

Transient Change in Discharging Probability of Neurons when the Inputs Operates more than Two Neurons.

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We introduced a mathematical method for processing of neural signal where there are more than two neurons. The basic concept derives from the queuing theory of counter proposed by Saaty (1960). The arriving impulses constitutes a Poisson distribution which is characterized by independently and identically distribution with mean of λ . The impulse departure from the processing neuron also constitutes a sequence of independent and identically distribution with mean of μ . For the initial condition, we set that there are i units preexisting in the system at $t=0$. The probabilistic differential-difference equations were solved by generating function technique. The present method will be available for analyzing the simultaneous neural processing system.

Independent Identical Distribution, Poisson input, Neural processing, Generator function.

入力が複数個の神経細胞に作用する場合の発射確率の過渡的変動

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神経細胞群がインパルス処理を行う過渡的過程において標的神経細胞が複数個ある場合の解析法を Saaty (1960) の研究にもとづいて紹介した。入力インパルスは均一かつ独立、無記憶型のポアソン型時系列(平均 λ) とした。入力を処理する神経細胞は2個ある場合を解析した。本研究では神経組織の任意の時刻においてすでに1個のインパルスが処理されている(処理確率: μ) 確率 $P_1(t)$ に関する連立微分差分確率方程式を母関数展開およびラプラス積分変換を用いて解いた。本研究では簡単のため処理する神経は2個とした。 $P_1(t)$ は初期条件によって大きく変化した。本研究方法をさらに発展させることで多数の神経細胞が同時に神経情報を処理する場合の時間経過を分析するのに有用である。

均一. 独立. 無記憶過程 ポアソン型時系列. 神経細胞活動. 連立微分差分確率方程式

1. Introduction.

Neural circuits can be characterized by its parallel information system. The neurons has an extensive soma to catch information carried by axon-dendrite synaptic connection. Such parallel processing system can be understood as many processing server system having Poisson input. The present paper introduce a mathematical method of Laplace transform of the transient probabilities of the ordered processing system with Poisson inputs multiple channels and exponential processing times. Explicit mathematical expressions are derived for the two neural processor system.

2. Mathematical process.

We analyze C-parallel-neuronal processing system with Poisson input and exponential processing time.

1. Impulses arrive by a Poisson distribution at the instants t_n ($n=1,2,3$). t_n is s renewal process. $T_n = t_n - t_{n-1}$ are independently and identically distributed according to $1 - \exp(-\lambda t)$.

An arriving impulse waits in a line and is allowed into processing at any one of c channels when the latter becomes idle.

On completing the processing, the impulses depart from the system. Let $\{s_n\}$ be a sequence of independent and identically distributed random variables that assume non negative exponential $1 - \exp(-\mu t)$.

The processing time courses are identical for processing channels. The processing $\{s_n\}$ is independent of input process $\{t_n\}$.

For the initial condition, we set that there are i units preexisting in the system at $t=0$. The forward equations are

$$\partial P_0(t) / \partial t = -\lambda P_0(t) + \mu P_1(t) \quad \text{-----(1-a)}$$

$$\partial P_n(t) / \partial t = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad \text{-----(1-b) } (1 \leq n < c)$$

$$\partial P_n(t) / \partial t = -(\lambda + c\mu) P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t) \quad \text{-----(1-b) } (c \leq n) \text{ (1-c)}$$

To solve this problem, we use the generating function

$$P(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n \quad P(z,0) = z^i$$

with initial condition $P_n(0) = \delta_{in}$, the kronecker symbol.

3. Results.

The computed results as a function of time for $P_1(t)$ changed more rapidly than those of $P_0(t)$ where there is no previous impulse on the neuron.

4. Conclusion.

A mathematical method of the transient probabilities of the ordered processing system of neural net with Poisson inputs multiple channels and exponential processing times will be available for analyzing actual neural activity for parallel information processing.

