

Transient Analysis of Genetic Expression by Diffusion Control Mechanism -II

H, Hirayama.*Y, Okita and **T, Kazui
Asahikawa Medical College hirayama@asahikawa-med.ac.jp
*Shizuoka University. **Hamamatsu Medical university.

We propose a mathematical method to analyze the transient change in the probability of gene regulation protein particle (a repressor protein) for the first time arrival at the target operator gene region of the DNA chain. The differential equation for the probabilistic behavior of the repressor molecular particle was diffusion type. The probability was expressed by the modified Bessel functions. By applying the Allen's approximation for the modified Bessel function, we obtained an approximation form of the first time arrival probability of the gene regulation protein particle immediately after the onset of the reaction. The impulse response of the probability showed considerable oscillation. The present result indicates the genetic expression particularly at its initial phase is unstable. The present method will be available for evaluating the temporal change in the probability of the gene regulation.

Gene regulation. Diffusion control. Repressor. Stochastic process. Laplace transform. Oscillation.

遺伝子発現制御における過渡的解析

平山博史, *沖田善光, 数井輝久

旭川市西神楽4-5 旭川医科大学 公衆衛生学講座
(電話0166-65-2111、内2411) E mail hirayama@asahikawa-med.ac.jp
* 静岡大学大学院電子科学研究施設. 浜松医科大学

遺伝子発現を制御する蛋白粒子レプレッサーの標的遺伝子領域への結合、解離状態を、Berg (1976) のモデルに基づいて確率的に記述した。DNA鎖を円柱座標系で記述し、結合、解離、再結合する確率を設定し、これらに対して、確率的拡散方程式を導出した。また生理的境界条件を設定し、有限空間での拡散を解析した。系の拡散型微分方程式を逆ラプラス変換を施すことによって、レプレッサー粒子が解離、結合する確率の過渡的過程を数値計算で算出した。生物実験で報告されている計測値（拡散係数、遺伝子の長さなど）を計算に利用した。制御タンパクは遺伝子制御のごく初期にはその結合、解離確率は非定常におおきく振動して不確定な挙動を示した。本研究は遺伝子発現制御における過渡的過程を解析するうえで有用である。

遺伝子発現. 制御蛋白粒子. 円柱座標系. 拡散. 解離. 結合反応. 逆ラプラス変換

1. Introduction.

Regulation of gene expression on the DNA is one of the most important issue in the life science. The present paper introduce recurrent binding model of the gene regulation protein particle, repressor protein for the target gene region on the DNA (the operator gene) which has been proposed by Berg (1976). Fig 1(left) shows the whole length $2L$ of the chain with radius b placed along the symmetry axis of a circular cylinder with radius R including medium. The sink is in the middle of the DNA chain. Irregular line indicates a possible trajectory for the particle before the final absorption.

2. Modeling

2-1. The basic differential equation

We take an initially homogeneous distribution of particles as repressors and sinks as operators in cylindrical coordinates. Every sink is imbedded in the middle of a long cylindrical chain as a DNA molecule of length $2L$ and radius b . A particle moves with a diffusion constant D in the medium and every time it faces to the chain, the particle is absorbed into the chain and moves along the chain with the a one dimensional diffusion constant D_1 .

By the stochastic process for short time duration dt , with the probability of λdt , the particle will leave the chain unless it has met the sink in which case it would be permanently absorbed. The diffusional flow of particles into the chain along, off and again onto the chain and finally into the sink is a complicated coupled multiple dimensional diffusion. For the simplicity, we separate the movement of the particle as

$u(z,t) dz$: The probability that the particles is in the space interval of $(z, z+dz)$ on the chain at the time t .

$G(z,t) dz dt$: The probability that a particle for the first time arrivals at the chain in the interval $(z, z+dz)$ in the time interval $(t, t+dt)$

$F(z, z', t-t') dz dt$: The probability that a particle leaving the chain from z' at time t' again returns to the chain in the interval $(z, z+dz)$ and in the time interval $(t, t+dt)$. For the symmetry reasons, analysis was confined only for the half plane $0 < z < L$ of the chain. The diffusion equation along the chain becomes

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial z^2} - \lambda u + \lambda \int dz' \int dt' F(z, z', t-t') u(z', t') + G(z, t) \quad (1)$$

The third term describes the return to the chain of a particle that left it at some earlier time. The last term is the arrival of an uncorrelated particle.

2-2. The boundary and initial conditions.

The boundary condition at $z=0$ for an ideal sink

$$u(0, t) = 0 \quad (2)$$

At the ends of the chain the choice of boundary conditions is not crucial and the reflecting boundaries are used

$$\frac{\partial u}{\partial z} = 0 \quad (z = \pm L) \quad (3)$$

For no initial particle on the chain, the initial conditions are

$$u(z, 0) = 0 \quad (4)$$

The flux of particles into the sink is

$$\Phi(t) = D_1 \frac{\partial u}{\partial z} (z=0) \quad (5)$$

Introducing cylindrical coordinates r and z with the z axis along the chain. For symmetry reasons, the angular

coordinate does not enter. Assuming that a particle which dissociates from the chain at $z=z'$ and time $t=0$ is lifted out the distance $r=a$ from the chain axis to start anew. The mathematical question is

" When and where the particle will return to the chain "

2-3. The method of solution.

The solution is given by the diffusion equation for the concentration $c(r, z, t)$ of particles

$$\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial z^2} - \frac{1}{D} \frac{\partial c}{\partial t} = 0 \quad (3-7)$$

The boundary conditions are

$$\frac{\partial c}{\partial z} (at \ z=L, z=0) = 0 \quad (7), (8),$$

$$k * c(r=b, z, t) = \frac{\partial c}{\partial r} (r=b, z, t) \quad (9) \text{ and}$$

$$\frac{\partial c}{\partial r} (r=R) = 0 \quad (10)$$

and the initial condition

$$c(r, z, t=0) = \delta(r-a) \delta(z-z') / (2a\pi) \quad (11)$$

$2\pi D b k$ would be the association rate onto the chains for a homogeneous distribution of particles.

$$c(r=b, z, t) = 0 \quad (12)$$

The particle flux, $F(z, z', t)$ back to the chain which enters equation (1) is now given by

$$F(z, z', t) = 2\pi * b * D * \frac{\partial c}{\partial r} = 2\pi * b * D * k * c(r=b, z, t) \quad (13)$$

This can be calculated after solving equation (6) to (12).

For the initial conditions, we assume that one particle distributes homogeneously in the cylinder.

$$c(r, z, t=0) = 1 / (\pi (R^2 - b^2) L) \quad (14)$$

As the solution with (12) is a fundamental solution,

$$G(z, t) = \int_0^L dz' \int_b^R da \ 2\pi a F(z, z', t) / [\pi (R^2 - b^2) L] \quad (3-15)$$

The understanding of equation (3-15) is following.

On the homogeneous initial condition (13), the particle start in the distance interval $(a, a+da)$ from the chain by the probability $2\pi a da / (\pi (R^2 - b^2))$. With the probability dz' / L , the particle will start in the interval $(z', z'+dz')$. Integrating the fundamental solution over all z' and a gives the solution G . By using the Laplace transformation, the solution of equations (6) to (11) is achieved through separating the variables

$$c(r, z, t) = v(z, t) w(r, t) \quad (16)$$

The axis (z) dependent part is found to

$$v(z, t) = 1/L + 2/L \sum \cos(n\pi z' / L) \cos(n\pi z / L) \exp(-D t (n\pi / L)^2) \quad (17)$$

The Laplace form of $\Phi(t)$ of particles into the sink is

$$\Phi(s) = L G(s) / [1 + 2 / \{ \lambda [1 - \phi(s)] + s \} \sum_{n=1}^{\infty} \{ D1(n\pi/L)^2 + s + \lambda [1 - \phi(s + D n^2 \pi^2 / L^2)] \}] \quad (18)$$

where

$$\psi(s) = k [I_1(qR) K_0(qa) + K_1(qR) I_0(qa)] / \{ I_1(qR) [k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)] \}$$

and

$$G(s) = 2 b k [I_1(qR) K_1(qb) - K_1(qR) I_1(qb)] / \{ I_1(qR) [k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)] \} L (R^2 - b^2) q$$

where $q = \sqrt{s/D}$. s is Laplace operator. The temporal change of the associated process is determined by the Laplace inverted transform of the $\Phi(s)$.

By expanding the modified Bessel functions on the numerator and denominator by Allen's approximation method,

$$G(s) = \sum_{n=0}^{26} N_n q^{2n} / \sum_{n=0}^{28} D_n q^{2n}$$

In summary, the we set following assumptions.

