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Linear Systems Analysis of an Electrical Circuit Model of Asymmetric Respiratory System

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We introduced linear systems analysis for respiratory system. The bronchial systems were described by an electrical circuit model which temporal changes of airway driving pressure and airway flow rate were described by five simultaneous linear differential equations. To elucidate the mechanical differences between the right and left lobes, we have separated the electrical system to two parts each of which is further consisted of the proximal and distal subdivisions. The resistance and capacitance were set to describe the airway resistance and bronchial compliance. The linear systems analysis disclosed that the system was stable. The singular values were changed significantly when the proximal resistance and compliance were changed. The linear system analysis will be available for evaluating the mechanical changes in proximal respiratory system.

Bronchial system. Electrical circuit. Left and right lobes. Proximal bronchi. Linear system analysis

左右非対称な気道力学特性を有する呼吸器系の電気回路モデルによる線形システム解析

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呼吸気道系におけるシステム特性を解析した。気道系を電気的等価回路で記述し、系の過渡的挙動を微分方程式で表わし、生理的に計測された系の力学特性を代入して、システムの特性を評価する方法を提唱した。左右の気道で異なる病態が生じた場合を考慮して、気道抵抗、キャパシタンスは左右で異なる値に設定できるように、電気回路モデルをより精密詳細にした。また中枢気道部と末梢気道部とを分離し、太い気管と細い気管との特性を明確にした。線形システム解析では、系は安定であった。系の特異値は中枢部の気道抵抗が増大しキャパシタンスが低下した場合大きく変化したが末梢部のそれらの変化はあまり反映されなかった。　肺における左右葉の中枢側における機械力学的変化に関しては線形システムで解析評価が可能であると推定された。

気道.　電気的等価回路.　左右葉.　中枢気道.　末梢気道.　線形システム解析

1. Introduction.

Respiratory system is the first step for gas exchange in the biological system. We introduce linear systems analysis for the repiratory system described by an equivalent electrical circuit.

2. Modeling.

Fig 1 is the model of right and left lungs which were comprised of proximal and peripheral of bronchus. The proximal one was consisted of the parallel combination of compliance Ca with series combination of Rr and Cr. The peripheral bronchus was expressed by resistance Rr1. The pressure across the compliance Cr and CL were set to be X2(t) and X3(t) respectively.

3. Results.

The linear systems analysis disclosed that the system is stable. Fig 2 shows the sigma values

$$C(j\omega I - A)^{-1} B$$

where A is the eigen matrix, C is out put matrix and B is control matrix when the system was expresed by

$$X' = AX + BU, \quad Y = CX$$

The standard values of system parameters are

Rr1 =4, Rr=3, R=2(H2Osec/L), Cr = 0.007,Ca=0.07, C=0.5(L/H2O)

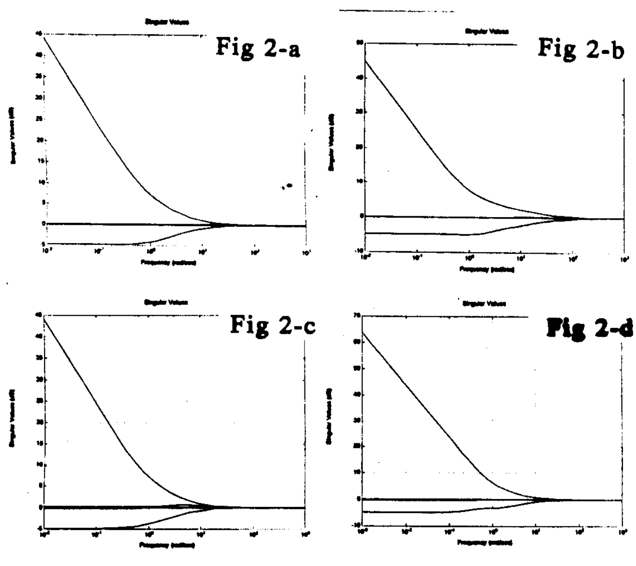
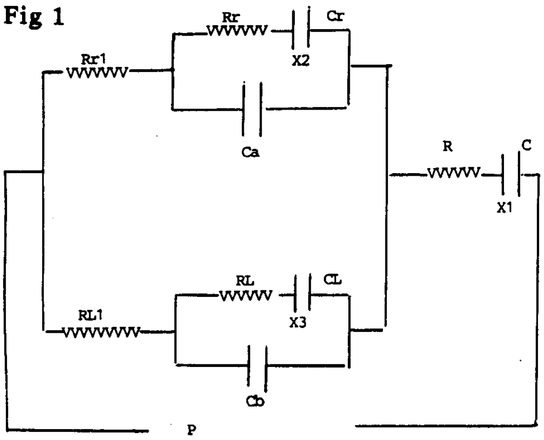


Fig 2-a Fig 2-b Fig 2-c Fig 2-d

Fig 1



[MATHEMATICAL EXPANSION]

1. For the right respiratory tract.

The flow at Cr is Cr X2'(t). Thus the pressure at Rr is Rr (Cr X2'(t)). Since the Cr-Rr and Ca is parallel, the pressure at Ca is equivalent to X2(t) + Rr Cr X2'(t) and the airway flow at Ca is

$$Ca \; d\,(X2(t) + Rr\,Cr\,X2'(t)) / dt$$
$$= Ca\,(X2'(t) + Rr\,Cr\,X2''(t)).$$

The airway flow rate at the Rr1 is the sum of flow at Rr-Cr and Ca. Thus Cr X2'(t) + Ca X2'(t) + Ca Rr Cr X2"(t). The pressure at Rr1 is thus

$$Rr1\,[X2'(t)\,(Cr + Ca) + Ca\,Rr\,Cr\,X2''(t)].$$

Consequently the total pressure at the right respiratory tract is

$$Rr1\,[X2'(t)\,(Cr + Ca) + Ca\,Rr\,Cr\,X2''(t)] + X2(t) + Rr\,Cr\,X2'(t).$$

By setting

$a1 = Rr\ Cr$, $a2 = Ca$, $a3 = Ca\ Rr\ Cr = a2\ a1$

$a4 = Cr + Ca$, $a5 = Ca\ Rr\ Cr$,

$a6 = Rr1\,(Cr + Ca) = Rr1\ a4$,

$a7 = Rr1\ Ca\ Rr\ Cr = Rr1\ a5$, $a8 = a1 + a6$

Then

The Pressure across Ca = X2(t) + a1 X2'(t)

The Flow at Ca = a2 X2'(t) + a3 X2"(t)

The Flow at Rr1 = a4 X2'(t) + a5 X2"(t)

The pressure at Rr1 = a6 X2'9t) + a7 X2"(t)

Thus the total pressure at right lung is

$$X2(t) + a8\,X2'(t) + a7\,X2''(t)$$

2. The left respiratory system.

By the same consideration the airway pressure and flow in the left bronchial system are by setting

$b1 = Rl\ Cl$, $b2 = Cb$, $b3 = Cb\ Rl\ Cl$

$b4 = Cl + Cb$, $b5 = Cb\ Rl\ Cl$, $b6 = Rl1\ b4$

$b7 = Rl1\ b5$, $b8 = b1 + b6$

The pressure across the Cb = X3(t) + b1 X3'(t)

The flow at Cb = b2 X3'(t) + b3 X3"(t)

The flow at Rl1 = b4 X3'(t) + b5 X3"(t)

The pressure at Rl1 = b6 X3'(t) + b7 X3"(t)

Then the total pressure on the left bronchus is

$$X3 + b8\,X3'(t) + b7\,X3''(t)$$

Since the pressure on the right and left bronchus is equivalent

$$X2(t) + a8\,X2'(t) + a7\,X2''(t) = X3(t) + b8\,X3'(t) + b7\,X3''(t) \quad \text{--------------(1)}$$

The total flow volume is equivalent to the C X1'(t)

$$a4\,X2'(t) + a5\,X2''(t) + b4\,X3'(t) + b5\,X3''(t) = C\,X1'(t) \quad \text{-------------(2)}$$

The total driving pressure P(t) is

$$P(t) = X2(t) + a8\,X2'(t) + a7\,X2''(t) + R\,(a4\,X2'(t) + a5\,X2''(t) + b4\,X3'(t) + b5\,X3''(t)) + X1(t) \quad \text{-----(3)}$$