5. Reference

1. Saaty, T.L. Operations research. 1960. pp 755-772.

6. Appendix.

We introduce a method proposed by Saaty. The Laplace transform of $P_n(t)$, the probability of n impulses waiting to be processed in the system at the instance t . Then, we proceed to the mathematical process of $P_1(t)$ and $P_0(t)$.

$$P_n^*(s) = \frac{\mu}{\lambda} \left[\sum_{j=0}^{c-1} (c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j-2}} \frac{1 - (\alpha_2/\alpha_1)^{n-j+1}}{1 - \alpha_2/\alpha_1} - (c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j-3}} \frac{1 - (\alpha_2/\alpha_1)^{n-j}}{1 - \alpha_2/\alpha_1} \right] \frac{1}{c\mu\alpha_2^{n-1-3}} \frac{1 - (\alpha_2/\alpha_1)^{n-i}}{1 - \alpha_2/\alpha_1} \quad (n \geq c) \quad (24)$$

We replace $P_j^*(s)$, for $0 \leq j \leq c-1$ by the expression that we determined. We also have, for $n \leq c-2$,

$$P_n^*(s) = \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu + 1)} \left\{ (c-1) P_{c-1}^*(s) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu} \right)^{n-k} e^{-\lambda/\mu} \cdot \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} - \frac{\lambda}{\mu} P_{c-2}^*(s) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu} \right)^{n-k} \cdot e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, (c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \right\} \quad (21)$$

Thus,

$$P_{c-1}^*(s) = \frac{\alpha_2^{i+1}}{1 - \alpha_2} \left/ \left(\alpha_2^{c-1} + \sum_{n=0}^{c-3} (c-n) \alpha_2^n \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu + 1)} \cdot \left\{ (c-1) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu} \right)^{n-k} e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} - \frac{\lambda}{\mu} P_{c-2}^*(s) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu} \right)^{n-k} e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, (c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \right\} \right) \right. \quad (22)$$

This expression must now be substituted in (20) and (21) to obtain $P_n^*(s)$ for $0 \leq n \leq c-2$ explicitly. Returning to (5) we note that the remaining $P_n^*(s)$ may be obtained as the coefficients of z^n . To do this we write the denominator in the form $-\lambda(z - \alpha_1)(z - \alpha_2)$ and apply Leibnitz's theorem to the product $(z - \alpha_1)^{-1} (z - \alpha_2)^{-1}$.

Since the invert Laplace transform for α_1 are

$$L^{-1}[s \alpha_1^{-n}] = (2\sqrt{2\lambda\mu})^{-n} [(2\sqrt{2\lambda\mu})^{-n} n/t^*]$$

$$* I_{n-1}((2\sqrt{2\lambda\mu})t) - n(n+1)/t^2 I_n((2\sqrt{2\lambda\mu})t)]$$

$$L^{-1}(\alpha_1^{-n}) = (2\sqrt{2\lambda\mu})^{-n} n/t I_n((2\sqrt{2\lambda\mu})t)$$

Then

$$\begin{aligned} L^{-1}[(s + \lambda) \alpha_1^{-(k+i+1)}] &= L^{-1}[s \alpha_1^{-(k+i+1)}] + \lambda L^{-1}[\alpha_1^{-(k+i+1)}] \\ &= (2\sqrt{2\lambda\mu})^{-(k+i+1)} [(2\sqrt{2\lambda\mu})^{-(k+i+1)} (k+i+1)/t^* \\ &\quad * I_{k+i}((2\sqrt{2\lambda\mu})t) - (k+i+1)(k+i+2)/t^2 \\ &\quad * I_{k+i+1}((2\sqrt{2\lambda\mu})t)] \\ &\quad + \lambda (2\sqrt{2\lambda\mu})^{-(k+i+1)} (k+i+1)/t I_{k+i+1}((2\sqrt{2\lambda\mu})t) \end{aligned}$$