1. A closed cylindrical cell around each chain.
2. Straight chains. As most of the particles dissociating from the chains will be reassociated fast, after small displacements only, they will experience the chains as approximately straight. Those straying too far will lose their correlations to the chains.
3. A particle dissociating from the chain is lifted out a distance $a-b$ from the chain where it starts again. There by $a-b$ is introduced as a parameter of the order of one molecular diameter a is approximately twice the b . We can let $a \rightarrow b$ and have association governed by the traditional boundary condition (3-10) or let k infinite and have the association entirely diffusion limited.
4. Only one particle per a sink.

The parameters were

k : microscopic non specify association rate constant $\gg 2 \pi D1$ (diffusion control limited)
 $= 10^3 / M$ (F.C. Collins and G E. Kimball 1949)

$$D = 6.5 \times 10^{-7} \text{ cm}^2/\text{s} \quad \text{--} \quad 5 \times 10^{-7} \text{ cm}^2/\text{s} \quad (\text{Berg 1978})$$

$$D1 = 3.5 \times 10^{-9} \text{ cm}^2/\text{s}$$

$$L = 1.7/2 \times 10^{-3} \text{ cm}$$

$$b = 5 \times 10^{-7} \text{ cm}$$

$$a = 10 \times 10^{-7} \text{ cm}$$

Fig 1

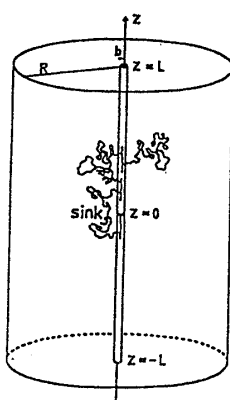
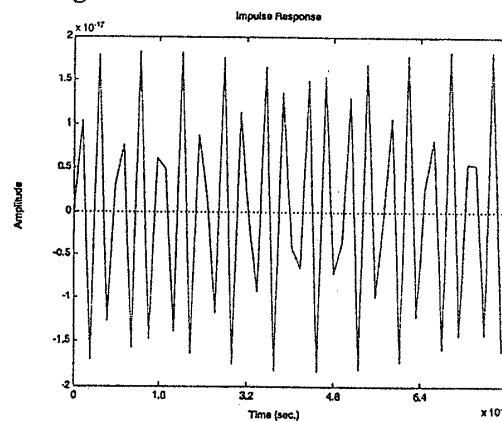


Fig 1(left) shows the whole length $2L$ of the chain with radius b placed along the symmetry axis of a circular cylinder with radius R including medium. The sink is in the middle of the DNA chain. Irregular line indicates a possible trajectory for the particle before the final absorption.

Fig 2



4. Results.

Fig 2 shows fluctuational temporal change in the impulse response of modified probability that a particle for the first time arrivals at the chain in the interval $(z, z+dz)$ in the time interval $(t, t+dt)$, namely

$$G(s) L (R^2 - q^2) / (2 b k) \quad \text{at } k = 0.0001.$$

The computation has continued until $t = 0.008$ sec. The impulse response has oscillated significantly with several different frequency components.

5. Conclusion.

Gene control by repressor particle must be unstable and the impulse response will be oscillative.

6. Reference.

1. Berg, O.G. and Blomberg, C. " Association kinetics with coupled diffusional flows. Biophysical . Chemistry. vol 4. pp 367-381. 1976.
2. Carslaw H S and Jaeger J C. Conduction of heat in solids (Oxford Univ. London 1947).

APPENDIX 1.

Carslaw Jager pp 222. XI.

Theorem

For the infinite hollow cylinder $a < r < b$, $0 < z < L$. $r=a$ kept at temperature $f(z)$. Radiation at the other surfaces into medium at zero

$$v = 2 \sum (\alpha n \cos(\alpha n z) + h \sin(\alpha n z)) \phi(r; n) \int f(z) (\alpha n \cos(\alpha n z) + h \sin(\alpha n z)) dz / [(\alpha n^2 + h^2) L + 2 h] \phi(a; n)$$

where

$$\phi(r; n) = I_0(r \alpha n) [\alpha n K_1(b \alpha n) - h K_0(b \alpha n)] + K_0(r \alpha n) [\alpha n I_1(b \alpha n) + h I_0(b \alpha n)]$$

Proof

The basic differential equation of this system is

$$\partial^2 v / \partial r^2 + 1/r \partial v / \partial r = k \partial^2 v / \partial z^2$$

The boundary conditions are

$$v = f(z) \quad \text{at } r = a$$

$$\partial v / \partial r + h v = 0 \quad \text{at } r = b$$

$$-\partial v / \partial z + h v = 0 \quad \text{at } z = 0$$

$$\partial v / \partial z + h v = 0 \quad \text{at } z = L$$

The general form of the solution is

$$v = \sum [A_n I_0(r \alpha n) + B_n K_0(r \alpha n)] [C_n \cos(\alpha n z) + D_n \sin(\alpha n z)]. \quad \text{Putting } C_n / D_n = \alpha n / h,$$

$$v = \sum [A_n I_0(r \alpha n) + B_n K_0(r \alpha n)] C_n [\cos(\alpha n z) + h / \alpha n \sin(\alpha n z)].$$

Setting $A_n = A_n C_n$, $B_n = B_n C_n$ and

$$X_n = \cos(\alpha n z) + h / \alpha n \sin(\alpha n z)$$

$$v = \sum X_n [A_n I_0(r \alpha_n) + B_n K_0(r \alpha_n)]$$

From the boundary condition (2) at $r = b$,

$$\partial v / \partial r = \sum X_n [A_n \alpha_n I_1(r \alpha_n) - B_n \alpha_n K_1(r \alpha_n)]$$

Thus, we have

$$A_n \alpha_n I_1(b \alpha_n) - B_n \alpha_n K_1(b \alpha_n) + h [A_n I_0(b \alpha_n) + B_n K_0(b \alpha_n)] = 0. \text{ This is}$$

$$A_n [\alpha_n I_1(b \alpha_n) + h A_n I_0(b \alpha_n)] = B_n [\alpha_n K_1(b \alpha_n) - h K_0(b \alpha_n)]. \text{ For this simple, we express this}$$

$$A_n C_1 = B_n C_2. \text{ Substituting } B_n = A_n C_1 / C_2 \text{ to } v$$

$$v = \sum X_n A_n / C_2 [C_2 I_0(r \alpha_n) + C_1 K_0(r \alpha_n)]$$

Putting

$$C_2 I_0(r \alpha_n) + C_1 K_0(r \alpha_n) = [\alpha_n K_1(b \alpha_n) - h K_0(b \alpha_n)] I_0(r \alpha_n) + K_0(r \alpha_n) [\alpha_n I_1(b \alpha_n) + h I_0(b \alpha_n)] = \phi(r; n). \text{ Then we have}$$

$$v = \sum X_n A_n \phi(r; n) / C_2. \text{ Applying the boundary condition } v = f(z) \text{ at } r = a, \text{ we have}$$

$$f(z) = \sum X_n A_n \phi(a; n) / C_2. \text{ Multiplying } X_m(z) \text{ on both side of this and integrate } [0, L], \text{ we have}$$

$$\int f(z) X_m(z) dz = \sum A_n \phi(a; n) / C_2 \int X_n X_m dz$$

The right side of the equation is effective only for $m=n$

$$\int X_n^2 dz = [(\alpha^2 m^2 + h^2) L + 2h] / (2\alpha^2 m^2)$$

Therefore

$$\int f(z) X_m(z) dz = A_n \phi(a; n) / C_2 [(\alpha^2 m^2 + h^2) L + 2h] / (2\alpha^2 m^2). \text{ Thus, we have}$$

$$A_n = 2\alpha^2 m^2 C_2 / [(\alpha^2 m^2 + h^2) L + 2h] * \phi(a; n) \int f(z) X_n(z) dz$$

$$B_n = 2\alpha^2 m^2 C_1 / [(\alpha^2 m^2 + h^2) L + 2h] * \phi(a; n) \int f(z) X_n(z) dz$$

$$\int f(z) X_n(z) dz$$

Therefore

$$v = \sum X_n [A_n I_0(r \alpha_n) + B_n K_0(r \alpha_n)]$$

$$= 2 \sum \alpha^2 m^2 X_n * [C_2 I_0(r \alpha_n) + C_1 K_0(r \alpha_n)]$$

$$/ [(\alpha^2 m^2 + h^2) L + 2h] * \phi(a; n) \int f(z) X_n(z) dz$$

Consequently, we have A1.

APPENDIX 2.

We show the precise mathematical process for obtaining the r dependent part of the solution.