3. System equations.

Assuming that the P(t) is the unknown optimal input pressure, from the equations (1) and (3), X2"(t) and X3"(t) can be expressed as functions of (X2'(t), X2(t), X3'(t), X3(t), X(t), P). From the equation (1),

$$a7\,X2''(t) - b7\,X3''(t) = X3(t) + b8\,X3'(t) - X2(t) - a8\,X2'(t) \quad \text{-------(4)}$$

rearranging the equation (3),

$$P(t) = X2''(t)\,(a7 + R\,a5) + X2'(t)(a8 + R\,a4) + X2(t) + X3''(t)\,(R\,b5) + X3'(t)\,(R\,b4) + X1(t)$$

Setting

$a9 = a7 + R\ a5$, $a10 = a8 + R\ a4$,

$a11 = R\ b5$, $a12 = R\ b4$

$$P(t) = a9\,X2''(t) + a10\,X2'(t) + X2(t) + a11\,X3''(t) + a12\,X3'(t) + X1(t)$$

Thus

$$a9\,X2''(t) + a11\,X3''(t) = P(t) - a10\,X2'(t) - X2(t) - a12\,X3'(t) - X1(t) \quad \text{------------(5)}$$

From equations (4) and (5), eliminate X3"(t)

equation(4) a11 + equation (5) b7

(a11 a7 + b7 a9) X2"(t) = a11 (X3(t) + b8 X3'(t) - X2(t) - a8 X2'(t)) + b7 (P(t) - a10 X2'(t) - X2(t) - a12 X3'(t) - X1(t))

Rearranging the right side of above equation and setting

a13 = a11 a7 + b7 a9, a14= (- a11 a8 - b7 a10)/ a13
a15= (a11 b8 - b7 a12) / a13, a16=(-a11 - b7)/ a13
a17=(a11) / a13, a18= - b7 / a13, a19= b7 / a13

Then

X2"(t) = a14 X2'(t) + a15 X3'(t) + a16 X2(t) + a17 X3(t) + a18 X1(t) + a19 P(t) --------------(6)

Substituting the equation (6) into the equation (4),

X3"(t) = (X3(t) + b8 X3'(t) - X2(t) - a8 X2'(t) - a7 X2"(t)) / (- b7)
= [X3(t) + b8 X3'(t) - X2(t) - a8 X2' (t) - a7 (a14 X2'(t) + a15 X3'(t) + a16 X2(t) + a17 X3(t) + a18 X1(t) + a19 P(t))] / (- b7)
= [X3'(t) (b8 - a7 a15) + X2'(t)(- a8 - a7 a14) + X3(t) (1 - a7 a17) + X2(t) (-1 - a7 a16) + X1(t) (-a7 a18) + P(t) (-a7 a19)] / (- b7)

By setting

a20 = (b8 - a7 a15)/ (- b7), a21= (-a8 - a7 a14) / (-b7)
a22 = (1 - a7 a17) / (-b7), a23 = (-1 - a7 a16) / (-b7)
a24 = (-a7 a18) / (-b7), a25 = (-a7 a19) / (-b7)

X3"(t) = a20 X3'(t) + a21 X2'(t) + a22 X3(t) + a23 X2(t) + a24 X1(t) + a25 P(t)

Then the state equation for the X1'(t) is

X1'(t) = [a4 X2' + a5 X2" + b4 X3' + b5 X3"]/C
= [a4 X2'(t) + a5 (a14 X2' + a15 X3' + a16 X2 + a17 X3 + a18 X1 + a19 P) + b4 X3' + b5 (a20 X3' + a21 X2' + a22 X3 + a23 X2 + a24 X1 + a25 P)] / C

By setting

a26 = (a4 + a5 a14 + b5 a21) / C
a27 = (b4 + a5 a15 + b5 a22) / C
a28 = (a5 a16+b5 a23) / C, a29 = (a5 a17+ b5 a22) / C
a30 = (a5 a18 + b5 a24) /C, a31 = (a5 a19 +b5 a25) / C

Then

X1'(t) = a26 X2'(t) + a27 X3'(t) + a28 X2(t) + a29 X3(t) + a30 X1(t) + a31 P(t)

By putting

X2'(t) = X4(t) , X3'(t) = X5(t)

The system equations are

X1'(t) = a26 X4(t) + a27 X5(t) + a28 X2(t) + a29 X3(t) + a30 X1(t) + a31 P(t)
X2'(t) = X4(t), X3'(t) = X5(t)
X4'(t) = X2"(t) = a14 X4(t) + a15 X5(t) + a16 X2(t) + a17 X3(t) + a18 X1(t) + a19 P(t)
X5'(t) = X3"(t) =a20 X5(t) + a21 X4(t) + a22 X3(t) + a23 X2(t) + a24 X1(t) + a25 P(t)

APPENDIX I

For the optimal control, we set the total input pressure as the optimal control input.

1. The flow at Rr and Cr is

Cr X2' = Cr X4

2. The Pressure at Rr and Cr is

X2 + a1 X2' = X2 + a1 X4

3. The flow at Ca is

a2 X2' + a3 X2" (= X4')
= a2 X4 + a3 * (a14 X4 + a15 X5 + a16 X2 + a17 X3 + a18 X1 + a19 P)

putting

k1 = a3 a18 , k2 = a3 a16 , k3 = a3 a17 ,
k4 = a3 a14 + a2 , k5 = a3 a15, k6 = a3 a19

Thus flow is

k1 X1 + k2 X2 + k3 X3 + k4 x4 + k5 x5 + k6 P

4. The flow at Rr1 is

a4 X2' + a5 X2" (= X4')
= a4 X4 + a5*(a14 X4 + a15 X5 + a16 X2 + a17 X3 + a18 X1 + a19 P)

putting

k7 = a5 a18 , k8 = a5 a16 , k9 = a5 a17 ,
k10 = a5 a14 + a4 , k11 = a5 a15, k12 = a5 a19

Thus flow is

k7 x1 + k8 x2 + k9 x3 + k10 x4 + k11 x5 + k12 P

5. The pressure at Rr1

a6 X2' + a7 X2" (= X4')
= a6 X4 + a7*(a14 X4 + a15 X5 + a16 X2 + a17 X3 + a18 X1 + a19 P)

putting

k13 = a7 a18 , k14 = a7 a16 , k15 = a7 a17 ,
k16 = a7 a14 + a6 , k17 = a7 a15, k18 = a7 a19

Thus the pressure is

k13 x1 + k14 x2 + k15 x3 + k16 x4 + k17 x5 + k18 P

6. The pressure at right lung

X2 + a8 X2' + a7 X2" (= X4')
= X2 + a8 X4 + a7*(a14 X4 + a15 X5 + a16 X2 + a17 X3 + a18 X1 + a19 P)

putting

k19 = a7 a18 , k20 = a7 a16 + 1 , k21 = a7 a17 ,
k22 = a7 a14 + a8 , k23 = a7 a15, k24 = a7 a19

Thus the pressure is

k19 x1 + k20 x2 + k21 x3 + k22 x4 + k23 x5 + k24 P

7. The flow at Rl, Cl is Cl X3'

8. The pressure at Cb : X3 + Rl Cl X3' = X3 + b1 X3'

9. The flow at Cb is

b2 X3' + b3 X3" (= X5')
= b2 X5 + b3 (a20 X5 + a21 X4 + a22 X3 + a23 X2 + a24 X1 + a25 P)

putting

k25 = b3 a24 , k26 = b3 a23, k27 = b3 a22
k28 = b3 a21, k29 = b3 a20 + b2 , k30 = b3 a25

Then

k25 X1 + k26 X2 + k27 X3 + k28 X4 + k29 X5 + k30 P

10. The flow at Rl1 is

b4 X3' + b5 X3" (= X5')
= b4 X5 + b5*(a20 X5 + a21 X4 + a22 X3 + a23 X2 + a24 X1 + a25 P)

Putting

k31 = b5 a24 , k32 = b5 a23, k33 = b5 a22
k34 = b5 a21, k35 = b5 a20 + b4 , k36 = b5 a25

Then

k31 x1 + k32 x2 + k33 x3 + k34 x4 + k35 x5 + k36 P

11. The pressure at Rl1 is

b6 X3' + b7 X3" (= X5')
= b6 X5 + b7*(a20 X5 + a21 X4 + a22 X3 + a23 X2 + a24 X1 + a25 P)

Putting

k37 = b7 a24 , k38 = b7 a23, k39 = b7 a22
k40 = b7 a21, k41 = b7 a20 + b6 , k42 = b7 a25

Then

k37 x1 + k38 x2 + k39 x3 + k40 x4 + k41 x3 + k42 P

12. The pressure at left is

X3 + b8 X3' + b7 X3" (= X5')
= X3 + b8 X5 + b7*(a20 X5 + a21 X4 + a22 X3 + a23 X2 + a24 X1 + a25 P)