$$\begin{aligned} L^{-1}[(s + \lambda) \alpha_1^{-(m+k+i+2)}] &= L^{-1}[s \alpha_1^{-(m+k+i+2)}] + \lambda L^{-1}[\alpha_1^{-(m+k+i+2)}] \\ &= (2\sqrt{2\lambda\mu})^{-(m+k+i+2)} [(2\sqrt{2\lambda\mu})^{-(m+k+i+2)} (m+k+i+2)/t^* \\ &\quad * I_{k+m+i+1}((2\sqrt{2\lambda\mu})t) - (m+k+i+2)(m+k+i+3)/t^2 \\ &\quad * I_{k+m+i+2}((2\sqrt{2\lambda\mu})t)] \\ &\quad + \lambda (2\sqrt{2\lambda\mu})^{-(m+k+i+2)} (m+k+i+2)/t^* \\ &\quad * I_{k+m+i+2}((2\sqrt{2\lambda\mu})t) \end{aligned}$$

$$\begin{aligned} L^{-1}[(s + \lambda) \alpha_1^{-(m+k+i+1)}] &= L^{-1}[s \alpha_1^{-(m+k+i+1)}] + \lambda L^{-1}[\alpha_1^{-(m+k+i+1)}] \\ &= (2\sqrt{2\lambda\mu})^{-(m+k+i+1)} [(2\sqrt{2\lambda\mu})^{-(m+k+i+1)} (m+k+i+1)/t^* \\ &\quad * I_{k+m+i}((2\sqrt{2\lambda\mu})t) - (m+k+i+1)(m+k+i+2)/t^2 \\ &\quad * I_{k+m+i+1}((2\sqrt{2\lambda\mu})t)] \\ &\quad + \lambda (2\sqrt{2\lambda\mu})^{-(m+k+i+1)} (m+k+i+1)/t^* \\ &\quad * I_{k+m+i+1}((2\sqrt{2\lambda\mu})t) \end{aligned}$$

We can calculate the probability of $P_1(t)$

$$\begin{aligned} P_1(t) &= \frac{e^{-(\lambda+2\mu)t}}{\mu(\lambda+2\mu)} \left[\sum_{k=0}^{\infty} \rho^{-(k+i+1)} L^{-1}[(s+\lambda) \alpha_1^{-(k+i+1)}] \right. \\ &\quad + \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m (\frac{1}{2})^k \rho^{-(m+k+i+2)} L^{-1}[(s+\lambda) \alpha_1^{-(m+k+i+2)}] \\ &\quad \left. + \frac{\lambda-2\mu}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m (\frac{1}{2})^k \rho^{-(m+k+i+1)} L^{-1}[(s+\lambda) \alpha_1^{-(m+k+i+1)}] \right]. \end{aligned}$$

1. General solution in Laplace transform.

2. By assuming that there are I impulses preexisting in the system at $t=0$. The forward equations of the system are

$$\begin{aligned} P_0'(t) &= -(\lambda + c \cdot \mu) \cdot P_0 + c \mu P_0 + c \mu P_1(t) - (c-1) \mu \cdot P_1(t) \\ P_n'(t) &= -(\lambda + c \cdot \mu) \cdot P_n + (c-n) \mu \cdot P_n + \lambda \cdot P_{n-1} \\ &\quad + c \cdot \mu P_{n+1} - (c-n-1) \mu \cdot P_{n+1} \quad 1 \leq n \leq c \\ P_n'(t) &= -(\lambda + c \mu) \cdot P_n + \lambda \cdot P_{n-1} + c \cdot \mu \cdot P_{n+1} \quad n \geq c \quad (1) \end{aligned}$$