For $r < a$

$$w(r, s) = [I_1(qR) K_0(qa) + K_1(qR) I_0(qa)] / \{2 \pi D I_1(qR)$$

$$[k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)]$$

$$* \{ [k K_0(qb) + q K_1(qb)] I_0(qr) -$$

$$[k I_0(qb) - q I_1(qb)] K_0(qr) \} \quad (q = \sqrt{s/D}) \text{ -----A1}$$

From the equation (A-3), the z -independent flux $\psi(t)$ of particles onto the chain. Its Laplace transform is

$$\phi(s) = 2 \pi b D \frac{\partial w}{\partial r} \bigg|_{r=b} = k \frac{[I_1(qR) K_0(qa) + K_1(qR) I_0(qa)]}{\{I_1(qR) [k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)]\}} \text{ -----A2}$$

The z -dependent flux in the equation (12) is equal to

$$F(z, z', t) = \psi(t) v(z, t) \text{ -----A3}$$

Hence the Laplace transform of F is

$$F(z, z', s) = \phi(s) / L + 2/L \sum_{n=1}^{\infty} \phi(s + D(n\pi/L)^2) \cos(n\pi z'/L) \cos(n\pi z/L) \text{ -----A4}$$

The Laplace transform of flux $G(z, s)$ is calculated through the integration $F(z, z', s)$.

$$G(s) = 2 b k [I_1(qR) K_1(qb) - K_1(qR) I_1(qb)] / \{I_1(qR) [k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)]\} L (R^2 - b^2) q \} \text{ -----A5}$$

To obtain the transient change in $G(s)$, we seek invert Laplace transform of $G(s)$. For instance, we seek $t \rightarrow 0$ at immediately after the diffusion reaction has started. In the present work, we show an approximation method for $G(s)$ by expanding the **modified Bessel function** by the **Allen's approximation**.

APPENDIX 3.

$$\text{----- } G(s) \text{ -----}$$

since

$$qb = b * \sqrt{s/D} = b * 10^{-7} \sqrt{s} / (5 * 10^{-7})^{1/2}$$

$$= \sqrt{s} * 2.68 * 10^{-3.5}$$

$$qR = R * \sqrt{s/D} = b * 10^{-5} \sqrt{s} / (5 * 10^{-7})^{1/2}$$

$$= \sqrt{s} * 2.68 * 10^{-1.5}$$

Therefore, we can apply the Allen's approximation for $I_0(qb)$, $I_1(qR)$, $K_0(qb)$, $K_1(qb)$, $K_1(qR)$ and $I_1(qb)$. to satisfy $x \leq 3.75$

$$\sqrt{s} * 2.68 * 10^{-3.5} \leq 3.75 \text{ Hence}$$

$$s \leq 10^7 (3.75/2.68)^2$$

$$\sqrt{s} * 2.68 * 10^{-1.5} \leq 3.75 \text{ Hence}$$

$$s \leq 10^3 (3.75/2.68)^2$$

Hence, we have sufficiently large s

$$1) I_0(qb) = \sum a_k (qb/3.75)^{2k} = \sum a_k q^{2k} \quad (k=0 \text{ to } 6)$$

$$2) I_1(qb) = qb \sum a_k (qb/3.75)^{2k} = \sum b_k q^{2k+1} \quad (k=0 \text{ to } 6)$$

$$3) K_0(qb) = -\log((qb)/2) * I_0(qb) + \sum a_k (qb/2)^{2k} \quad (k=0 \text{ to } 6) \\ = -\log((qb)/2) * I_0(qb) + \sum c_k q^{2k} \quad (k=0 \text{ to } 6)$$

$$5) K_1(qb) = \log((qb)/2) * I_1(qb) + \sum a_k (qb/2)^{2k} / b \\ = \log((qb)/2) * I_1(qb) + \sum d_k q^{2k-1} \quad (k=0 \text{ to } 6)$$

7) Denominator of $G(s) / q$

$$= I_1(qR) [k K_0(qb) + q K_1(qb)] + K_1(qR) [k I_0(qb) - q I_1(qb)]$$

$$= I_1(qR) [k (-\log((qb)/2) * I_0(qb) + \sum c_k q^{2k})$$

$$+ q (\log((qb)/2) * I_1(qb) + \sum d_k q^{2k-1})]$$

$$+ (\log((qR)/2) * I_1(qR) + \sum e_k q^{2k-1}) [k I_0(qb) - q I_1(qb)]$$

$$= \log(qb/2) I_1(qR) [-k I_0(qb) + q I_1(qb)]$$

$$+ \log(qR/2) I_1(qR) [k I_0(qb) - q I_1(qb)]$$

$$+ k I_1(qR) \sum c_k q^{2k} + I_1(qR) \sum d_k q^{2k}$$

$$+ k I_0(qb) \sum e_k q^{2k-1} - I_1(qb) \sum e_k q^{2k}$$

The first and the second terms are

$$= -\log(b/R) I_1(qR) [k I_0(qb) - q I_1(qb)]$$

The third and the fourth terms are

$$= I_1(qR) [k \sum_{k=0}^6 c_k q^{2k} + \sum_{k=0}^6 d_k q^{2k}]$$

$$\text{setting } \sum_{L=0}^6 (k c_L + d_L) = \sum_{L=0}^6 h_L$$

$$= I_1(qR) \sum_{m=0}^6 h_m q^{2m}$$

Hence, the denominator is

$$= q [-\log(b/R) I_1(qR) [k I_0(qb) - q I_1(qb)]$$

$$+ I_1(qR) \sum_{m=0}^6 h_m q^{2m}$$

$$+ [k I_0(qb) - q I_1(qb)] \sum_{k=0}^6 e_k q^{2k-1}$$

These are even functions because q is multiplied

Denominator

$$\begin{aligned}
&= -\log(b/R) q I_1(qR) [k I_0(qb) - q I_1(qb)] \\
&+ q I_1(qR) \sum_{m=0}^{\infty} h_m q^{2m} + q [k I_0(qb) - q I_1(qb)] \sum_{k=0}^{\infty} e_k q^{2k-1} \\
&= -\log(b/R) \sum_{k=0}^{\infty} B_k q^{2k+2} [k I_0(qb) - \sum_{k=0}^{\infty} b_k q^{2k+2}] (k=0 \text{ to } 6) \\
&+ \sum_{k=0}^{\infty} B_k q^{2k+2} \sum_{m=0}^{\infty} h_m q^{2m} (k=0 \text{ to } 6) \\
&+ [k I_0(qb) - \sum_{k=0}^{\infty} b_k q^{2k+2}] \sum_{k=0}^{\infty} e_k q^{2k} (k=0 \text{ to } 6)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&k I_0(qb) - \sum_{k=0}^{\infty} b_k q^{2k+2} (k=0 \text{ to } 6) \\
&= k \sum_{L=0}^{\infty} a_L q^{2L} - \sum_{L=0}^{\infty} b_L q^{2L+2} (k=0 \text{ to } 6)
\end{aligned}$$

$$\begin{aligned}
&= k (a_0 + a_1 q^2 + a_2 q^4 + a_3 q^6 + a_4 q^8 + a_5 q^{10} + a_6 q^{12}) \\
&- (b_0 q^2 + b_1 q^4 + b_2 q^6 + b_3 q^8 + b_4 q^{10} + b_5 q^{12} + b_6 q^{14})
\end{aligned}$$

$$\begin{aligned}
&\text{setting } U_1 = (k a_1 - b_0), U_2 = (k a_2 - b_1) \\
&= \sum_{n=0}^{\infty} U_n q^{2n} (n=0 \text{ to } 14)
\end{aligned}$$

Then,

$$\begin{aligned}
\text{Deominator} &= -\log(b/R) \sum_{k=0}^{\infty} B_k q^{2k+2} \sum_{n=0}^{\infty} U_n q^{2n} \\
&+ \sum_{k=0}^{\infty} B_k q^{2k+2} \sum_{m=0}^{\infty} h_m q^{2m} + \sum_{n=0}^{\infty} U_n q^{2n} \sum_{k=0}^{\infty} e_k q^{2k} \\
&= [-\log(b/R) \sum_{n=0}^{\infty} U_n q^{2n} + \sum_{m=0}^{\infty} h_m q^{2m}] \sum_{k=0}^{\infty} B_k q^{2k+2} \\
&+ \sum_{n=0}^{\infty} U_n q^{2n} \sum_{k=0}^{\infty} e_k q^{2k}
\end{aligned}$$

The first term is

$$\begin{aligned}
&-\log(b/R) (U_0 + U_1 q^2 + U_2 q^4 + U_3 q^6 + U_4 q^8 \\
&+ U_5 q^{10} + U_6 q^{12} + U_7 q^{14}) \\
&+ h_0 + h_1 q^2 + h_2 q^4 + h_3 q^6 + h_4 q^8 + h_5 q^{10} + h_6 q^{12}
\end{aligned}$$

$$\begin{aligned}
&\text{setting } V_0 = h_0 - \log(b/R) U_0 \text{ ----} \\
&V_7 = h_7 - \log(b/R) U_7
\end{aligned}$$

$$= \sum_{n=0}^{\infty} V_n q^{2n}$$

The denominator is

$$\begin{aligned}
&= \sum_{n=0}^{\infty} V_n q^{2n} \sum_{k=0}^{\infty} B_k q^{2k+2} + \sum_{n=0}^{\infty} U_n q^{2n} \sum_{k=0}^{\infty} e_k q^{2k} \\
&= \sum_{n=0}^{\infty} Z_n q^{2n} (n=0 \text{ to } 14)
\end{aligned}$$