Putting

$k43 = b7\ a24,\ k44 = b7\ a23,\ k45 = b7\ a22 + 1$

$k46 = b7\ a21,\ k47 = b7\ a20 + b8,\ k48 = b7\ a25$

Then

$k43\ x1 + k44\ x2 + k45\ x3 + k46\ x4 + k47\ x5 + k48\ P$

13. The total flow CX1' is putting

$k49 = C\ a30,\ k50 = C\ a28,\ k51 = C\ a29$

$k52 = C\ a26,\ k53 = C\ a27,\ k54 = C\ a31$

Thus

$k49\ x1 + k50\ x2 + k51\ x3 + k52\ x4 + k53\ x5 + k54\ P$

For the eValuation of the optimality,we set following performance function as

$$J = \int \alpha 1 (Cr\ X4)^2 + \alpha 2 (X2 + a1\ X4)^2 + \alpha 3 (a2\ X4 + a3\ X4')^2 + \alpha 4 (a4\ X4 + a5\ X4')^2 + \alpha 5 (a6\ X4 + a7\ X4')^2 + \alpha 6 (X2 + a8\ X4 + a7\ X4')^2 + \beta 1 (Cl\ X3')^2 + \beta 2 (X3 + Rl\ Cl\ X3')^2 + \beta 3 (b2\ X3' + b3\ X3'')^2 + \beta 4 (b4\ X3' + b5\ X3'')^2 + \beta 5 (b6\ X3' + b7\ X3'')^2 + \beta 6 (X3 + b8\ X3' + b7\ X3'')^2 + \gamma (C\ X1')^2]\ dt$$

Expanding each term as

$\alpha 3 [k1\ x1 + k2\ x2 + k3\ x3 + k4\ x4 + k5\ x5 + k6\ P]^2$
$= \alpha 3 [k12\ x12 + k2^2 x2^2 + k3^2 x3^2 + k4^2 x4^2$
$+ k5^2 x5^2 + k6^2\ P2 + 2 (k1\ k2\ x1\ x2 + k1\ k3\ x1\ x3$
$+ k1\ k4\ x1\ x4 + k1\ k5\ x1\ x5 + k1\ k6\ x1\ P + k2\ k3\ x2\ x3$
$+ k2\ k4\ x2\ x4 + k2\ k5\ x2\ x5 + k2\ k6\ x2\ P + k3\ k4\ x3\ x4$
$+ k3\ k5\ x3\ x5 + k3\ k6\ x3\ P + k4\ k5\ x4\ x5 + k4\ k6\ x4\ P$
$+ k5\ k6\ x5\ P)]$

$\alpha 4 [k7\ x1 + k8\ x2 + k9\ x3 + k10\ x4 + k11\ x5 + k12\ P]^2$
$= \alpha 4 [k7^2 x1^2 + k8^2 x2^2 + k9^2 x3^2 + k10^2 x4^2$
$+ k11^2 x5^2 + k12^2 p^2 + 2 (k7\ k8\ x1\ x2 + k7\ k9\ x1\ x3$
$+ k7\ k10\ x1\ x4 + k7\ k11\ x1\ x5 + k7\ k12\ x1\ P$
$+ k8\ k9\ x2\ x3 + k8\ k10\ x2\ x4 + k8\ k11\ x2\ x5$
$+ k8\ k12\ x2\ P + k9\ k10\ x3\ x4 + k9\ k11\ x3\ x4$
$+ k9\ k12\ x3\ P + k10\ k11\ x4\ x5 + k10\ k12\ x4\ P$
$+ k11\ k12\ x5\ P)]$

$\alpha 5 [k13\ x1 + k14\ x2 + k15\ x3 + k16\ x4 + k17\ x5$
$+ k18\ P]^2$
$= \alpha 5 [k13^2 x1^2 + k14^2 x2^2 + k15^2 x3^2 + k16^2 x4^2$
$+ k17^2 x5^2 + k18^2 P^2 + 2 (k13\ k14\ x1\ x2$
$+ k13\ k15\ x1\ x3 + k13\ k16\ x1\ x4 + k13\ k17\ x1\ x5$
$+ k13\ k18\ x1\ P + k14\ k15\ x2\ x3 + k14\ k16\ x2\ x4$
$+ k14\ k17\ x2\ x5 + k14\ k18\ x2\ P + k15\ k16\ x3\ x4$
$+ k15\ k17\ x3\ x5 + k15\ k18\ x3\ P + k16\ k17\ x4\ x5$
$+ k16\ k18\ x4\ P + k17\ k18\ x5\ P)]$

$\alpha 6 [k19\ x1 + k20\ x2 + k21\ x3 + k22\ x4 + k23\ x5$
$+ k24\ P]^2$
$= \alpha 6 [k19^2 x1^2 + k20^2 x2^2 + k21^2 x3^2\ k22^2 x4^2$
$+ k23^2 x5^2 + k24^2 P^2 + 2 (k19\ k20\ x1\ x2$
$+ k19\ k21\ x1\ x3 + k19\ k22\ x1\ x4 + k19\ k23\ x1\ x5$
$+ k19\ k24\ x1\ P + k20\ k21\ x2\ x3 + k20\ k22\ x2\ x4$
$+ k20\ k23\ x2\ x5 + k20\ k24\ x2\ P + k21\ k22\ x3\ x4$
$+ k21\ k23\ x3\ x5 + k21\ k24\ x3\ P + k22\ k23\ x4\ x5$
$+ k22\ k24\ x4\ P + k23\ h24\ x5\ P)]$

$\beta 3 [k25\ x1 + k26\ x2 + k27\ x3 + k28\ x4 + k29\ x5$
$+ k30\ P]^2$
$= \beta 3 [k25^2 x1^2 + k26^2 x2^2 + k27^2 x3^2 + k28^2 x4^2$
$+ k29^2 x5^2 + k30^2 P^2 + 2 (k25\ k26\ x1\ x2$
$+ k25\ k27\ x1\ x3 + k25\ k28\ x1\ x4 + k25\ k29\ x1\ x5$
$+ k25\ k30\ x1\ P + k26\ k27\ x2\ x3 + k26\ k28\ x2\ x4$
$+ k26\ k29\ x2\ x5 + k26\ k30\ x2\ P + k27\ k28\ x3\ x4$
$+ k27\ k29\ x3\ x5 + k27\ k30\ x3\ P + k28\ k29\ x4\ x5$
$+ k28\ k30\ x4\ P + k29\ k30\ x5\ P)]$

$\beta 4 [k31\ x1 + k32\ x2 + k33\ x3 + k34\ x4 + k35\ x5$
$+ k36\ P]^2$
$= \beta 4 [k31^2 x1^2 + k32^2 x2^2 + k33^2 x3^2 + k34^2 x4^2$
$+ k35^2 x5^2 + k36^2 P^2 + 2 (k31\ k32\ x1\ x2$
$+ k31\ k33\ x1\ x3 + k31\ k34\ x1\ x4 + k31\ k35\ x1\ x5$
$+ k31\ k36\ x1\ P + k32\ k33\ x2\ x3 + k32\ k34\ x2\ x4$
$+ k32\ k35\ x2\ x5 + k32\ k36\ x2\ P + k33\ k34\ x3\ x4$
$+ k33\ k35\ x3\ x5 + k33\ k36\ x3\ P + k34\ k35\ x4\ x5$
$+ k34\ k36\ x4\ P + k35\ k36\ x5\ P)]$

$\beta 5 [k37\ x1 + k38\ x2 + k39\ x3 + k40\ x4 + k41\ x3$
$+ k42\ P]^2$
$= \beta 5 [k37^2 x1^2 + k38^2 x2^2 + k39^2 x3^2 + k40^2 x4^2$
$+ k41^2 x5^2 + k42^2 P^2 + 2 (k37\ k38\ x1\ x2$
$+ k37\ k39\ x1\ x3 + k37\ k40\ x1\ x4 + k37\ k41\ x1\ x5$
$+ k37\ k42\ x1\ P + k38\ k39\ x2\ x3 + k38\ k40\ x2\ x4$
$+ k38\ k41\ x2\ x5 + k38\ k42\ x2\ P + k39\ k40\ x3\ x4$
$+ k39\ k41\ x3\ x5 + k39\ k42\ x3\ P + k40\ k41\ x4\ x5$
$+ k40\ k42\ x4\ P + k41\ k42\ x5\ P)]$