Multiplying $z^0, z^n (1 \leq n \leq c-1), z^n (n \geq c)$ on each of them,

$$\begin{aligned} z^0 \cdot P_0'(t) &= -(\lambda + c \mu) \cdot z^0 \cdot P_0 + c \mu z^0 \cdot P_0 + c \mu z^0 \cdot P_1 - (c-1) \mu \cdot z^0 \cdot P_1 \\ \sum_{n=1}^{c-1} z^n \cdot P_n'(t) &= -(\lambda + c \mu) \cdot \sum_{n=1}^{c-1} z^n \cdot P_n(t) + \mu \cdot \sum_{n=1}^{c-1} (c-n) \cdot z^n \cdot P_n(t) \\ &\quad + \lambda \cdot \sum_{n=1}^{c-1} z^n \cdot P_{n-1} + c \mu \sum_{n=1}^{c-1} z^n \cdot P_{n+1} - \mu \cdot \sum_{n=1}^{c-1} (c-n-1) \cdot P_{n+1} \cdot z^n \\ \sum_{n=c}^{\infty} z^n \cdot P_n'(t) &= -(\lambda + c \mu) \cdot \sum_{n=c}^{\infty} z^n \cdot P_n + \lambda \sum_{n=c}^{\infty} z^n \cdot P_{n-1} + c \mu \sum_{n=c}^{\infty} z^n \cdot P_{n+1} \end{aligned}$$

We define the generating function

$$G(t) \cdot P(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n \quad (n=0 \text{ to infinity}) \quad (3)$$

Adding them by side by side, we have

$$\text{The left part} = \sum_{n=0}^{\infty} z^n \cdot P_n'(t) = \partial G(t) / \partial t$$

The first term in the right

$$= -(\lambda + c \mu) (z^0 P_0 + z P_1 + \dots) = -(\lambda + c \mu) \cdot G$$

$$\lambda \cdot \left[\sum_{n=1}^{c-1} z^n \cdot P_{n-1} + \sum_{n=c}^{\infty} z^n \cdot P_{n-1} \right] = \lambda \sum_{n=1}^{\infty} z^n \cdot P_{n-1} = \lambda (z \cdot P_0 + z^2 P_1 + \dots)$$

$$= \lambda z \cdot G \quad \text{and}$$

$$c \mu \cdot \left[z^0 \cdot P_1 + \sum_{n=1}^{c-1} z^n \cdot P_{n+1} + \sum_{n=c}^{\infty} z^n \cdot P_{n+1} \right] = c \cdot \mu \cdot z^{-1} [G - P_0]$$

The residual of the right side are

$$\begin{aligned} c \mu \cdot z^0 \cdot P_0 - (c-1) \mu \cdot z^0 \cdot P_1 + \mu \cdot \sum_{n=1}^{c-1} (c-n) \cdot z^n \cdot P_n - \mu \cdot \sum_{n=1}^{c-1} (c-n-1) \cdot z^n \cdot P_{n+1} \\ = c \cdot \mu \cdot P_0 - \mu (c-1) (1-z) P_1 - \mu (c-2) z (1-z) \cdot P_2 \\ - \mu (c-3) z^2 (1-z) \cdot P_3 + \dots - \mu (c-n) z^{n-1} (1-z) \cdot P_n \\ - \mu \cdot z^{c-2} (1-z) \cdot P_{c-1} = -\mu (1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n \end{aligned}$$

Hence,

$$\begin{aligned} G'(t) &= -(\lambda + c \mu) G + \lambda \cdot z \cdot G + c \cdot \mu \cdot z^{-1} (G - P_0) \\ &\quad - \mu (1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n \end{aligned}$$

For this, we have Laplace transform

$$\begin{aligned} L[G(z, t)] &= G^*(s) \quad G(z, t=0) = z^i \\ s \cdot G^*(s) - z^i &= -(\lambda + c \mu) \cdot G^* + \lambda \cdot z \cdot G^* + c \mu (G^* - P_0^*) / z \\ &\quad - \mu (1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n^* \end{aligned}$$

Therefore,

$$\begin{aligned} G^*(s) [s + (\lambda + c \mu) - \lambda \cdot z - c \cdot \mu / z] \\ = z^i - c \cdot \mu \cdot P_0^* / z - \mu (1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n^* \quad (5) \end{aligned}$$