Numerator of G(s)

$$\begin{aligned}
&= I_1(qR) K_1(qb) - K_1(qR) I_1(qb) \\
&= I_1(qR) [\log(qb/2) I_1(qb) + \sum_{k=0}^{\infty} d_k q^{2k-1} \\
&- [\log(qR/2) I_1(qR) + \sum_{k=0}^{\infty} e_k q^{2k-1}]] I_1(qb) \\
&= I_1(qR) I_1(qb) [\log(qb/2) - \log(qR/2)] \\
&+ I_1(qR) \sum_{k=0}^{\infty} d_k q^{2k-1} - I_1(qb) \sum_{k=0}^{\infty} e_k q^{2k-1} \\
&= \log(b/R) I_1(qR) I_1(qb) \\
&+ I_1(qR) \sum_{k=0}^{\infty} d_k q^{2k-1} - I_1(qb) \sum_{k=0}^{\infty} e_k q^{2k-1} \\
&= \log(b/R) \sum_{k=0}^{\infty} B_k q^{2k+1} \sum_{k=0}^{\infty} b_k q^{2k+1} \\
&+ \sum_{k=0}^{\infty} B_k q^{2k+1} \sum_{k=0}^{\infty} d_k q^{2k-1} - \sum_{k=0}^{\infty} b_k q^{2k+1} \sum_{k=0}^{\infty} e_k q^{2k-1} \\
&= \sum_{k=0}^{\infty} B_k q^{2k} [\log(b/R) \sum_{k=0}^{\infty} b_k q^{2k+2} + \sum_{k=0}^{\infty} d_k q^{2k} \\
&- \sum_{k=0}^{\infty} b_k q^{2k} \sum_{k=0}^{\infty} e_k q^{2k}
\end{aligned}$$

We expand the first term in the right

$$\begin{aligned}
&\log(b/R) \sum_{k=0}^{\infty} b_k q^{2k+2} + \sum_{k=0}^{\infty} d_k q^{2k} \\
&= \log(b/R) * (b_0 q^2 + b_1 q^4 + b_2 q^6 + b_3 q^8 + b_4 q^{10} \\
&+ b_5 q^{12} + b_6 q^{14}) * (d_0 + d_1 q^2 + d_2 q^4 + d_3 q^6 + d_4 q^8 \\
&+ d_5 q^{10} + d_6 q^{12}) \\
&= d_0 + (d_1 + \log(b/R) b_0) q^2 + (d_2 + \log(b/R) b_1) q^4 \\
&+ (d_3 + \log(b/R) b_2) q^6 + (d_4 + \log(b/R) b_3) q^8 \\
&+ (d_5 + \log(b/R) b_4) q^{10} + (d_6 + \log(b/R) b_5) q^{12} \\
&+ \log(b/R) b_6 q^{14} \\
&= W_0 + W_1 q^2 + W_2 q^4 + W_3 q^6 + W_4 q^8 + W_5 q^{10} \\
&+ W_6 q^{12} + W_7 q^{14} \\
&= \sum_{n=0}^{\infty} W_n q^{2n}
\end{aligned}$$

The numerator

$$\begin{aligned}
&= \sum_{k=0}^{\infty} B_k q^{2k} \sum_{n=0}^{\infty} W_n q^{2n} - \sum_{k=0}^{\infty} b_k q^{2k} \sum_{k=0}^{\infty} e_k q^{2k} \\
&= \sum_{k=0}^{\infty} Y_k q^{2k} (k=0 \text{ to } 13) \\
&***** \\
&\phi(s) = k [I_1(qR) K_0(qa) + K_1(qR) I_0(qa)] * \\
&/ ([I_1(qR) [k K_0(qb) + q K_1(qb)] \\
&+ K_1(qR) [k I_0(qb) - q I_1(qb)]])
\end{aligned}$$

To equate the denominator of $\phi(s)$ to that of $G(s)$, we multiply q on both the numerator and denominator.

$$\begin{aligned}
\phi(s) &= qk [I_1(qR) K_0(qa) + K_1(qR) I_0(qa)] * \\
&/ q ([I_1(qR) [k K_0(qb) + q K_1(qb)] \\
&+ K_1(qR) [k I_0(qb) - q I_1(qb)]])
\end{aligned}$$

Hence the difference between $G(s)$ is the numerator only. By the Allen's approximation, we have

$$\begin{aligned}
K_0(qa) &= -\log((qa)/2) * I_0(qa) + \sum_{k=0}^{\infty} a_k (a/2)^{2k} q^{2k} \\
&\text{putting } a_k (a/2)^{2k} = c_{ak} \\
&= -\log((qa)/2) * I_0(qa) + \sum_{k=0}^{\infty} c_{ak} q^{2k} (k=0 \text{ to } 6)
\end{aligned}$$

$$\begin{aligned}
I_0(qa) &= \sum_{k=0}^{\infty} a_k (a/3.75)^{2k} q^{2k} (k=0 \text{ to } 6) \\
&\text{putting } a_k (a/3.75)^{2k} = a_{ak} \\
&= \sum_{k=0}^{\infty} a_{ak} q^{2k} (k=0 \text{ to } 6)
\end{aligned}$$

Then, thenumerator is

$$\begin{aligned}
&= q I_1(qR) K_0(qa) + q K_1(qR) I_0(qa) \\
&= q I_1(qR) [-\log((qa)/2) * I_0(qa) + \sum_{k=0}^{\infty} c_{ak} q^{2k}] \\
&+ q [\log((qR)/2) * I_1(qR) + \sum_{k=0}^{\infty} e_k q^{2k-1}] * I_0(qa) \\
&= -q I_1(qR) \log((qa)/2) * I_0(qa) + q I_1(qR) \sum_{k=0}^{\infty} c_{ak} q^{2k} \\
&+ q \log((qR)/2) * I_1(qR) I_0(qa) + \sum_{k=0}^{\infty} e_k q^{2k-1} I_0(qa) \\
&= q I_1(qR) * I_0(qa) (-\log(qa/2) + \log(qR/2)) \\
&+ q \sum_{k=0}^{\infty} B_k q^{2k+1} \sum_{k=0}^{\infty} c_{ak} q^{2k} + \sum_{k=0}^{\infty} e_k q^{2k} \sum_{k=0}^{\infty} a_{ak} q^{2k}
\end{aligned}$$

$$\text{where } I_1(qR) = \sum_{k=0}^{\infty} B_k q^{2k+1}$$

$$I_0(qa) = \sum_{k=0}^{\infty} a_{ak} q^{2k}$$

$$\begin{aligned}
&= q \sum_{k=0}^{\infty} B_k q^{2k+1} \sum_{k=0}^{\infty} a_{ak} q^{2k} \log(R/a) \\
&+ \sum_{k=0}^{\infty} B_k q^{2k+2} \sum_{k=0}^{\infty} c_{ak} q^{2k} + \sum_{k=0}^{\infty} e_k q^{2k} \sum_{k=0}^{\infty} a_{ak} q^{2k} \\
&= \sum_{k=0}^{\infty} B_k q^{2k+2} [\sum_{k=0}^{\infty} a_{ak} q^{2k} \log(R/a) + \sum_{k=0}^{\infty} c_{ak} q^{2k}] \\
&+ \sum_{k=0}^{\infty} e_k q^{2k} \sum_{k=0}^{\infty} a_{ak} q^{2k}
\end{aligned}$$

where..