$\beta 6 [k43\ x1 + k44\ x2 + k45\ x3 + k46\ x4 + k47\ x5$
$+ k48\ P]^2$
$= \beta 6 [k43^2 x1^2 + k44^2 x2^2 + k45^2 x3^2 + k46^2 x4^2$
$+ k47^2 x5^2 + k48^2 P^2 + 2 (k43\ k44\ x1\ x2$
$+ k43\ k45\ x1\ x3 + k43\ k46\ x1\ x4 + k43\ k47\ x1\ x5$
$+ k43\ k48\ x1\ P + k44\ k45\ x2\ x3 + k44\ k46\ x2\ x4$
$+ k44\ k47\ x2\ x5 + k44\ k48\ x2\ P + k45\ k46\ x3\ x4$
$+ k45\ k47\ x3\ x5 + k45\ k48\ x3\ P + k46\ k47\ x4\ x5$
$+ k46\ k48\ x4\ P + k47\ k48\ x5\ P)]$

$\gamma [k49\ x1 + k50\ x2 + k51\ x3 + k52\ x4 + k53\ x5 + k54\ P]^2$
$= \gamma [k49^2 x1^2 + k50^2 x2^2 + k51^2 x3^2 + k52^2 x4^2$
$+ k53^2 x5^2 + k54^2 P^2 + 2 (k49\ k50\ x1\ x2$
$+ k49\ k51\ x1\ x3 + k49\ k52\ x1\ x4 + k49\ k53\ x1\ x5$
$+ k49\ k54\ x1\ P + k50\ k51\ x2\ x3 + k50\ k52\ x2\ x4$
$+ k50\ k53\ x2\ x5 + k52\ k54\ x2\ P + k51\ k52\ x3\ x4$
$+ k51\ k53\ x3\ x5 + k51\ k54\ x3\ P + k52\ k53\ x4\ x5$
$+ k52\ k54\ x4\ P + k53\ k54\ x5\ P)]$

Gathering the similar terms and rearrange

$x1^2 [\alpha 3\ k1^2 + \alpha 4\ k7^2 + \alpha 5\ k13^2 + \alpha 6\ k19^2$
$+ \beta 3\ k25^2 + \beta 4\ k31^2 + \beta 5\ k37^2 + \beta 6\ k43^2$
$+ \gamma\ 49^2]$: d1

$x2^2 [\alpha 3\ k2^2 + \alpha 4\ k8^2 + \alpha 5\ k14^2 + \alpha 6\ k20^2$
$+ \alpha 2 + \beta 3\ k26^2 + \beta 4\ k32^2 + \beta 5\ k38^2 + \beta 6\ k44^2$
$+ \gamma\ 50^2]$: d2

$x3^2 [\alpha 3\ k3^2 + \alpha 4\ k9^2 + \alpha 5\ k15^2 + \alpha 6\ k21^2 + \beta 2$
$+ \beta 3\ k27^2 + \beta 4\ k33^2 + \beta 5\ k39^2 + \beta 6\ k45^2$
$+ \gamma\ 51^2]$: d3

$x4^2 [\alpha 3\ k4^2 + \alpha 4\ k10^2 + \alpha 5\ k16^2 + \alpha 6\ k22^2$
$+ \alpha 1\ Cr^2 + \alpha 2\ a1^2 + \beta 3\ k28^2 + \beta 4\ k34^2 + \beta 5\ k40^2$
$+ \beta 6\ k46^2 + \gamma\ 52^2]$: d4

$x5^2$ [α3 $k5^2$ + α4 $k11^2$ + α5 $k17^2$ + α6 $k23^2$
+ β1 Cl^2 + β2 $b1^2$ + β3 $k29^2$ + β4 $k35^2$ + β5 $k41^2$
+ β6 $k47^2$ + γ 53^2] : d5

P^2 [α3 $k6^2$ + α4 $k12^2$ + α5 $k18^2$ + α6 $k24^2$
+ β3 $k30^2$ + β4 $k36^2$ + β5 $k42^2$ + β6 $k48^2$
+ γ 54^2 + δ] : d6

2 x1 x2 [α3 k1 k2 + α4 k7 k8 + α5 k13 k14
+ α6 k19 k20 + β3 k25 k26 + β4 k31 k32
+ β5 k37 k38 + β6 k43 k44 + γ k49 k50] : d7

2 x1 x3 [α3 k1 k3 + α4 k7 k9 + α5 k13 k15
+ α6 k19 k21 + β3 k25 k27 + β4 k31 k33
+ β5 k37 k39 + β6 k43 k45 + γ k49 k51] : d8

2 x1 x4 [α3 k1 k4 + α4 k7 k10 + α5 k13 k16
+ α6 k19 k22 + β3 k25 k28 + β4 k31 k34
+ β5 k37 k40 + β6 k43 k46 + γ k49 k52] : d9

2 x1 x5 [α3 k1 k5 + α4 k7 k11 + α5 k13 k17
+ α6 k19 k23 + β3 k25 k29 + β4 k31 k35
+ β5 k37 k41 + β6 k43 k47 + γ k49 k53] : d10

2 x1 P [α3 k1 k6 + α4 k7 k12 + α5 k13 k18
+ α6 k19 k24 + β3 k25 k30 + β4 k31 k36
+ β5 k37 k42 + β6 k43 k48 + γ k49 k54] : d11

2 x2 x3 [α3 k2 k3 + α4 k8 k9 + α5 k14 k15
+ α6 k20 k21 + β3 k26 k27 + β4 k32 k33
+ β5 k38 k39 + β6 k44 k45 + γ k50 k51] : d12

2 x2 x4 [α3 k2 k4 + α4 k8 k10 + α5 k14 k16
+ α6 k20 k22 + β3 k26 k28 + β4 k32 k34 + β5 k38 k40
+ β6 k44 k46 + α2 a1 + γ k50 k52] : d13

2 x2 x5 [α3 k2 k5 + α4 k8 k11 + α5 k14 k17
+ α6 k20 k23 + β3 k26 k29 + β4 k32 k35
+ β5 k38 k41 + β6 k44 k47 + γ k50 k53] : d14

2 x2 P [α3 k2 k6 + α4 k8 k12 + α5 k14 k18
+ α6 k20 k24 + β3 k26 k30 + β4 k32 k36
+ β5 k38 k42 + β6 k44 k48 + γ k50 k54] : d15

2 x3 x4 [α3 k3 k4 + α4 k9 k10 + α5 k15 k16
+ α6 k21 k22 + β3 k27 k28 + β4 k33 k34
+ β5 k39 k40 + β6 k45 k46 + γ k51 k52] : d16

2 x3 x5 [α3 k3 k5 + α4 k9 k11 + α5 k15 k17
+ α6 k21 k23 + β3 k27 k29 + β4 k33 k35
+ β5 k39 k41 + β6 k45 k47 + β2 b1 + γ k51 k53] : d17

2 x3 P [α3 k3 k6 + α4 k9 k12 + α5 k15 k18
+ α6 k21 k24 + β3 k27 k30 + β4 k33 k36
+ β5 k39 k42 + β6 k45 k48 + γ k51 k54] : d18

2 x4 x5 [α3 k4 k5 + α4 k10 k11 + α5 k16 k17
+ α6 k22 k23 + β3 k28 k29 + β4 k34 k35
+ β5 k40 k41 + β6 k46 k47 + γ k52 k53] : d19

2 x4 P [α3 k4 k6 + α4 k10 k12 + α5 k16 k18
+ α6 k22 k24 + β3 k28 k30 + β4 k34 k36
+ β5 k40 k42 + β6 k46 k48 + γ k52 k54] : d20

2 x5 P [α3 k5 k6 + α4 k11 k12 + α5 k17 k18
+ α6 k23 k24 + β3 k29 k30 + β4 k35 k36
+ β5 k41 k42 + β6 k47 k48 + γ k53 k54] : d21

Hamiltonian is

H = d1 $x1^2$ + d2 $x2^2$ + d3 $x3^2$ + d4 $x4^2$ + d5 $x5^2$ + d6 P^2
+ d7 x1 x2 + d8 x1 x3 + d9 x1 x4 + d10 x1 x5 + d11 x1 P
+ d12 x2 x3 + d13 x2 x4 + d14 x2 x5 + d15 x2 P
+ d16 x3 x4 + d17 x3 x5 + d18 x3 P + d19 x4 x5 + d20 x4 P
+ d21 x5 P
+ q1 (a30 x1 + a28 x2+ a29 x3 + a26 x4 + a27 x5 + a31 P)
+ q2 (x4) + q3 (x5)
+ q4 (a18 x1 + a16 x2 + a17 x3 + a14 x4 + a15 x5 + a19 P)
+ q5 (a24 x1 + a23 x2 + a22 x3 + a21 x4 + a20 x5 + a25 P)

This Hamiltonian will be added by some other similar terms when we set P = X6 , x6' = U as an optimal control input. The mathematical procedure is given in the next.