This is the same to

$$P^*(z, s) = \frac{\left[z^{i+1} - \mu (1-z) \sum_{n=0}^{c-1} (c-n) z^n \cdot P_n^*(s) \right]}{-\lambda (z - \alpha_1) (z - \alpha_2)}$$

The root of the denominator is

$$\alpha_{\pm} = \frac{\lambda + c \mu + s \pm \sqrt{(\lambda + c \mu + s)^2 - 4 \lambda c \mu}}{2 \cdot \lambda} \quad (6)$$

Since P_n^* must exist inside and on $|z| = 1$ and since

$|\alpha_2| < 1$, then numerator must have $(z - \alpha_2)$ as a factor. Thus

$$\sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot P_n^* = \alpha_2^{i+1} [\mu (1 - \alpha_2)] \quad (7)$$

2. In order to determine $P_n^* (0 \leq n \leq c-1)$ and thus be able to determine P_n^* for $n \geq c$ explicitly, we have to obtain c equations involving the c unknowns $P_n^* (0 \leq n \leq c-1)$. Once, P_n^* was determined, then by taking the inverse Laplace transforms, $P_n(t)$ can be determined. These $c-1$ equations are

$$\begin{aligned} P_0'(t) &= -\lambda \cdot P_0 + \mu \cdot P_1 \\ P_n'(t) &= -(\lambda + n \cdot \mu) \cdot P_n + \lambda \cdot P_{n-1} + (n+1) \mu \cdot P_{n+1} \quad (8) \end{aligned}$$

Here setting

$$Q(z, t) = \sum_{n=0}^{c-2} z^n \cdot P_n(t) \quad (9)$$

We assume that $\bar{t} \rightarrow c-2$ at $t=0$.

By multiplying z^0 and z^n on both sides of these

equations, we have the relations for $\sum_{n=0}^{c-2} P_n(t) \cdot z^n$

$$\begin{aligned} z^0 \cdot P_0'(t) &= -\lambda \cdot z^0 \cdot P_0 + \mu \cdot z^0 \cdot P_1 \\ \sum_{n=1}^{c-2} z^n \cdot P_n'(t) &= -\sum_{n=1}^{c-2} (\lambda + n \mu) z^n \cdot P_n + \lambda \sum_{n=1}^{c-2} z^n \cdot P_{n-1} \\ &\quad + \mu \sum_{n=1}^{c-2} (n+1) \cdot P_{n+1} \cdot z^n \end{aligned}$$

Adding side by side

$$\begin{aligned} \sum_{n=0}^{c-2} z^n \cdot P_n'(t) &= -\lambda \cdot (z^0 \cdot P_0 + \sum_{n=1}^{c-2} z^n \cdot P_n) \\ &\quad + \lambda \cdot (z^1 \cdot P_0 + z^2 \cdot P_1 + \dots + z^{c-2} \cdot P_{c-3}) \\ &\quad - \mu \cdot [1 \cdot z^1 \cdot P_1 + 2 \cdot z^2 \cdot P_2 + 3 \cdot z^3 \cdot P_3 + (c-2) \cdot z^{c-2} \cdot P_{c-2}] \\ &\quad + \mu \cdot [1 \cdot P_1 \cdot z^0 + 2 \cdot z \cdot P_2 + 3 \cdot z^2 \cdot P_3 + \dots + (c-1) \cdot z^{c-2} \cdot P_{c-1}] \end{aligned}$$

Then, we have

$$\frac{\partial Q}{\partial z} = P_1 + 2zP_2 + 3z^2P_3 + \dots + (c-2)z^{c-3}P_{c-2}$$

