1}^1-F_{n+1}^1$

$-2(n-1)(n+2)H_n^1+(n-2)(n-1)H_{n-1}^1$

$+(n+2)(n+3)H_{n+1}^1$

$+2(2n+1)B_n^1-2(n-1)B_{n-1}^1$

$-2(n+2)B_{n+1}^1=0$

Substitute equations (31)(32)(33) into D_n F_n H_n

$\frac{10cG}{\sinh\alpha}\sin^2x\Big(\frac{\lambda_{n-1}}{2n-1}-\frac{\lambda_{n+1}}{2n+3}\Big)\mathrm{cosech}(n+\frac{1}{2})\alpha$

$-10k_nB_n^1+10(k_n-1)(n-1)\frac{B_{n-1}^1}{2n-1}$

$+10(k_n+1)(n+2)\frac{B_{n+1}^1}{2n+3}$

$-(n-1)\frac{2cG}{\sinh\alpha}\sin^2x\Big(\frac{\lambda_{n-2}}{2n-3}-\frac{\lambda_n}{2n+1}\Big)\mathrm{cosech}(n-\frac{1}{2})\alpha$

$+(n-1)\cdot 2k_{n-1}B_{n-1}^1-(n-1)\cdot 2\cdot(k_{n-1}-1)(n-2)\frac{B_{n-2}^1}{2n-3}$

$-(n-1)\cdot 2\cdot(k_{n-1}+1)(n+1)\frac{B_n^1}{2n+1}$

$+(n+2)\frac{2cG}{\sinh\alpha}\sin^2x\Big(\frac{\lambda_n}{2n+1}-\frac{\lambda_{n+2}}{2n+5}\Big)\mathrm{cosech}(n+\frac{3}{2})\alpha$

$-(n+2)\cdot 2k_{n+1}B_{n+1}^1+2(n+2)(k_{n+1}-1)n\frac{B_n^1}{2n+1}$

$+2(n+2)(k_{n+1}+1)(n+3)\frac{B_{n+2}^1}{2n+5}$

$+\frac{4cG}{\sinh\alpha}\Big[\sin^2x\ \Big\{(n-1)n\frac{\lambda_{n-1}}{2n-1}$

$-(n+1)(n+2)\frac{\lambda_{n+1}}{2n+3}\Big\}$

$-\cos^2x(n+\frac{1}{2})(\lambda_{n-1}-\lambda_{n+1})\Big]\mathrm{cosech}(n+\frac{1}{2})\alpha$

$-2(k_n-1)(n-1)n\frac{B_{n-1}^1}{2n-1}$

$+2(k_n+1)(n+1)(n+2)\frac{B_{n+1}^1}{2n+3}$

$-\frac{2cG}{\sinh\alpha}\Big[\sin^2x\ \Big\{(n-2)(n-1)\frac{\lambda_{n-2}}{2n-3}$

$-(n)(n+1)\frac{\lambda_n}{2n+1}\Big\}$

$-\cos^2x(n-\frac{1}{2})(\lambda_{n-2}-\lambda_n)\Big]\mathrm{cosech}(n-\frac{1}{2})\alpha$

$+(k_{n-1}-1)(n-2)(n-1)\frac{B_{n-2}^1}{2n-3}$

$-(k_{n-1}+1)\cdot n\cdot(n+1)\frac{B_n^1}{2n+1}$

$-\frac{2cG}{\sinh\alpha}\Big[\sin^2x\ \Big\{n(n+1)\frac{\lambda_n}{2n+1}$

$-(n+2)(n+3)\frac{\lambda_{n+2}}{2n+5}\Big\}$

$-\cos^2x(n+\frac{3}{2})(\lambda_n-\lambda_{n+2})\Big]\mathrm{cosech}(n+\frac{3}{2})\alpha$