APPENDIX II.

For the second case where we set P = x6 and putting X6' = U as an optimal control. Then the state equations are

X1' = a30 X1 + a28 X2 + a29 X3 + a26 X4 + a27 X5
+ a31 X6
X2' = X4 X3' = X5
X4' = a14 X4 + a15 X5 + a16 X2 + a17 X3 + a18 X1
+ a19 X6
X5' = a20 X5 + a21 X4 + a22 X3 + a23 X2 + a24 X1
+ a25 X6
X6' = U

The terms that have to be newly involved other than the previous expansion in APPENDIX 1 in the prformance function relating to the second order differentiation of the airway flow rate are

λ1 (Cr X4' $)^2$,
λ2 (a2 X2" (= a2 X4') + a3 X4"$)^2$
λ3 (a4 X4' + a5 X4" $)^2$ for the left lobe and
μ1 (CL X5' $)^2$
μ2 (b2 X5' + b3 X5" $)^2$
μ3 (b4 X5' + b5 X5" $)^2$ for the right lobe
ε1 (C X1" $)^2$, δ δ U^2

Hence, the newly added terms in the criterion function are

$X4'^2$ [λ1 Cr^2 + λ2 $a2^2$ + λ3 $a4^2$] : ξ1
$X4''^2$ [λ2 $a3^2$ + λ3 $a5^2$] : ξ2
2 X4' X4" (λ2 a2 a3 + λ3 a4 a5) : ξ3
$X5'^2$ [μ1 CL^2 + μ2 $b2^2$ + μ3 $b4^2$] : η1
$X5''^2$ [μ2 $b3^2$ + μ3 $b5^2$] : η2
2 X5' X5" (μ2 b2 b3 + μ3 b4 b5) : η3
$X1''^2$ [ε1 C^2] (= Kp) + δ δ U^2

Expanding each term such as

ξ1 (X'4 $)^2$
= ξ1 (a18 X1 + a16 X2 + a17 X3 + a14 X4 + a15 X5
+ a19 X6 $)^2$
= ξ1 [($a18^2$ $X1^2$ + $a16^2$ $X2^2$
+ $a17^2$ $X3^2$$a14^2$ $X4^2$ + $a15^2$ $X5^2$ +$a19^2$ $X6^2$
+ 2 (a18 a16 X1 X2 + a18 a17 X1 X3 + a18 a14 X1 X4
+ a18 a15 X1 X5 + a18 a19 X1 X6 + a16 a17 X2 X3
+ a16 a14 X2 X4 + a16 a15 X2 X5 + a16 a19 X2 X6
+ a17 a14 X3 X4 + a17 a15 X3 X5 + a17 a19 X3 X6
+ a14 a15 X4 X5 + a14 a19 X4 X6 + a15 a19 X5 X6)]

$$
\begin{aligned}
&\eta 1 (X'5)^2 \\
&= \eta 1 (a24\,X1 + a23\,X2 + a22\,X3 + a21\,X4 + a20\,X5 \\
&\quad + a25\,X6)^2 \\
&= \eta 1 [a24^2 X1^2 + a23^2 X2^2 + a22^2 X3^2 \\
&\quad + a24^2 X4^2 + a20^2 X5^2 + a25^2 X6^2 + 2 (a24\,a23\,X1\,X2 \\
&\quad + a24\,a22\,X2\,X3 + a24\,a21\,X1\,X4 + a24\,a20\,X1\,X5 \\
&\quad + a24\,a25\,X1\,X6 + a23\,a22\,X2\,X3 + a23\,a21\,X2\,X4 \\
&\quad + a23\,a20\,X2\,X5 + a23\,a25\,X2\,X6 + a22\,a21\,X3\,X4 \\
&\quad + a22\,a20\,X3\,X5 + a22\,a25\,X3\,X6 + a21\,a20\,X4\,X5 \\
&\quad + a21\,a25\,X4\,X5 + a20\,a25\,X5\,X6)]
\end{aligned}
$$

$$
\begin{aligned}
X1'' &= a30 (a30\,X1 + a28\,X2 + a29\,X3 + a26\,X4 \\
&\quad + a27\,X5 + a31\,X6) + a28 (X4) + a29 (X5) \\
&\quad + a26 (a18\,X1 + a16\,X2 + a17\,X3 + a14\,X4 + a15\,X5 \\
&\quad + a19\,X6) + a27 (a24\,X1 + a23\,X2 + a22\,X3 + a21\,X4 \\
&\quad + a20\,X5 + a25\,X6) + a31 (u)
\end{aligned}
$$

By setting

L1 : X1 (a30 a30 + a26 a18 + a27 a24)
L2 : X2 (a30 a28 + a26 a16 + a27 a23)
L3 : X3 (a30 a29 + a26 a17 + a27 a22)
L4 : X4 (a30 a26 + a26 a14 + a27 a21 + a28)
L5 : X5 (a30 a27 + a26 a25 + a27 a20 + + a29)
L6 : X6 (a30 a31 + a26 a19 + a27 a25)
L7 : u (a31)

Then

$$
\begin{aligned}
X1'' &= L1\,X1 + L2\,X2 + L3\,X3 + L4\,X4 + L5\,X5 \\
&\quad + L6\,X6 + L7\,u
\end{aligned}
$$

$$
\begin{aligned}
X4'' &= a18 (a30\,X1 + a28\,X2 + a29\,X3 + a26\,X4 \\
&\quad + a27\,X5 + a31\,X6) + a16 (X4) + a17 (X5) \\
&\quad + a14 (a18\,X1 + a16\,X2 + a17\,X3 + a14\,X4 + a15\,X5 \\
&\quad + a19\,X6) + a15 (a24\,X1 + a23\,X2 + a22\,X3 + a21\,X4 \\
&\quad + a20\,X5 + a25\,X6) + a19 (u)
\end{aligned}
$$

By setting

L8 : X1 (a18 a30 + a14 a18 + a25 a24)
L9 : X2 (a18 a28 + a14 a16 + a15 a23)
L10 : X3 (a18 a29 + a14 a17 + a15 a22)
L11 : X4 (a18 a26 + a14 a14 + a15 a21 + a16)
L12 : X5 (a18 a27 + a14 a15 + a15 a20 + a17)
L13 : X6 (a18 a31 + a14 a19 + a15 a25)
L14 : u (a19)

Then

$$
\begin{aligned}
X4'' &= L8\,X1 + L9\,X2 + L10\,X3 + L11\,X4 + L12\,X5 \\
&\quad + L13\,X6 + L14\,u
\end{aligned}
$$

$$
\begin{aligned}
X5'' &= a24 (a30\,X1 + a28\,X2 + a29\,X3 + a26\,X4 \\
&\quad + a27\,X5 + a31\,X6) + a23 (X4) + a22 (X5) \\
&\quad + a21 (a18\,X1 + a16\,X2 + a17\,X3 + a14\,X4 + a15\,X5 \\
&\quad + a19\,X6) + a20 (a24\,X1 + a23\,X2 + a22\,X3 \\
&\quad + a21\,X4 + a20\,X5 + a25\,X6) + a25 (u)
\end{aligned}
$$

By setting

L15 : X1 (a24 a30 + a21 a18 + a20 a24)
L16 : X2 (a24 a28 + a21 a16 + a20 a23)
L17 : X3 (a24 a29 + a21 a17 + a20 a22)
L18 : X4 (a24 a26 + a21 a14 + a20 a21 + a23)
L19 : X5 (a24 a27 + a21 a28 + a20 a20 + a22)
L20 : X6 (a24 a31 + a21 a19 + a20 a25)
L21 : u (a25)

Then

$$
\begin{aligned}
X5'' &= L15\,X1 + L15\,X2 + L17\,X3 + L18\,X4 \\
&\quad + L19\,X5 + L20\,X6 + L21\,u
\end{aligned}
$$