Therefore

$$\begin{aligned} Q' &= -\lambda \cdot Q + \lambda \cdot z \cdot Q - \mu z \cdot \partial Q / \partial z + \mu \cdot \partial Q / \partial z \\ &\quad - \lambda \cdot z^{c-1} P_{c-2} + \mu (c-1) z^{c-2} P_{c-1} \\ \lambda (z \cdot P_0 + z^2 \cdot P_1 + \dots + z^{c-2} \cdot P_{c-3}) &= \lambda z \cdot Q - \lambda z^{c-1} \cdot P_{c-2} \\ \mu [1 \cdot z^0 \cdot P_1 + 2 \cdot z \cdot P_2 + 3 \cdot z^2 \cdot P_3 + \dots + (c-2) \cdot z^{c-3} \cdot P_{c-2} + (c-1) z^{c-2} \cdot P_{c-1}] \\ &= \mu \cdot \partial Q / \partial z + \mu (c-1) z^{c-2} \cdot P_{c-1} \end{aligned}$$

As a result, we have

$$-\frac{\partial Q}{\partial t} + \mu \cdot \frac{\partial Q}{\partial z} \cdot (1-z) = \lambda \cdot Q \cdot (1-z) + \lambda z^{c-1} \cdot P_{c-2} - \mu \cdot (c-1) z^{c-2} \cdot P_{c-1}$$

The characteristic equation is

$$\frac{dt}{dz} = \frac{dz}{\lambda(1-z) \cdot Q + \lambda \cdot z^{c-1} \cdot P_{c-2} - \mu(c-1) z^{c-2} \cdot P_{c-1}} \quad (11)$$

1. For the first two equations, we have

$$z = 1 - c_1 e^{\mu t} \quad (12)$$

2. Substitute this to the third equation (13)

$$\frac{dt}{dz} = \frac{dQ}{(-1) \cdot \lambda \cdot c_1 e^{\mu t} \cdot Q + \lambda (1 - c_1 e^{\mu t})^{c-2} \cdot P_{c-2} - \mu (c-1) (1 - c_1 e^{\mu t})^{c-2} \cdot P_{c-1}}$$

For this, putting

$$f(x) = \lambda c_1 e^{\mu x} \quad \text{then,} \quad \int_0^t f(x) dx = \frac{\lambda c_1}{\mu} e^{\mu t}$$

Putting

$$g(x) = \mu (c-1) (1 - c_1 e^{\mu x})^{c-2} \cdot P_{c-1} - \lambda (1 - c_1 e^{\mu x})^{c-1} \cdot P_{c-2}$$

we have

$$\begin{aligned} Q(t) &= e^{-\lambda c_1 / \mu e^{\mu t}} \left[\int_0^t [g(x) e^{\lambda c_1 / \mu e^{\mu x}}] dx + c \right] \\ &= e^{-\lambda c_1 / \mu e^{\mu t}} \cdot \int_0^t \left[\mu (c-1) (1 - c_1 e^{\mu x})^{c-2} \cdot P_{c-1} - \lambda (1 - c_1 e^{\mu x})^{c-1} \cdot P_{c-2} \right] e^{\lambda c_1 / \mu e^{\mu x}} \\ &\quad + c_2 (e^{-\lambda c_1 / \mu e^{\mu t}}) \quad (14) \end{aligned}$$

3. Since $c_2 = f(c_1) = f[(1-z)e^{-\mu}]$ is unknown function

For $i > c-2$: $Q(z,0) = 0$ and

For $i \leq c-2$: $Q(z,0) = z^i$

$$f(1-z) = z^i e^{\lambda \mu (1-z)}$$

Then, setting $y = 1-z$, we have

$$f(y) = (1-y)^i e^{\lambda \mu y} \quad \text{and}$$

$$f[(1-z)e^{-\mu}] = [1 - (1-z)e^{-\mu}] e^{\lambda \mu (1-z)e^{-\mu}}$$

Therefore, in equation (14), setting $c_1 = (1-z)e^{-\mu}$

and, we get $c_2 = f(c_1) = [1 - (1-z)e^{-\mu}] e^{\lambda \mu (1-z)e^{-\mu}}$

Substitute this to equation (14)

$$Q = e^{-\lambda \mu (1-z)} \int_0^1 \mu(c-1) \{1 - (1-z)e^{-\mu(1-t)}\}^{c-2} P_{c-1}(x) \\ - \lambda \{1 - (1-z)e^{-\mu(1-t)}\}^{c-1} P_{c-2}(x) e^{\lambda \mu (1-z)e^{-\mu(1-t)}} dt \\ + e^{-\lambda \mu (1-z)(1-e^{-\mu})} \cdot [1 - (1-z)e^{-\mu}] \quad \text{--- (15)}$$