$$[\sum_{k=0}^{\infty} a_{ak} q^{2k} \log(R/a) + \sum_{k=0}^{\infty} c_{ak} q^{2k}]$$

$$\begin{aligned}
&= (\log(R/a) aa0 + ca0) q^0 + (\log(R/a) aa1 + ca1) q^2 \\
&\quad + (\log(R/a) aa2 + ca2) q^4 + (\log(R/a) aa3 + ca3) q^6 \\
&\quad + (\log(R/a) aa4 + ca4) q^8 + (\log(R/a) aa5 + ca5) q^{10} \\
&\quad + (\log(R/a) aa6 + ca6) q^{12} \\
&= m0 + m1 q^2 + m2 q^4 + m3 q^6 + m4 q^8 + m5 q^{10} \\
&\quad + m6 q^{12} \\
&\text{Then, the numerator is expanded as} \\
&\quad = \sum B_k q^{2k+2} \sum m_k q^{2k} + \sum e_k q^{2k} \sum a_{ak} q^{2k} \\
&\quad (B0 q^2 + B1 q^4 + B2 q^6 + B3 q^8 + B4 q^{10} + B5 q^{12} + B6 q^{14}) \\
&\quad (m0 + m1 q^2 + m2 q^4 + m3 q^6 + m4 q^8 + m5 q^{10} + m6 q^{12}) \\
&\quad + \\
&\quad (e0 + e1 q^2 + e2 q^4 + e3 q^6 + e4 q^8 + e5 q^{10} + e6 q^{12}) \\
&\quad (aa0 + aa1 q^2 + aa2 q^4 + aa3 q^6 + aa4 q^8 + aa5 q^{10} + aa6 q^{12}) \\
&\quad \text{*****} \\
&\quad (B0 q^2 + B1 q^4 + B2 q^6 + B3 q^8 + B4 q^{10} + B5 q^{12} + B6 q^{14}) \\
&\quad (m0 + m1 q^2 + m2 q^4 + m3 q^6 + m4 q^8 + m5 q^{10} + m6 q^{12}) \\
&= \\
&\quad B0 m0 q^2 + B0 m1 q^4 + B0 m2 q^6 + B0 m3 q^8 + B0 m4 q^{10} \\
&\quad \quad + B0 m5 q^{12} + B0 m6 q^{14} \\
&\quad + B1 m0 q^4 + B1 m1 q^6 + B1 m2 q^8 + B1 m3 q^{10} + B1 m4 q^{12} \\
&\quad \quad + B1 m5 q^{14} + B1 m6 q^{16} \\
&\quad + B2 m0 q^6 + B2 m1 q^8 + B2 m2 q^{10} + B2 m3 q^{12} + B2 m4 q^{14} \\
&\quad \quad + B2 m5 q^{16} + B2 m6 q^{18} \\
&\quad + B3 m0 q^8 + B3 m1 q^{10} + B3 m2 q^{12} + B3 m3 q^{14} + B3 m4 q^{16} \\
&\quad \quad + B3 m5 q^{18} + B3 m6 q^{20} \\
&\quad + B4 m0 q^{10} + B4 m1 q^{12} + B4 m2 q^{14} + B4 m3 q^{16} + B4 m4 q^{18} \\
&\quad \quad + B4 m5 q^{20} + B4 m6 q^{22} \\
&\quad + B5 m0 q^{12} + B5 m1 q^{14} + B5 m2 q^{16} + B5 m3 q^{18} + B5 m4 q^{20} \\
&\quad \quad + B5 m5 q^{22} + B5 m6 q^{24} \\
&\quad + B6 m0 q^{14} + B6 m1 q^{16} + B6 m2 q^{18} + B6 m3 q^{20} + B6 m4 q^{22} \\
&\quad \quad + B6 m5 q^{24} + B6 m6 q^{26}
\end{aligned}$$

$$\begin{aligned}
&= (e0 + e1 q^2 + e2 q^4 + e3 q^6 + e4 q^8 + e5 q^{10} + e6 q^{12}) \\
&\quad (aa0 + aa1 q^2 + aa2 q^4 + aa3 q^6 + aa4 q^8 + aa5 q^{10} + aa6 q^{12}) \\
&= \\
&\quad e0 aa0 + e0 aa1 q^2 + e0 aa2 q^4 + e0 aa3 q^6 + e0 aa4 q^8 \\
&\quad \quad + e0 aa5 q^{10} + e0 aa6 q^{12} \\
&\quad + e1 aa0 q^2 + e1 aa1 q^4 + e1 aa2 q^6 + e1 aa3 q^8 + e1 aa4 q^{10} \\
&\quad \quad + e1 aa5 q^{12} + e1 aa6 q^{14} \\
&\quad + e2 aa0 q^4 + e2 aa1 q^6 + e2 aa2 q^8 + e2 aa3 q^{10} + e2 aa4 q^{12} \\
&\quad \quad + e2 aa5 q^{14} + e2 aa6 q^{16} \\
&\quad + e3 aa0 q^6 + e3 aa1 q^8 + e3 aa2 q^{10} + e3 aa3 q^{12} + e3 aa4 q^{14} \\
&\quad \quad + e3 aa5 q^{16} + e3 aa6 q^{18} \\
&\quad + e4 aa0 q^8 + e4 aa1 q^{10} + e4 aa2 q^{12} + e4 aa3 q^{14} + e4 aa4 q^{16} \\
&\quad \quad + e4 aa5 q^{18} + e4 aa6 q^{20} \\
&\quad + e5 aa0 q^{10} + e5 aa1 q^{12} + e5 aa2 q^{14} + e5 aa3 q^{16} + e5 aa4 q^{18} \\
&\quad \quad + e5 aa5 q^{20} + e5 aa6 q^{22} \\
&\quad + e6 aa0 q^{12} + e6 aa1 q^{14} + e6 aa2 q^{16} + e6 aa3 q^{18} + e6 aa4 q^{20} \\
&\quad \quad + e6 aa5 q^{22} + e6 aa6 q^{24} \\
&\quad \text{*****}
\end{aligned}$$

Hence the numerator is

$$\begin{aligned}
&= e0 aa0 + q^2 (e0 aa1 + e1 aa0 + B0 m0) \\
&\quad + q^4 (e0 aa2 + e1 aa1 + e2 aa0 + B0 m1 + B1 m0) \\
&\quad + q^6 (e0 aa3 + e1 aa2 + e2 aa1 + e3 aa0 + B0 m2 \\
&\quad \quad + B1 m1 + B2 m0) \\
&\quad + q^8 (e0 aa4 + e1 aa3 + e2 aa2 + e3 aa1 + e4 aa0 \\
&\quad \quad + B0 m3 + B1 m2 + B2 m1 + B3 m0)
\end{aligned}$$

$$\begin{aligned}
&\quad + q^{10} (e0 aa5 + e1 aa4 + e2 aa3 + e3 aa2 + e4 aa1 + e5 aa0 \\
&\quad \quad + B0 m4 + B1 m3 + B2 m2 + B3 m1 + B4 m0) \\
&\quad + q^{12} (e0 aa6 + e1 aa5 + e2 aa4 + e3 aa3 + e4 aa2 + e5 aa1 \\
&\quad \quad + e6 aa0 \\
&\quad \quad + B0 m5 + B1 m4 + B2 m3 + B3 m2 + B4 m1 + B5 m0) \\
&\quad + q^{14} (e1 aa6 + e2 aa5 + e3 aa4 + e4 aa3 + e5 aa2 + e6 aa1 \\
&\quad \quad + B0 m6 + B1 m5 + B2 m4 + B3 m3 + B4 m2 + B5 m1 \\
&\quad \quad + B6 m0) \\
&\quad + q^{16} (e2 aa6 + e3 aa5 + e4 aa4 + e5 aa3 + e6 aa2 \\
&\quad \quad + B1 m6 + B2 m5 + B3 m4 + B4 m3 + B5 m2 + B6 m1) \\
&\quad + q^{18} (e3 aa6 + e4 aa5 + e5 aa4 + e6 aa3 \\
&\quad \quad + B2 m6 + B3 m5 + B4 m4 + B5 m3 + B6 m2) \\
&\quad + q^{20} (e4 aa6 + e5 aa5 + e6 aa4 \\
&\quad \quad + B3 m6 + B4 m5 + B5 m4 + B6 m3) \\
&\quad + q^{22} (e5 aa6 + e6 aa5 + B4 m6 + B5 m5 + B6 m4) \\
&\quad + q^{24} (e6 aa6 + B5 m6 + B6 m5) \\
&\quad + q^{26} (B6 m6)
\end{aligned}$$

$$\begin{aligned}
w(r,s) &= q [I_1(qR) K_0(qa) + K_1(qR) I_0(qa)]^* \\
&\quad \{ [k K_0(qb) + q K_1(qb)] I_0(qr) \\
&\quad \quad - [k I_0(qb) - q I_1(qb)] K_0(qr) \} \\
&\quad / (2 \pi D q [I_1(qR) [k K_0(qb) + q K_1(qb)] \\
&\quad \quad + K_1(qR) [k I_0(qb) - q I_1(qb)]])
\end{aligned}$$

Hence the denominator of $w(r,s)$ is identical with that of $G(s)$. The numerator of $w(r,s)$ is the product of

$$\begin{aligned}
&\phi(s) \text{ and} \\
&\quad \{ [k K_0(qb) + q K_1(qb)]^* I_0(qr) \\
&\quad \quad - [k I_0(qb) - q I_1(qb)]^* K_0(qr) \}
\end{aligned}$$