The expansion of powers are

$$
\begin{aligned}
&Kp (X1'')^2 \\
&= Kp [L1\,X1 + L2\,X2 + L3\,X3 + L4\,X4 + L5\,X5 \\
&\quad + L6\,X6 + L7\,u]^2 \\
&= Kp [L1^2 X1^2 + L2^2 X2^2 + L3^2 X3^2 + L4^2 X4^2 \\
&\quad + L5^2 X5^2 + L6^2 X6^2 + L7^2 u^2 + 2 (L1\,L2\,X1\,X2 \\
&\quad + L1\,L3\,X1X3 + L1\,L4\,X1X4 + L1\,L5\,X1X5 \\
&\quad + L1\,L6\,X1\,X6 + L1\,L7\,X1\,u + L2\,L3\,X2\,X3 \\
&\quad + L2\,L4\,X2\,X4 + L2\,L5\,X2\,X5 + L2\,L6\,X2\,X6 \\
&\quad + L2\,L7\,X2\,u + L3\,L4\,X3\,X4 + L3\,L5\,X3\,X5 \\
&\quad + L3\,L6\,X3\,X6 + L3\,L7\,X3\,u + L4\,L5\,X4\,X5 \\
&\quad + L4\,L6\,X4\,X6 + L4\,L7\,X4\,u + L5\,L6\,X5\,u + L6\,L7\,X6\,u
\end{aligned}
$$

$$
\begin{aligned}
&\xi 2 (X4'')^2 \\
&= \xi 2 [L8\,X1 + L9\,X2 + L10\,X3 + L11\,X4 + L12\,X5 \\
&\quad + L13\,X6 + L14\,u]^2 \\
&= \xi 2 [L8^2 X1^2 + L9^2 X2^2 + L10^2 X3^2\,L11^2 X4^2 \\
&\quad + L12^2 X5^2 + L13^2 X6^2 + L14^2 u^2 \\
&\quad + 2 (L8\,L9\,X1\,X2 + L8\,L10\,X1\,X3 + L8\,L11\,X1\,X4 \\
&\quad + L8\,L12\,X1\,X5 + L8\,L13\,X1\,X6 + L8\,L14\,X\,u \\
&\quad + L9\,L10\,X2\,X3 + L9\,L11\,X2\,X4 + L9\,L12\,X2\,X5 \\
&\quad + L9\,L13\,X2\,X6 + L9\,L14\,X2\,u + L10\,L11\,X3\,X4 \\
&\quad + L10\,L12\,X3\,X5 + L10\,L13\,X3\,X6 + L10\,L14\,X3\,u \\
&\quad + L11\,L12\,X4\,X5 + L11\,L13\,X4\,X6 + L11\,L14\,X4 \\
&\quad + L12\,L13\,X5\,X6 + L12\,L14\,X5\,u + L13\,L14\,X6\,u
\end{aligned}
$$

$$
\begin{aligned}
&\eta 2 (X5'')^2 \\
&= [L15\,X1 + L16\,X2 + L17\,X3 + L18\,X4 + L19\,X5 \\
&\quad + L20\,X6 + L21\,u] \\
&= [L15^2 X1^2 + L10^2 X2^2 + L17^2 X3^2 \\
&\quad + L18^2 X4^2 + L19^2 X5^2 + L20^2 X6^2 + L21^2 u^2 \\
&\quad + 2 (L15\,L16\,X1\,X2 + L15\,L17\,X1\,X3 + L15\,L18\,X1\,X4 \\
&\quad + L15\,L19\,X1\,X5 + L15\,L20\,X1\,X6 + L15\,L21\,X1\,u \\
&\quad + L16\,L17\,X2\,X3 + L16\,L18\,X2\,X4 + L16\,L19\,X2\,X5 \\
&\quad + L16\,L20\,X2\,X6 + L16\,L21\,X2\,u + L17\,L18\,X3\,X4 \\
&\quad + L17\,L19\,X3\,X5 + L17\,L20\,X3\,X6 + L17\,L21\,X3\,u \\
&\quad + L15\,L19\,X4\,X5 + L18\,L20\,X4\,X6 + L18\,L21\,X4\,u \\
&\quad + L19\,L20\,X5\,X6 + L19\,L21\,X5\,u + L20\,L21\,X6\,u
\end{aligned}
$$

$$
\begin{aligned}
&\xi 3\,X'4\,X''4 \\
&= \xi 3 (a18\,X1 + a16\,X2 + a17\,X3 + a14\,X4 + a15\,X5 \\
&\quad + a19\,X6) * (L8\,X1 + L9\,X2 + L10\,X3 + L11\,X4 \\
&\quad + L12\,X5 + L13\,X6 + L14\,u) \\
&= \xi 3 [X1^2 (a18\,L8) + X2^2 (a16\,L9) + X3^2 (a17\,L10) \\
&\quad + X4^2 (a14\,L11) + X5^2 (a15\,L12) + X6^2 (a19\,L13) \\
&\quad + X1\,X2 [L8\,a16 + a18\,L9] + X1\,X3 [L8\,a17 + a18\,L10] \\
&\quad + X1\,X4 [L8\,a14 + a18\,L11] \\
&\quad + X1\,X5 [L8\,a15 + a18\,L12] + X1\,X6 [L8\,a19 + a18\,L13] \\
&\quad + X1\,u [a18\,L14] \\
&\quad + X2\,X3 [L9\,a17 + a16\,L10] + X2\,X4 [L9\,a14 + a16\,L11] \\
&\quad + X2\,X5 [L9\,a15 + a16\,L12] \\
&\quad + X2\,X6 [L9\,a19 + a16\,L13] + X2\,u [a16\,L14] \\
&\quad + X3\,X4 [L10\,a14 + a17\,L11] \\
&\quad + X3\,X5 [L10\,a15 + a17\,L12] + X3\,X6 [L10\,L19 \\
&\quad + a17\,L13] + X3\,u [a17\,L14] \\
&\quad + X4\,X5 [L11\,a15 + a14\,L12] + X4\,X6 [L11\,a19 \\
&\quad + a14\,L13] + X4\,u [a14\,L14] \\
&\quad + X5\,X6 [L12\,a19 + a15\,L13] + X5\,u [a15\,L14] \\
&\quad + X6\,u [a19\,L14]]
\end{aligned}
$$

putting

$L8\,a16 + a18\,L9 = L27$
$L8\,a17 + a18\,L10 = L28$
$L8\,a14 + a18\,L11 = L29$
$L8\,a15 + a18\,L12 = L30$
$L8\,a19 + a18\,L13 = L31$
$+ a18\,L14 = L32$

$L9\,a17 + a16\,L10 = L33$
$L9\,a14 + a16\,L11 = L34$
$L9\,a15 + a16\,L12 = L35$
$L9\,a19 + a16\,L13 = L36$
$a16\,L14 = L37$

$L10\,a14 + a17\,L11 = L38$
$L10\,a15 + a17\,L12 = L39$
$L10\,L19 + a17\,L13 = L40$
$a17\,L14 = L41$
$L11\,a15 + a14\,L12 = L42$
$L11\,a19 + a14\,L13 = L43$
$a14\,L14 = L44$
$L12\,a19 + a15\,L13 = L45$
$a15\,L14 = L46$
$a19\,L14 = L47$

We have

$\xi 3\, X'4\, X''4$
$= \xi 3\, [\, X1^2 (a18\,L8) + X2^2 (a16\,L9) + X3^2 (a17\,L10)$
$+ X4^2 (a14\,L11) + X5^2 (a15\,L12) + X6^2 (a19\,L13)$
$+ L27\,X1\,X2 + L28\,X1\,X3 + L29\,X1\,X4$
$+ L30\,X1\,X5 + L31\,X1\,X6 + L32\,X1\,u$
$+ L33\,X2\,X3 + L34\,X2\,X4 + L35\,X2\,X5$
$+ L36\,X2\,X6 + L37\,X2\,u + L38\,X3\,X4$
$+ L39\,X3\,X5 + L40\,X3\,X6 + L41\,X3\,u$
$+ L42\,X4\,X5 + L43\,X4\,X6 + L44\,X4\,u$
$+ L45\,X5\,X6 + L46\,X5\,u + L47\,X6\,u$