For the first term in the integration of (15)

$$L\left[(1-e^{-t})^{c-1} \cdot (1-\lambda e^{-t})^\mu e^{\mu t}\right] = \frac{\Gamma(v) \Gamma(s)}{\Gamma(v+s)} \Phi_1[s, \mu, v, \lambda, a]$$

Setting following in the above

$$v = 1, \quad \lambda = 1-z, \quad \mu = 2-c, \quad a = \lambda/\mu(1-z)$$

$$L\left[(1-(1-z)e^{-t})^{c-1} e^{\lambda \mu (1-z)e^{-t}}\right] = \frac{\Gamma(1) \Gamma(s)}{\Gamma(s+1)} \Phi_1[s, 2-c, 1, 1-z, \lambda(1-z)/\mu]$$

Hence, putting t to μt , then $s \rightarrow s/\mu$

$$L[F_{c-2}(t-x)] \\ = L\left[(1-(1-z)e^{-t})^{c-1} e^{\lambda \mu (1-z)e^{-t}}\right] = \frac{1}{\mu} \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 2-c, 1, 1-z, \lambda/\mu(1-z)]$$

Similarly for the second term in (15) $P_{c-2}(x)$

$$L[F_{c-1}(t-x)] \\ = L\left[(1-(1-z)e^{-t})^{c-1} e^{\lambda \mu (1-z)e^{-t}}\right] = \frac{1}{\mu} \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 1-c, 1, 1-z, \lambda/\mu(1-z)]$$

Hence, about

$$L[F_{c-2}(t-x)] \quad \text{and} \quad L[P_{c-1}(x)]$$

$$L[F_{c-1}(t-x)] \quad \text{and} \quad L[P_{c-2}(x)]$$

integration with respect to dx

$$L[Q(z,t)] = e^{-\lambda \mu (1-z)} \left\{ \mu(c-1) \cdot L[P_{c-1}(t)] \cdot \Phi_1[s/\mu, 2-c, 1, 1-z, \lambda(1-z)/\mu] \right. \\ = e^{-\lambda \mu (1-z)} \left\{ \frac{\mu(c-1)}{\mu} \cdot L[P_{c-1}(t)] \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 2-c, 1, 1-z, \frac{\lambda}{\mu}(1-z)] \right. \\ \left. - \frac{\lambda}{\mu} \cdot L[P_{c-2}(t)] \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 1-c, 1, 1-z, \frac{\lambda}{\mu}(1-z)] \right\} \quad \text{--- (17)}$$

In the following, we proceed by the steps of

□ Integration of $L[P_{c-2}(t)]$

□ Express this by $P_{c-1}(s)$

□ (Substitute them into the equation (17))

1. Laplace transform of $L[P_{c-2}(t)]$

From equation (9)

$$Q(z,t) = \sum_{n=0}^{c-2} P_n(t) \cdot z^n \quad \text{and} \quad \frac{\partial^{c-2} Q}{\partial z^{c-2}} = (c-2)! P_{c-2}(s)$$

Hence,

$$L[P_{c-2}(t)] = \frac{1}{(c-2)!} \frac{\partial^{c-2} L[Q(z,t)]}{\partial z^{c-2}} \quad (z=0)$$