$+(k_{n+1}-1)\cdot n(n+1)\frac{B_n^1}{2n+1}$

$-(k_{n+1}+1)(n+2)(n+3)\frac{B_{n+2}^1}{2n+5}$

$+2(n-1)(n+2)\frac{2cG\sin^2x}{\sinh\alpha}$

$\Big(\frac{\lambda_{n-1}}{2n-1}-\frac{\lambda_{n+1}}{2n+3}\Big)\mathrm{cosech}(n+\frac{1}{2})\alpha$

$-2(n-1)(n+2)(k_n-1)\frac{B_{n-1}^1}{2n-1}$

$+2(n-1)(n+2)(k_n+1)\frac{B_{n+1}^1}{2n+3}$

$-(n-2)(n-1)2cG\frac{\sin^2x}{\sinh\alpha}\Big(\frac{\lambda_{n-2}}{2n-3}-\frac{\lambda_n}{2n+1}\Big)$

$\mathrm{cosech}(n-\frac{1}{2})\alpha$

$+(n-2)(n-1)(k_{n-1}-1)\frac{B_{n-2}^1}{2n-3}$

$-(n-2)(n-1)(k_{n-1}+1)\frac{B_n^1}{2n+1}$

$-(n+2)(n+3)\frac{2cG\sin^2x}{\sinh\alpha}$

$\Big(\frac{\lambda_n}{2n+1}-\frac{\lambda_{n+2}}{2n+5}\Big)\mathrm{cosech}(n+\frac{3}{2})\alpha$

$+(n+2)(n+3)(k_{n+1}-1)\frac{B_n^1}{2n+1}$

$-(n+2)(n+3)(k_{n+1}+1)\frac{B_{n+2}^1}{2n+5}$

$+2(2n+1)B_n^1-2(n-1)B_{n-1}^1-2(n+2)B_{n+1}^1$

① The coefficients of Bn-2

$B_{n-2}\Big(-(n-1)2(k_{n-1}-1)\frac{(n-2)}{(2n-3)}$

$+(k_{n-1}-1)(n-2)(n-1)\frac{1}{(2n-3)}$

$+(n-2)(n-1)(k_{n-1}-1)\frac{1}{(2n-3)}\Big)=0$

② The coefficients of Bn+2

$B_{n+2}\Big(2(n+2)(k_{n+1}+1)(n+3)\frac{1}{(2n+5)}$

$-(k_{n+1}+1)(n+2)(n+3)\frac{1}{(2n+5)}$

$-(n+2)(n+3)(k_{n+1}+1)\frac{1}{(2n+5)}\Big)=0$

③ The coefficients of Bn-1

$B_{n-1}\Big(10(k_n-1)\frac{(n-1)}{(2n-1)}+(n-1)\cdot 2\cdot k_{n-1}$

$-2(k_n-1)(n-1)\frac{n}{(2n-1)}$

$-2(n-1)(n+2)(k_n-1)\frac{1}{(2n-1)}-2(n-1)\Big)$

$$=B_{n-1}\frac{(n-1)}{(2n-1)}\Big[10(k_n-1)+2\cdot k_{n-1}\cdot(2n-1)$$
$$-2(k_n-1)n-2(n+2)(k_n-1)-2\cdot(2n-1)\Big]$$
$$=B_{n-1}\frac{(n-1)\cdot 2}{(2n-1)}\Big[(2n-1)(k_{n-1}-1)$$
$$-(k_n-1)(2n-3)\Big]$$

④ The coefficients of Bn+1

$$B_{n+1}\Big[\frac{10(k_n+1)(n+2)}{(2n+3)}-(n+2)\cdot k_{n+1}$$
$$+\frac{2(k_n+1)(n+1)(n+2)}{(2n+3)}+\frac{2(n-1)(n+2)(k_n+1)}{(2n+3)}$$
$$-2(n+2)\Big]$$
$$=B_{n+1}\frac{(n+2)}{(2n+3)}\cdot 2\cdot\Big[(k_n+1)(2n+5)$$
$$-(2n+3)(k_{n+1}+1)\Big]$$