Therefore, we firstly calculate

$$\begin{aligned}
&\{ [k K_0(qb) + q K_1(qb)]^* I_0(qr) \\
&\quad \quad - [k I_0(qb) - q I_1(qb)]^* K_0(qr) \}
\end{aligned}$$

and then, multiply the numerator of $\phi(s)$

Expanding the Modified bessel function by Allen's numerical approximation as

$$\begin{aligned}
1) \quad I_0(qb) &= \sum a_k q^{2k} \quad (k=0 \text{ to } 6) \\
1-b) \quad I_0(qr) &= \sum a_k (r/3.75)^{2k} q^{2k} \quad (k=0 \text{ to } 6) \\
&\quad \text{putting} \quad a_k (r/3.75)^{2k} = ark \\
&= \sum ark q^{2k} \quad (k=0 \text{ to } 6) \\
2) \quad I_1(qb) &= \sum bk q^{2k+1} \quad (k=0 \text{ to } 6) \\
3) \quad K_0(qb) &= -\log((qb)/2) * I_0(qb) + \sum ck q^{2k} \quad (k=0 \text{ to } 6) \\
3-b) \quad K_0(qr) &= -\log((qr)/2) * I_0(qr) + \sum ak (r/2)^{2k} q^{2k} \\
&\quad \text{putting} \quad ak (r/2)^{2k} = crk \\
&= -\log((qr)/2) * I_0(qr) + \sum crk q^{2k} \quad (k=0 \text{ to } 6) \\
4) \quad K_1(qb) &= \log((qb)/2) * I_1(qb) + \sum dk q^{2k-1} \quad (k=0 \text{ to } 6)
\end{aligned}$$

Then we approximate each term as

$$\begin{aligned}
1. \quad \{ [k K_0(qb) + q K_1(qb)]^* I_0(qr) \\
&= \{ k (-\log((qb)/2) * I_0(qb) + \sum ck q^{2k}) (k=0 \text{ to } 6)
\end{aligned}$$

$$\begin{aligned}
& + q (\log((qb)/2) * I1(qb) + \sum dk q^{2k-1}) (k=0 \text{ to } 6) \} I0(qr) \\
& = - k \log((qb)/2) * I0(qb) * I0(qr) + k \sum ck q^{2k} I0(qr) \\
& + q * \log((qb)/2) * I1(qb) * I0(qr) + \sum dk q^{2k} I0(qr) \\
& \quad (k=0 \text{ to } 6)
\end{aligned}$$

$$\begin{aligned}
& 2. \{ k I0(qb) - q I1(qb) \} K0(qr) \\
& = k I0(qb) (- \log((qr)/2) * I0(qr) + \sum crk q^{2k}) \\
& - q I1(qb) (- \log((qr)/2) * I0(qr) + \sum crk q^{2k}) \\
& = - k I0(qb) \log((qr)/2) * I0(qr) + k I0(qb) \sum crk q^{2k} \\
& + q I1(qb) \log((qr)/2) * I0(qr) - q I1(qb) \sum crk q^{2k}
\end{aligned}$$

Therefore

$$\begin{aligned}
& k I0(qb) I0(qr) [- \log(qb/2) + \log(qr/2)] \\
& + q I1(qb) I0(qr) [\log(qb/2) - \log(qr/2)] \\
& + k [\sum ck q^{2k} I0(qr) - I0(qb) \sum crk q^{2k}] \\
& + I0(qr) \sum dk q^{2k} + q I1(qb) \sum crk q^{2k} \\
& = k I0(qb) I0(qr) \log(r/b) + q I1(qb) I0(qr) \log(b/r) \\
& + k [\sum ck q^{2k} \sum ark q^{2k} - \sum ak q^{2k} \sum crk q^{2k}] \\
& + \sum ark q^{2k} \sum dk q^{2k} + q \sum bk q^{2k+1} \sum crk q^{2k} \\
& = \log(r/b) [k \sum ak q^{2k} - \sum bk q^{2k+2}] \sum ark q^{2k} \\
& + k [\sum ck q^{2k} \sum ark q^{2k} - \sum ak q^{2k} \sum crk q^{2k}] \\
& + \sum ark q^{2k} \sum dk q^{2k} \\
& + \sum bk q^{2k+2} \sum crk q^{2k}
\end{aligned}$$

$$\begin{aligned}
& = \sum ak q^{2k} [k \log(r/b) \sum ark q^{2k} - k \sum rck q^{2k}] \\
& + \sum ark q^{2k} [k \sum ck q^{2k} + \sum dk q^{2k}] \\
& + \sum bk q^{2k+2} [- \log(r/b) \sum ark q^{2k} + \sum crk q^{2k}]
\end{aligned}$$

Putting

$$\begin{aligned}
& k (\log(r/b) ar0 - cr0) : g0 \\
& k (\log(r/b) ar1 - cr1) : g1 \\
& k (\log(r/b) ar2 - cr2) : g2 \\
& k (\log(r/b) ar3 - cr3) : g3 \\
& k (\log(r/b) ar4 - cr4) : g4 \\
& k (\log(r/b) ar5 - cr5) : g5 \\
& k (\log(r/b) ar6 - cr6) : g6
\end{aligned}$$

Putting

$$\begin{aligned}
& (k c0 + d0) : f0 \\
& (k c1 + d1) : f1 \\
& (k c2 + d2) : f2 \\
& (k c3 + d3) : f3 \\
& (k c4 + d4) : f4 \\
& (k c5 + d5) : f5 \\
& (k c6 + d6) : f6
\end{aligned}$$

Putting

$$\begin{aligned}
& (- \log(r/b) ar0 + cr0) : p0 \\
& (- \log(r/b) ar1 + cr1) : p1 \\
& (- \log(r/b) ar2 + cr2) : p2 \\
& (- \log(r/b) ar3 + cr3) : p3 \\
& (- \log(r/b) ar4 + cr4) : p4 \\
& (- \log(r/b) ar5 + cr5) : p5 \\
& (- \log(r/b) ar6 + cr6) : p6
\end{aligned}$$

Then, the second factor of the numerator is

$$\begin{aligned}
& = \sum ak q^{2k} \sum gk q^{2k} + \sum ark q^{2k} \sum fk q^{2k} \\
& + \sum bk q^{2k+2} \sum pk q^{2k} \\
& = (a0 + a1 q^2 + a2 q^4 + a3 q^6 + a4 q^8 + a5 q^{10} + a6 q^{12}) \\
& (g0 + g1 q^2 + g2 q^4 + g3 q^6 + g4 q^8 + g5 q^{10} + g6 q^{12})
\end{aligned}$$

$$\begin{aligned}
& + (ar0 + ar1 q^2 + ar2 q^4 + ar3 q^6 + ar4 q^8 + ar5 q^{10} + ar6 q^{12}) \\
& (f0 + f1 q^2 + f2 q^4 + f3 q^6 + f4 q^8 + f5 q^{10} + f6 q^{12}) \\
& + (b0 q^2 + b1 q^4 + b2 q^6 + b3 q^8 + b4 q^{10} + b5 q^{12} + b6 q^{14}) \\
& (p0 + p1 q^2 + p2 q^4 + p3 q^6 + p4 q^8 + p5 q^{10} + p6 q^{12})
\end{aligned}$$

Expansion of each product is

$$\begin{aligned}
& (a0 + a1 q^2 + a2 q^4 + a3 q^6 + a4 q^8 + a5 q^{10} + a6 q^{12}) \\
& (g0 + g1 q^2 + g2 q^4 + g3 q^6 + g4 q^8 + g5 q^{10} + g6 q^{12}) \\
& = a0 g0 + a0 g1 q^2 + a0 g2 q^4 + a0 g3 q^6 + a0 g4 q^8 \\
& \quad + a0 g5 q^{10} + a0 g6 q^{12} \\
& + a1 g0 q^2 + a1 g1 q^4 + a1 g2 q^6 + a1 g3 q^8 + a1 g4 q^{10} \\
& \quad + a1 g5 q^{12} + a1 g6 q^{14} \\
& + a2 g0 q^4 + a2 g1 q^6 + a2 g2 q^8 + a2 g3 q^{10} + a2 g4 q^{12} \\
& \quad + a2 g5 q^{14} + a2 g6 q^{16} \\
& + a3 g0 q^6 + a3 g1 q^8 + a3 g2 q^{10} + a3 g3 q^{12} + a3 g4 q^{14} \\
& \quad + a3 g5 q^{16} + a3 g6 q^{18} \\
& + a4 g0 q^8 + a4 g1 q^{10} + a4 g2 q^{12} + a4 g3 q^{14} + a4 g4 q^{16} \\
& \quad + a4 g5 q^{18} + a4 g6 q^{20} \\
& + a5 g0 q^{10} + a5 g1 q^{12} + a5 g2 q^{14} + a5 g3 q^{16} + a5 g4 q^{18} \\
& \quad + a5 g5 q^{20} + a5 g6 q^{22} \\
& + a6 g0 q^{12} + a6 g1 q^{14} + a6 g2 q^{16} + a6 g3 q^{18} + a6 g4 q^{20} \\
& \quad + a6 g5 q^{22} + a6 g6 q^{24}
\end{aligned}$$