$\eta 3\, X'5\, X''5$
$= \eta 3\, (a24\,X1 + a23\,X2 + a22\,X3 + a21\,X4 + a20\,X5$
$+ a25\,X6)\ (L15\,X1 + L16\,X2 + L17\,X3 + L18\,X4$
$+ L19\,X5 + L20\,X6 + L21\,u)$
$= \eta 3\, [\, X1^2 (a24\,L15) + X2^2 (a23\,L16) + X3^2 (a22\,L17)$
$+ X4^2 (a21\,L18) + X5^2 (a20\,L19) + X6^2 (a25\,L20)$
$+ X1\,X2\,[L15\,a23 + a24\,L16] + X1\,X3\,[L15\,a22$
$+ a24\,L17] + X1\,X4\,[L15\,a21 + a24\,L18]$
$+ X1\,X5\,[L15\,a20 + a24\,L19] + X1\,X6\,[L15\,a25$
$+ a24\,L20] + X1\,u\,[a24\,L21]$
$+ X2\,X3\,[L16\,a22 + a23\,L17] + X2\,X4\,[L16\,a21$
$+ a23\,L18] + X2\,X5\,[L16\,a20 + a23\,L19]$
$+ X2\,X6\,[L16\,a25 + a23\,L20] + X2\,u\,[a23\,L21]$
$+ X3\,X4\,[L17\,a21 + a22\,L18]$
$+ X3\,X5\,[L17\,a20 + a22\,L19] + X3\,X6\,[L17\,L25$
$+ a22\,L20] + X3\,u\,[a22\,L21]$
$+ X4\,X5\,[L18\,a20 + a21\,L19] + X4\,X6\,[L19\,a25$
$+ a21\,L20] + X4\,u\,[a21\,L21]$
$+ X5\,X6\,[L19\,a25 + a20\,L20] + X5\,u\,[a20\,L21]$
$+ X6\,u\,[a25\,L21]$

Setting

$L15\,a23 + a24\,L16 = L54$
$L15\,a22 + a24\,L17 = L55$
$L15\,a21 + a24\,L18 = L56$
$L15\,a20 + a24\,L19 = L57$
$L15\,a25 + a24\,L20 = L58$
$a24\,L21 = L59$

$L16\,a22 + a23\,L17 = L60$
$L16\,a21 + a23\,L18 = L61$
$L16\,a20 + a23\,L19 = L62$
$L16\,a25 + a23\,L20 = L63$
$a23\,L21 = L64$

$L17\,a21 + a22\,L18 = L65$
$L17\,a20 + a22\,L19 = L66$
$L17\,a25 + a22\,L20 = L67$
$a22\,L21 = L68$

$L18\,a20 + a21\,L19 = L69$
$L18\,a25 + a21\,L20 = L70$
$a21\,L21 = L71$

$L19\,a25 + a20\,L20 = L72$
$a20\,L21 = L73$
$a25\,l21 = L74$

Then,we have

$= \eta 3\, [\, X1^2 (a24\,L15) + X2^2 (a23\,L16) + X3^2 (a22\,L17)$
$+ X4^2 (a21\,L18) + X5^2 (a20\,L19) + X6^2 (a25\,L20)$
$+ L54\,X1\,X2 + L55\,X1\,X3 + L56\,X1\,X4$
$+ L57\,X1\,X5 + L58\,X1\,X6 + L59\,X1\,u$
$+ L60\,X2\,X3 + L61\,X2\,X4 + L62\,X2\,X5$
$+ L63\,X2\,X6 + L64\,X2\,u + L65\,X3\,X4$
$+ L66\,X3\,X5 + L67\,X3\,X6 + L68\,X3\,u$
$+ L69\,X4\,X5 + L70\,X4\,X6 + L71\,X4\,u$
$+ L72\,X5\,X6 + L73\,X5\,u + L74\,X6\,u$

Thus the additionaL terms are

$X1^2 (K\,L1^2 + \xi 2\,L8^2 + \eta 2\,L15^2 + \xi 3\,L21 + \eta 3\,L48$
$+ \xi 1\,aL8^2 + \eta 1\,a24^2)\ :\ dd1$
$X2^2 (K\,L2^2 + \xi 2\,L9^2 + \eta 2\,L16^2 + \xi 3\,L22 + \eta 3\,L49$
$+ \xi 1\,aL6^2 + \eta 1\,a23^2)\ :\ dd2$
$X3^2 (K\,L3^2 + \xi 2\,L10^2 + \eta 2\,L17^2 + \xi 3\,L23 + \eta 3\,L50$
$+ \xi 1\,aL7^2 + \eta 1\,a22^2)\ :\ dd3$
$X4^2 (K\,L4^2 + \xi 2\,L11^2 + \eta 2\,L18^2 + \xi 3\,L24 + \eta 3\,L51$
$+ \xi 1\,aL4^2 + \eta 1\,a21^2)\ :dd4$
$X5^2 (K\,L5^2 + \xi 2\,L12^2 + \eta 2\,L19^2 + \xi 3\,L25 + \eta 3\,L52$
$+ \xi 1\,aL5^2 + \eta 1\,a20^2)\ :dd5$
$X6^2 (K\,L6^2 + \xi 2\,L13^2 + \eta 2\,L20^2 + \xi 3\,L26 + \eta 3\,L53$
$+ \xi 1\,aL9^2 + \eta 1\,a25^2) : dd6$
$u^2 (K\,L7^2 + \xi 2\,L14^2 + \eta 2\,L21^2\ ) : ddU$

$2\,X1\,X2\ \ (K\,L1\,L2 + \xi 2\,L8\,L9 + \eta 2\,L15\,L16$
$+ \xi 3\,L27 + \eta 3\,L54 + \xi 1\,aL8\,a16 + \eta 1\,a24\,a23) : dd7$
$2\,X1\,X3\ \ (K\,L1\,L3 + \xi 2\,L8\,L10 + \eta 2\,L15\,L17$
$+ \xi 3\,L28 + \eta 3\,L55 + \xi 1\,aL8\,a17 + \eta 1\,a24\,a22) : dd8$
$2\,X1\,X4\ \ (K\,L1\,L4 + \xi 2\,L8\,L11 + \eta 2\,L15\,L18$
$+ \xi 3\,L29 + \eta 3\,L56 + \xi 1\,aL8\,a14 + \eta 1\,a24\,a21) : dd9$
$2\,X1\,X5\ \ (K\,L1\,L5 + \xi 2\,L8\,L12 + \eta 2\,L15\,L19$
$+ \xi 3\,L30 + \eta 3\,L57 + \xi 1\,aL8\,a15 + \eta 1\,a24\,a20) : dd10$
$2\,X1\,X6\ \ (K\,L1\,L6 + \xi 2\,L8\,L13 + \eta 2\,L15\,L20$
$+ \xi 3\,L31 + \eta 3\,L58 + \xi 1\,aL8\,a19 + \eta 1\,a24\,a25) : dd11$

$2\,X1\,u\;(K\,L1\,L7 + \xi 2\,L8\,L14 + \eta 2\,L15\,L21$
$+ \xi 3\,L32 + \eta 3\,L59) : ddUx1$

$2\,X2\,X3\;(K\,L2\,L3 + \xi 2\,L9\,L10 + \eta 2\,L16\,L17$
$+ \xi 3\,L33 + \eta 3\,L60 + \xi 1\,aL6\,a17 + \eta 1\,a23\,a22) : dd12$

$2\,X2\,X4\;(K\,L2\,L4 + \xi 2\,L9\,L11 + \eta 2\,L16\,L18$
$+ \xi 3\,L34 + \eta 3\,L61 + \xi 1\,aL6\,a14 + \eta 1\,a23\,a21) : dd13$

$2\,X2\,X5\;(K\,L2\,L5 + \xi 2\,L9\,L12 + \eta 2\,L16\,L19$
$+ \xi 3\,L35 + \eta 3\,L62 + \xi 1\,aL6\,a15 + \eta 1\,a23\,a20) : dd14$

$2\,X2X6\;(K\,L2\,L6 + \xi 2\,L9\,L13 + \eta 2\,L16\,L20$
$+ \xi 3\,L36 + \eta 3\,L63 + \xi 1\,aL6\,a19 + \eta 1\,a23\,a25) : dd15$

$2\,X2\,u\;(K\,L2\,L7 + \xi 2\,L9\,L14 + \eta 2\,L16\,L21$
$+ \xi 3\,L37 + \eta 3\,L64) : ddux2$

$2\,X3\,X4\;(K\,L3\,L4 + \xi 2\,L10\,L11 + \eta 2\,L17\,L18$
$+ \xi 3\,L38 + \eta 3\,L65 + \xi 1\,aL7\,a14 + \eta 1\,a22\,a21) : dd16$

$2\,X3\,X5\;(K\,L3\,L5 + \xi 2\,L10\,L12 + \eta 2\,L17\,L19$
$+ \xi 3\,L39 + \eta 3\,L66 + \xi 1\,aL7\,a15 + \eta 1\,a22\,a20) : dd17$

$2\,X3\,X6\;(K\,L3\,L6 + \xi 2\,L10\,L13 + \eta 2\,L17\,L20$
$+ \xi 3\,L40 + \eta 3\,L67 + \xi 1\,aL7\,a19 + \eta 1\,a22\,a25) : dd18$

$2\,X3\,u\;(K\,L3\,L7 + \xi 2\,L10\,L14 + \eta 2\,L17\,L21$
$+ \xi 3\,L41 + \eta 3\,L68) : ddUx3$