⑤ The coefficients of Bn

$$B_n^1\Big(-10k_n-(n-1)\cdot 2\cdot(k_{n-1}+1)(n+1)\frac{1}{(2n+1)}$$
$$+2(n+2)(k_{n+1}-1)\frac{n}{2n+1}$$
$$-(k_{n-1}+1)n(n+1)\frac{1}{(2n+1)}$$
$$+(k_{n+1}-1)n(n+1)\frac{1}{(2n+1)}$$
$$-(n-2)(n-1)(k_{n-1}+1)\frac{1}{(2n+1)}$$
$$+(n+2)(n+3)(k_{n+1}-1)\frac{1}{(2n+1)}+2(2n+1)\Big)$$
$$=\frac{B_n^1}{(2n+1)}\Big(-10k_n\cdot(2n+1)$$
$$-2(n-1)(n+1)(k_{n-1}+1)$$
$$+2(n+2)n(k_{n+1}-1)-n(n+1)(k_{n-1}+1)$$
$$+n(n+1)(k_{n+1}-1)-(n-2)(n-1)(k_{n-1}+1)$$
$$+(n+2)(n+3)(k_{n+1}-1)$$
$$+2(2n+1)^2\Big)$$
$$=\frac{B_n^1}{(2n+1)}\Big[-10k_n(2n+1)-(k_{n-1}-1)2n(2n-1)$$
$$-2\cdot 2n(2n-1)+2(2n+1)^2$$
$$+(k_{n+1}+1)2(2n+3)(n+1)$$
$$-2\cdot 2(2n+3)(n+1)\Big]$$

$$=\frac{2B_n^1 n}{(2n+1)}\Big[(2n-3)(k_n-1)-(2n-1)(k_{n-1}-1)\Big]$$
$$+\frac{2B_n^1}{(2n+1)}(n+1)\Big[-(2n+5)(k_n+1)+(k_{n+1}+1)(2n+3)\Big]$$
$$+\frac{2B_n^1}{(2n+1)}\Big[(2n^2-3n)+(2n^2+7n+5)$$
$$-4n^2+2n-2(2n^2+5n+3)$$
$$+(4n^2+4n+1)\Big]$$
$$=\frac{2B_n^1 n}{(2n+1)}\Big[(2n-3)(k_n-1)-(2n-1)(k_{n-1}-1)\Big]$$
$$+\frac{2B_n^1}{(2n+1)}(n+1)\Big[-(2n+5)(k_n+1)+(k_{n+1}+1)(2n+3)\Big]$$

Then, the reight side of the equation is

$$\frac{cG}{\sinh\alpha}\sin^2 x\Big[10\Big(\frac{\lambda_{n-1}}{2n-1}-\frac{\lambda_{n+1}}{2n+3}\Big)\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha$$
$$-(n-1)2\Big(\frac{\lambda_{n-2}}{2n-3}-\frac{\lambda_n}{2n+1}\Big)\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha$$
$$+(n+2)2\Big(\frac{\lambda_n}{2n+1}-\frac{\lambda_{n+2}}{2n+5}\Big)\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha$$
$$+\Big\{4(n-1)n\frac{\lambda_{n-1}}{2n-1}-4(n+1)(n+2)\frac{\lambda_{n+1}}{2n+3}\Big\}\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha$$
$$+\Big\{-2(n-2)(n-1)\frac{\lambda_{n-2}}{2n-3}+2n(n+1)\frac{\lambda_n}{2n+1}\Big\}\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha$$
$$+\Big\{-2n(n+1)\frac{\lambda_n}{2n+1}+2(n+2)(n+3)\frac{\lambda_{n+2}}{2n+5}\Big\}\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha$$
$$+4(n-1)(n+2)\Big(\frac{\lambda_{n-1}}{2n-1}-\frac{\lambda_{n+1}}{2n+3}\Big)\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha$$
$$-2(n-2)(n-1)\Big(\frac{\lambda_{n-2}}{2n-3}-\frac{\lambda_n}{2n+1}\Big)\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha$$
$$-2(n+2)(n+3)\Big(\frac{\lambda_n}{2n+1}-\frac{\lambda_{n+2}}{2n+5}\Big)\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha\Big]$$