$$\begin{aligned}
& (ar0 + ar1 q^2 + ar2 q^4 + ar3 q^6 + ar4 q^8 + ar5 q^{10} + ar6 q^{12}) \\
& (f0 + f1 q^2 + f2 q^4 + f3 q^6 + f4 q^8 + f5 q^{10} + f6 q^{12}) \\
& = ar0 f0 + ar0 f1 q^2 + ar0 f2 q^4 + ar0 f3 q^6 + ar0 f4 q^8 \\
& \quad + ar0 f5 q^{10} + ar0 f6 q^{12} \\
& + ar1 f0 q^2 + ar1 f1 q^4 + ar1 f2 q^6 + ar1 f3 q^8 + ar1 f4 q^{10} \\
& \quad + ar1 f5 q^{12} + ar1 f6 q^{14} \\
& + ar2 f0 q^4 + ar2 f1 q^6 + ar2 f2 q^8 + ar2 f3 q^{10} + ar2 f4 q^{12} \\
& \quad + ar2 f5 q^{14} + ar2 f6 q^{16} \\
& + ar3 f0 q^6 + ar3 f1 q^8 + ar3 f2 q^{10} + ar3 f3 q^{12} + ar3 f4 q^{14} \\
& \quad + ar3 f5 q^{16} + ar3 f6 q^{18} \\
& + ar4 f0 q^8 + ar4 f1 q^{10} + ar4 f2 q^{12} + ar4 f3 q^{14} + ar4 f4 q^{16} \\
& \quad + ar4 f5 q^{18} + ar4 f6 q^{20} \\
& + ar5 f0 q^{10} + ar5 f1 q^{12} + ar5 f2 q^{14} + ar5 f3 q^{16} + ar5 f4 q^{18} \\
& \quad + ar5 f5 q^{20} + ar5 f6 q^{22} \\
& + ar6 f0 q^{12} + ar6 f1 q^{14} + ar6 f2 q^{16} + ar6 f3 q^{18} + ar6 f4 q^{20} \\
& \quad + ar6 f5 q^{22} + ar6 f6 q^{24}
\end{aligned}$$

$$\begin{aligned}
& (b0 q^2 + b1 q^4 + b2 q^6 + b3 q^8 + b4 q^{10} + b5 q^{12} + b6 q^{14}) \\
& (p0 + p1 q^2 + p2 q^4 + p3 q^6 + p4 q^8 + p5 q^{10} + p6 q^{12}) \\
& = b0 p0 q^2 + b0 p1 q^4 + b0 p2 q^6 + b0 p3 q^8 + b0 p4 q^{10} \\
& \quad + b0 p5 q^{12} + b0 p6 q^{14} \\
& + b1 p0 q^4 + b1 p1 q^6 + b1 p2 q^8 + b1 p3 q^{10} + b1 p4 q^{12} \\
& \quad + b1 p5 q^{14} + b1 p6 q^{16} \\
& + b2 p0 q^6 + b2 p1 q^8 + b2 p2 q^{10} + b2 p3 q^{12} + b2 p4 q^{14} \\
& \quad + b2 p5 q^{16} + b2 p6 q^{18} \\
& + b3 p0 q^8 + b3 p1 q^{10} + b3 p2 q^{12} + b3 p3 q^{14} + b3 p4 q^{16} \\
& \quad + b3 p5 q^{18} + b3 p6 q^{20} \\
& + b4 p0 q^{10} + b4 p1 q^{12} + b4 p2 q^{14} + b4 p3 q^{16} + b4 p4 q^{18} \\
& \quad + b4 p5 q^{20} + b4 p6 q^{22} \\
& + b5 p0 q^{12} + b5 p1 q^{14} + b5 p2 q^{16} + b5 p3 q^{18} + b5 p4 q^{20} \\
& \quad + b5 p5 q^{22} + b5 p6 q^{24} \\
& + b6 p0 q^{14} + b6 p1 q^{16} + b6 p2 q^{18} + b6 p3 q^{20} + b6 p4 q^{22} \\
& \quad + b6 p5 q^{24} + b6 p6 q^{26}
\end{aligned}$$

Associating these terms

$$\begin{aligned}
 &= a_0 g_0 + a_0 f_0 + \\
 &+ q^2 (a_0 g_1 + a_1 g_0 + a_0 f_1 + a_1 f_0 + b_0 p_0) \\
 &+ q^4 (a_0 g_2 + a_1 g_1 + a_2 g_0 \\
 &\quad + a_0 f_2 + a_1 f_1 + a_2 f_0 + b_0 p_1 + b_1 p_0) \\
 &+ q^6 (a_0 g_3 + a_1 g_2 + a_2 g_1 + a_3 g_0 \\
 &\quad + a_0 f_3 + a_1 f_2 + a_2 f_1 + a_3 f_0 + b_0 p_2 \\
 &\quad + b_1 p_1 + b_2 p_0) \\
 &+ q^8 (a_0 g_4 + a_1 g_3 + a_2 g_2 + a_3 g_1 + a_4 g_0 \\
 &\quad + a_0 f_4 + a_1 f_3 + a_2 f_2 + a_3 f_1 + a_4 f_0 \\
 &\quad + b_0 p_3 + b_1 p_2 + b_2 p_1 + b_3 p_0) \\
 &+ q^{10} (a_0 g_5 + a_1 g_4 + a_2 g_3 + a_3 g_2 + a_4 g_1 + a_5 g_0 \\
 &\quad + a_0 f_5 + a_1 f_4 + a_2 f_3 + a_3 f_2 + a_4 f_1 + a_5 f_0 \\
 &\quad + b_0 p_4 + b_1 p_3 + b_2 p_2 + b_3 p_1 + b_4 p_0) \\
 &+ q^{12} (a_0 g_6 + a_1 g_5 + a_2 g_4 + a_3 g_3 + a_4 g_2 + a_5 g_1 \\
 &\quad + a_6 g_0 \\
 &\quad + a_0 f_6 + a_1 f_5 + a_2 f_4 + a_3 f_3 + a_4 f_2 + a_5 f_1 \\
 &\quad + a_6 f_0 \\
 &\quad + b_0 p_5 + b_1 p_4 + b_2 p_3 + b_3 p_2 + b_4 p_1 + b_5 p_0) \\
 &+ q^{14} (a_1 g_6 + a_2 g_5 + a_3 g_4 + a_4 g_3 + a_5 g_2 + a_6 g_1 \\
 &\quad + a_1 f_6 + a_2 f_5 + a_3 f_4 + a_4 f_3 + a_5 f_2 + a_6 f_1 \\
 &\quad + b_0 p_6 + b_1 p_5 + b_2 p_4 + b_3 p_3 + b_4 p_2 + b_5 p_1 \\
 &\quad + b_6 p_0) \\
 &+ q^{16} (a_2 g_6 + a_3 g_5 + a_4 g_4 + a_5 g_3 + a_6 g_2 \\
 &\quad + a_2 f_6 + a_3 f_5 + a_4 f_4 + a_5 f_3 + a_6 f_2 \\
 &\quad + b_1 p_6 + b_2 p_5 + b_3 p_4 + b_4 p_3 + b_5 p_2 + b_6 p_1) \\
 &+ q^{18} (a_3 g_6 + a_4 g_5 + a_5 g_4 + a_6 g_3 \\
 &\quad + a_3 f_6 + a_4 f_5 + a_5 f_4 + a_6 f_3 \\
 &\quad + b_2 p_6 + b_3 p_5 + b_4 p_4 + b_5 p_3 + b_6 p_2) \\
 &+ q^{20} (a_4 g_6 + a_5 g_5 + a_6 g_4 \\
 &\quad + a_4 f_6 + a_5 f_5 + a_6 f_4 \\
 &\quad + b_3 p_6 + b_4 p_5 + b_5 p_4 + b_6 p_3) \\
 &+ q^{22} (a_5 g_6 + a_6 g_5 \\
 &\quad + a_5 f_6 + a_6 f_5 + b_4 p_6 + b_5 p_5 + b_6 p_4) \\
 &+ q^{24} (a_6 g_6 + a_6 f_6 + b_5 p_6 + b_6 p_5) \\
 &+ q^{26} (b_6 p_6)
 \end{aligned}$$

$$= \sum N_m q^{2m}$$

+++++

The numerator of $w(s)$

$$= (\sum N_m q^{2m}) (\sum L_m q^{2m}) \quad (m=0 \text{ to } 13)$$

*Because the numerator of $\phi(s)$

$$= \sum L_m q^{2m} \quad (m=0 \text{ to } 13)$$

$$L_0 = e_0 a a_0 \quad \text{-----} \quad L_{13} = B_6 m_6$$

$$\begin{aligned}
 &(N_0 + N_1 q^2 + N_2 q^4 + N_3 q^6 + N_4 q^8 + N_5 q^{10} + N_6 q^{12} \\
 &+ N_7 q^{14} + N_8 q^{16} + N_9 q^{18} + N_{10} q^{20} + N_{11} q^{22} \\
 &+ N_{12} q^{24} + N_{13} q^{26}) \\
 &(L_0 + L_1 q^2 + L_2 q^4 + L_3 q^6 + L_4 q^8 + L_5 q^{10} + L_6 q^{12} \\
 &+ L_7 q^{14} + L_8 q^{16} + L_9 q^{18} + L_{10} q^{20} + L_{11} q^{22} + L_{12} q^{24} \\
 &+ L_{13} q^{26})
 \end{aligned}$$