$2\,X4\,X5\;(K\,L4\,L5 + \xi 2\,L11\,L12 + \eta 2\,L18\,L19$
$+ \xi 3\,L42 + \eta 3\,L69 + \xi 1\,aL4\,a15 + \eta 1\,a21\,a20) : dd19$

$2\,X4\,X6\;(K\,L4\,L6 + \xi 2\,L11\,L13 + \eta 2\,L18\,L20$
$+ \xi 3\,L43 + \eta 3\,L70 + \xi 1\,aL4\,a19 + \eta 1\,a21\,a25) : dd20$

$2\,X4\,u\;(K\,L4\,L7 + \xi 2\,L11\,L14 + \eta 2\,L18\,L21$
$+ \xi 3\,L44 + \eta 3\,L71) : dduX4$

$2\,X5\,X6\;(K\,L5\,L6 + \xi 2\,L12\,L13 + \eta 2\,L19\,L20$
$+ \xi 3\,L45 + \eta 3\,L72 + \xi 1\,aL5\,a19 + \eta 1\,a20\,a25) : dd21$

$2\,X5\,u\;(K\,L5\,L7 + \xi 2\,L12\,L14 + \eta 2\,L19\,L21$
$+ \xi 3\,L46 + \eta 3\,L73) : ddux5$

$2\,X6\,u\;(K\,L6\,L7 + \xi 2\,L13\,L14 + \eta 2\,L20\,L21$
$+ \xi 3\,L47 + \eta 3\,L74) : ddux6$

Summing with the previous terms and setting
$m1 = d1 + dd1$ ----$mn = dn + dd\,n$ ---- $m21 = d21 + dd21$
$m22 = ddu,\; m23 = ddux1,\; m24 = ddux2$
$m25 = ddux3,\; m26 = ddux4,\; m27 = ddux5,\; m28 = ddux6$

The HamiLtonian function is

$$
\begin{aligned}
H =\; & m1\,X1^2 + m2\,X2^2 + m3\,X3^2 + m4\,X4^2 + m5\,X5^2 \\
& + m6\,X6^2 + m7\,X1\,X2 + m8\,X1\,X3 + m9\,X1\,X4 \\
& + m10\,X1\,X5 + m11\,X1\,X6 + m12\,X2\,X3 + m13\,X2\,X4 \\
& + m14\,X2\,X5 + m15\,X2\,X6 + m16\,X3\,X4 + m17\,X3\,X5 \\
& + m18\,X3\,X6 + m19\,X4\,X5 + m20\,X4\,X6 + m21\,X5\,X6 \\
& + m22\,U^2 + m23\,X1\,U + m24\,X2\,U + m25\,X3\,U \\
& + m26\,X4\,U + m27\,X5\,U + m28\,X6\,U \\
& + p1\,(a30\,X1 + a28\,X2 + a29\,X3 + a26\,X4 + a27\,X5 \\
& \quad + a31\,X6) \\
& + p2\;X4 + p3\,X5 \\
& + p4\,(a18\,X1 + a16\,X2 + a17\,X3 + a14\,X4 + a15\,X5 \\
& \quad + a19\,X6) \\
& + p5\,(a24\,X1 + a23\,X2 + a22\,X3 + a21\,X4 + a20\,X5 \\
& \quad + a25\,X6) + p6\,U
\end{aligned}
$$

The optimaL controL is obtained by differentiating H with respect to U.

$$
\begin{aligned}
& 2\,m22\,U + m23\,X1 + m24\,X2 + m25\,X3 + m26\,X4 \\
& + m27\,X5 + m28\,X6 + p6 = 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
U =\; & n1\,X1 + n2\,X2 + n3\,X3 + n4\,X4 + n5\,X5 + n6\,X6 \\
& + n7\,p6
\end{aligned}
$$

Then the optimaL state equation is modified onLy for X6'. The optimized differentiaL equations for the co-state variabLes are on the other hand,

$$
\begin{aligned}
p1' = & - dH/dX1 = -(2\,m1\,X1 + m7\,X2 + m8\,X3 + m9\,X4 \\
& + m10\,X5 + m11\,X6 + m23\,U + a30\,p1 + a18\,p4 \\
& + a24\,p5)
\end{aligned}
$$

Substituting U to above equation, the optimized form is

$$
\begin{aligned}
p1' =\; & X1\,(-2\,m1 - m23\,n1) + X2\,(-m7 - m23\,n2) \\
& + X3\,(-m8 - m23\,n3) + X4\,(-m9 - m23\,n4) \\
& + X5\,(-m10 - m23\,n5) + X6\,(-m11 - m23\,n6) \\
& + p1\,(-a30) + p4\,(-a18) + p5\,(-a24) \\
& + p6\,(-m23\,n7)
\end{aligned}
$$

By simiLar procedure optimized differentiaL

$$
\begin{aligned}
p2' = & - dH/dX2 = -(2\,m2\,X2 + m7\,X1 + m12\,X3 \\
& + m13\,X4 + m14\,X5 + m15\,X6 + m24\,U + a28\,p1 \\
& + a16\;p4 + a23\,p5) \\
= \; & X1\,(-m7 - m24\,n1) + X2\,(-2m2 - m24\,n2) \\
& + X3\,(-m12 - m24\,n3) + X4\,(-m13 - m24\,n4) \\
& + X5\,(-m14 - m24\,n5) + X6\,(-m15 - m24\,n6) \\
& + p1\,(-a28) + p4\,(-a16) + p5\,(-a23) + p6(-m24\,n7)
\end{aligned}
$$

$$
\begin{aligned}
p3' = & - dH/dX3 = -(2\,m3\,X3 + m8\,X1 + m12\,X2 \\
& + m16\,X4 + m17\,X5 + m18\,X6 + m25\,U + a29\,p1 \\
& + a17\;p4 + a22\,p5) \\
= \; & X1\,(-m8 - m25\,n1) + X2\,(-m12 - m25\,n2) \\
& + X3\,(-2m3 - m25\,n3) + X4\,(-m16 - m25\,n4) \\
& + X5\,(-m17 - m25\,n5) + X6\,(-m18 - m25\,n6) \\
& + p1\,(-a29) + p4\,(-a17) + p5\,(-a22) + p6(-m25\,n7)
\end{aligned}
$$

$$
\begin{aligned}
p4' = & - dH/dX4 = -(2\,m4\,X4 + m9\,X1 + m13\,X2 \\
& + m16\,X3 + m19\,X5 + m20\,X6 + m26\,U + a26\,p1 \\
& + p2 + a14\;p4 + a21\,p5) \\
= \; & X1\,(-m9 - m26\,n1) + X2\,(-m13 - m26\,n2) \\
& + X3\,(-m16 - m26\,n3) + X4\,(-2m4 - m26\,n4) \\
& + X5\,(-m19 - m26\,n5) + X6\,(-m20 - m26\,n6) \\
& + p1\,(-a26) + p2\,(-1) + p4\,(-a14) \\
& + p5\,(-a21) + p6\,(-m26\,n7)
\end{aligned}
$$

$$
\begin{aligned}
p5' = & - dH/dX5 = -(2\,m5\,X5 + m10\,X1 + m14\,X2 \\
& + m17\,X3 + m19\,X4 + m21\,X6 + m27\,U + a27\,p1 \\
& + p3 + a15\;p4 + a20\,p5) \\
= \; & X1\,(-m10 - m27\,n1) + X2\,(-m14 - m27\,n2) \\
& + X3\,(-m17 - m27\,n3) + X4\,(-m19 - m27\,n4) \\
& + X5\,(-2m5 - m27\,n5) + X6\,(-m21 - m27\,n6) \\
& + p1\,(-a27) + p3(-1) + p4\,(-a15) \\
& + p5\,(-a20) + p6\,(-m27\,n7)
\end{aligned}
$$

$$
\begin{aligned}
p6' = & - dH/dX6 = -(2\,m6\;X6 + m11\,X1 + m145X2 \\
& + m18\,X3 + m20\,X4 + m21\,X5 + m28\,U + a31\,p1 \\
& + a19\;p4 + a25\,p5) \\
= \; & X1\,(-m11 - m28\,n1) + X2\,(-m15 - m28\,n2) \\
& + X3\,(-m18 - m28\,n3) + X4\,(-m20 - m28\,n4) \\
& + X5\,(-m21 - m28\,n5) + X6\,(-2\,m6 - m28\,n6) \\
& + p1\,(-a31) + p4\,(-a19) + p5\,(-a25) + p6\,(-m28\,n7)
\end{aligned}
$$