Then,

$$\frac{cG}{\sinh\alpha}\sin^2 x\Big[$$
$$2\,\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha\Big\{-\frac{(n-1)\lambda_{n-2}}{(2n-3)}+\frac{(n-1)\lambda_n}{(2n+1)}$$
$$-\frac{(n-2)(n-1)\lambda_{n-2}}{(2n-3)}+\frac{n(n+1)\lambda_n}{(2n+1)}$$
$$-\frac{(n-2)(n-1)\lambda_{n-2}}{(2n-3)}+\frac{(n-2)(n-1)\lambda_n}{(2n+1)}\Big\}$$
$$+2\,\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha\Big\{\frac{5\lambda_{n-1}}{(2n-1)}-\frac{5\lambda_{n+1}}{(2n+3)}$$
$$+\frac{2(n-1)n\lambda_{n-1}}{(2n-1)}-\frac{2(n+1)(n+2)\lambda_{n+1}}{(2n+3)}$$
$$+\frac{2(n-1)(n+2)\lambda_{n-1}}{(2n-1)}-\frac{2(n-1)(n+2)\lambda_{n+1}}{(2n+3)}\Big\}$$
$$+2\,\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha\Big\{\frac{(n+2)\lambda_n}{(2n+1)}-\frac{(n+2)\lambda_{n+2}}{(2n+5)}$$
$$-\frac{n(n+1)\lambda_n}{(2n+1)}+\frac{(n+2)(n+3)\lambda_{n+2}}{(2n+5)}$$
$$-\frac{(n+2)(n+3)\lambda_n}{(2n+1)}+\frac{(n+2)(n+3)\lambda_{n+2}}{(2n+5)}\Big\}$$

$$\frac{2cG}{\sinh\alpha}\sin^2 x\Big[$$
$$\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha\Big\{-\frac{\lambda_{n-2}}{(2n-3)}((n-1)+2(n^2-3n+2))$$
$$+\frac{\lambda_n}{(2n+1)}((n-1)+n^2+n+n^2-3n+2)\Big\}$$
$$+\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha\Big\{\frac{\lambda_{n-1}}{(2n-1)}(5+2n^2-2n+2n^2+2n-4)$$
$$-\frac{\lambda_{n+1}}{(2n+3)}(5+2(n^2+3n+2)+2(n^2+n-2))\Big\}$$
$$+\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha\Big\{\frac{\lambda_n}{(2n+1)}(n+2-n^2-n-n^2-5n-6)$$
$$+\frac{\lambda_{n+2}}{(2n+5)}(-n-2+2(n^2+5n+6))\Big\}$$

$$\frac{2cG}{\sinh\alpha}\sin^2 x\Big[\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha\Big\{-\lambda_{n-2}(n-1)$$
$$+\frac{\lambda_n}{(2n+1)}((n-1)^2+n(n+1))\Big\}$$
$$+\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha\Big\{\lambda_{n-1}(4n^2+1)$$
$$-\frac{\lambda_{n+1}}{(2n+3)}(4(n+1)^2+1)\Big\}$$
$$+\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha\Big\{\frac{\lambda_n}{(2n+1)}(-(n+2)^2-n(n+1))$$
$$+\lambda_{n+2}(n+2)\Big\}$$

Finally the terms multiplised by $\cos^2 x$ are

$$\frac{cG}{\sinh\alpha}\Big[-4\cos^2 x\Big(n+\frac{1}{2}\Big)\big(\lambda_{n-1}-\lambda_{n+1}\big)\mathrm{cosech}\Big(n+\frac{1}{2}\Big)\alpha$$
$$+2\cos^2 x\Big(n-\frac{1}{2}\Big)\big(\lambda_{n-2}-\lambda_n\big)\mathrm{cosech}\Big(n-\frac{1}{2}\Big)\alpha$$
$$+2\cos^2 x\Big(n+\frac{3}{2}\Big)\big(\lambda_n-\lambda_{n+2}\big)\mathrm{cosech}\Big(n+\frac{3}{2}\Big)\alpha\Big]$$