$$\begin{aligned}
 &= N_0 L_0 + N_0 L_1 q^2 + N_0 L_2 q^4 + N_0 L_3 q^6 + N_0 L_4 q^8 \\
 &\quad + N_0 L_5 q^{10} + N_0 L_6 q^{12} + N_0 L_7 q^{14} \\
 &\quad + N_0 L_8 q^{16} + N_0 L_9 q^{18} + N_0 L_{10} q^{20} + N_0 L_{11} q^{22} \\
 &\quad + N_0 L_{12} q^{24} + N_0 L_{13} q^{26} \\
 &+ N_1 L_0 q^2 + N_1 L_1 q^4 + N_1 L_2 q^6 + N_1 L_3 q^8 + N_1 L_4 q^{10} \\
 &\quad + N_1 L_5 q^{12} + N_1 L_6 q^{14} + N_1 L_7 q^{16} + N_1 L_8 q^{18} \\
 &\quad + N_1 L_9 q^{20} + N_1 L_{10} q^{22} + N_1 L_{11} q^{24} \\
 &\quad + N_1 L_{12} q^{26} + N_1 L_{13} q^{28}
 \end{aligned}$$

$$\begin{aligned}
 &+ N_2 L_0 q^4 + N_2 L_1 q^6 + N_2 L_2 q^8 + N_2 L_3 q^{10} + N_2 L_4 q^{12} \\
 &\quad + N_2 L_5 q^{14} + N_2 L_6 q^{16} + N_2 L_7 q^{18} + N_2 L_8 q^{20} \\
 &\quad + N_2 L_9 q^{22} + N_2 L_{10} q^{24} + N_2 L_{11} q^{26} \\
 &\quad + N_2 L_{12} q^{28} + N_2 L_{13} q^{30} \\
 &+ N_3 L_0 q^6 + N_3 L_1 q^8 + N_3 L_2 q^{10} + N_3 L_3 q^{12} \\
 &\quad + N_3 L_4 q^{14} + N_3 L_5 q^{16} + N_3 L_6 q^{18} + N_3 L_7 q^{20} \\
 &\quad + N_3 L_8 q^{22} + N_3 L_9 q^{24} + N_3 L_{10} q^{26} + N_3 L_{11} q^{28} \\
 &\quad + N_3 L_{12} q^{30} + N_3 L_{13} q^{32} \\
 &+ N_4 L_0 q^8 + N_4 L_1 q^{10} + N_4 L_2 q^{12} + N_4 L_3 q^{14} \\
 &\quad + N_4 L_4 q^{16} + N_4 L_5 q^{18} + N_4 L_6 q^{20} \\
 &\quad + N_4 L_7 q^{22} + N_4 L_8 q^{24} + N_4 L_9 q^{26} + N_4 L_{10} q^{28} \\
 &\quad + N_4 L_{11} q^{30} + N_4 L_{12} q^{32} + N_4 L_{13} q^{34} \\
 &+ N_5 L_0 q^{10} + N_5 L_1 q^{12} + N_5 L_2 q^{14} + N_5 L_3 q^{16} \\
 &\quad + N_5 L_4 q^{18} + N_5 L_5 q^{20} + N_5 L_6 q^{22} \\
 &\quad + N_5 L_7 q^{24} + N_5 L_8 q^{26} + N_5 L_9 q^{28} + N_5 L_{10} q^{30} \\
 &\quad + N_5 L_{11} q^{32} + N_5 L_{12} q^{34} + N_5 L_{13} q^{36} \\
 &+ N_6 L_0 q^{12} + N_6 L_1 q^{14} + N_6 L_2 q^{16} + N_6 L_3 q^{18} \\
 &\quad + N_6 L_4 q^{20} + N_6 L_5 q^{22} + N_6 L_6 q^{24} \\
 &\quad + N_6 L_7 q^{26} + N_6 L_8 q^{28} + N_6 L_9 q^{30} + N_6 L_{10} q^{32} \\
 &\quad + N_6 L_{11} q^{34} + N_6 L_{12} q^{36} + N_6 L_{13} q^{38} \\
 &+ N_7 L_0 q^{14} + N_7 L_1 q^{16} + N_7 L_2 q^{18} + N_7 L_3 q^{20} \\
 &\quad + N_7 L_4 q^{22} + N_7 L_5 q^{24} + N_7 L_6 q^{26} \\
 &\quad + N_7 L_7 q^{28} + N_7 L_8 q^{30} + N_7 L_9 q^{32} + N_7 L_{10} q^{34} \\
 &\quad + N_7 L_{11} q^{36} + N_7 L_{12} q^{38} + N_7 L_{13} q^{40} \\
 &+ N_8 L_0 q^{16} + N_8 L_1 q^{18} + N_8 L_2 q^{20} + N_8 L_3 q^{22} \\
 &\quad + N_8 L_4 q^{24} + N_8 L_5 q^{26} + N_8 L_6 q^{28} \\
 &\quad + N_8 L_7 q^{30} + N_8 L_8 q^{32} + N_8 L_9 q^{34} + N_8 L_{10} q^{36} \\
 &\quad + N_8 L_{11} q^{38} + N_8 L_{12} q^{40} + N_8 L_{13} q^{42}
 \end{aligned}$$

$$\begin{aligned}
 &+ N_9 L_0 q^{18} + N_9 L_1 q^{20} + N_9 L_2 q^{22} + N_9 L_3 q^{24} \\
 &\quad + N_9 L_4 q^{26} + N_9 L_5 q^{28} + N_9 L_6 q^{30} \\
 &\quad + N_9 L_7 q^{32} + N_9 L_8 q^{34} + N_9 L_9 q^{36} + N_9 L_{10} q^{38} \\
 &\quad + N_9 L_{11} q^{40} + N_9 L_{12} q^{42} + N_9 L_{13} q^{44} \\
 &+ N_{10} L_0 q^{20} + N_{10} L_1 q^{22} + N_{10} L_2 q^{24} + N_{10} L_3 q^{26} \\
 &\quad + N_{10} L_4 q^{28} + N_{10} L_5 q^{30} + N_{10} L_6 q^{32} \\
 &\quad + N_{10} L_7 q^{34} + N_{10} L_8 q^{36} + N_{10} L_9 q^{38} \\
 &\quad + N_{10} L_{10} q^{40} + N_{10} L_{11} q^{42} + N_{10} L_{12} q^{44} \\
 &\quad + N_{10} L_{13} q^{46} \\
 &+ N_{11} L_0 q^{22} + N_{11} L_1 q^{24} + N_{11} L_2 q^{26} + N_{11} L_3 q^{28} \\
 &\quad + N_{11} L_4 q^{30} + N_{11} L_5 q^{32} + N_{11} L_6 q^{34} \\
 &\quad + N_{11} L_7 q^{36} + N_{11} L_8 q^{38} + N_{11} L_9 q^{40} \\
 &\quad + N_{11} L_{10} q^{42} + N_{11} L_{11} q^{44} + N_{11} L_{12} q^{46} \\
 &\quad + N_{11} L_{13} q^{48} \\
 &+ N_{12} L_0 q^{24} + N_{12} L_1 q^{26} + N_{12} L_2 q^{28} + N_{12} L_3 q^{30} \\
 &\quad + N_{12} L_4 q^{32} + N_{12} L_5 q^{34} + N_{12} L_6 q^{36} \\
 &\quad + N_{12} L_7 q^{38} + N_{12} L_8 q^{40} + N_{12} L_9 q^{42} \\
 &\quad + N_{12} L_{10} q^{44} + N_{12} L_{11} q^{46} + N_{12} L_{12} q^{48} \\
 &\quad + N_{12} L_{13} q^{50} \\
 &+ N_{13} L_0 q^{26} + N_{13} L_1 q^{28} + N_{13} L_2 q^{30} + N_{13} L_3 q^{32} \\
 &\quad + N_{13} L_4 q^{34} + N_{13} L_5 q^{36} + N_{13} L_6 q^{38} \\
 &\quad + N_{13} L_7 q^{40} + N_{13} L_8 q^{42} + N_{13} L_9 q^{44} \\
 &\quad + N_{13} L_{10} q^{46} + N_{13} L_{11} q^{48} + N_{13} L_{12} q^{50} \\
 &\quad + N_{13} L_{13} q^{52}
 \end{aligned}$$

$$= (\sum X_m q^{2m}) \quad (m=0 \text{ to } 26)$$