In the equation (17), there are products of

$$g(z) = e^{-\lambda/\mu(1-z)}$$

$$h(z) = \Phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)]$$

の積の形をしているので

$$\frac{\partial^{(n)} [f \cdot g]}{\partial x^n} = \sum_{r=0}^n C_r \cdot \frac{\partial^{(n-r)} f(x)}{\partial x^{(n-r)}} \cdot \frac{\partial^{(r)} g(x)}{\partial x^r}$$

$$\frac{\partial^{c-2}}{\partial z^{c-2}} \left[e^{\lambda/\mu(1-z)} \cdot \Phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] \right] \\ = \sum_{k=0}^{c-2} C_k \cdot \frac{\partial^{(k)} \Phi_1}{\partial z^{(k)}} \cdot \frac{\partial^{(c-2-k)} e^{\lambda/\mu(1-z)}}{\partial z^{(c-2-k)}} \quad (z=0)$$

$$D^{(c-k-2)} e^{\lambda/\mu(1-z)} = (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu}$$

Therefore, z differentiation in the first term of $Q(z,t)$

$$\frac{\partial^{c-2} L[Q(z,t)]}{\partial z^{c-2}} \Big|_{(z=0)} = \sum_{k=0}^{c-2} C_k \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c) \cdot P_{c-1}^*(s) \cdot (c-1)}{\partial z^{(k)}} \quad (z=0)$$

In the second term

$$\frac{\partial^{c-2} L[Q(z,t)]}{\partial z^{c-2}} \Big|_{(z=0)} = - \sum_{k=0}^{c-2} C_k \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c) \cdot P_{c-1}^*(s) \cdot (\lambda/\mu)}{\partial z^{(k)}} \quad (z=0)$$

Hence,

$$P_{c-2}^*(s) = \frac{1}{(c-2)!} \frac{\partial^{c-2} Q^*(z,s)}{\partial z^{c-2}} \Big|_{(z=0)} \\ = \frac{1}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right. \\ \left. - \left(\frac{\lambda}{\mu} \right) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right\}$$

Solving this with respect to $P_{c-2}^*(s)$

$$P_{c-2}^*(s) \left[1 + \left(\frac{\lambda}{\mu} \right) \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \cdot e^{-\lambda/\mu} \cdot \frac{\Gamma(s/\mu)}{(c-2)! \Gamma(s/\mu+1)} \right] \\ = \frac{1}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} e^{-\lambda/\mu} \cdot (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}}$$

Hence, we have the next equation (20)

$$P_{c-2}^*(s) = \frac{(c-1)}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} P_{c-1}^*(s) \cdot e^{-\lambda/\mu} \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \\ \cdot \left[1 + \frac{\Gamma(s/\mu)}{(c-2)!} \cdot \frac{(\lambda/\mu) e^{-\lambda/\mu}}{\Gamma(s/\mu+1)} \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right]^{-1}$$

2. Next we seek $P_n^*(s)$, since

$$\frac{\partial^{(n)} Q^*(z,s)}{\partial z^{(n)}} = n! P_n(s) \Big|_{(z=0)}$$

Then, the differentiation for $Q(z,s)$ is turned from

$c-2$ to n . Similarly for the terms of $n \leq c-2$,

$$P_n^*(s) = \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right. \\ \left. - \left(\frac{\lambda}{\mu} \right) \cdot P_{c-2}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right\} \quad (21)$$

Here utilizing the equation (7)

$$\sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot P_n^* = \alpha_2^{c+1} \Gamma(\mu(1-\alpha_2))$$

Multiplying on the both sides of (21) on $(c-n) \cdot \alpha_2^n$ and

summing $\sum_{n=0}^{c-1}$

$$\sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot P_n^*(s) \\ = \sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right. \\ \left. - \frac{\lambda}{\mu} \cdot P_{c-2}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu} \right)^{c-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right\}$$

Since

$$\Phi_1(2-c) = f_1[\Gamma(m+2-c)/\Gamma(2-c)]$$

Then, for the terms of $n = c-2$, $c-1$ are dropped and the series until $c-3$ is

$$\frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} = \sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \cdot P_{c-1}^*(s)$$

$$\left\{ (c-1) \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right.$$

$$\left. - \frac{\lambda}{\mu} \cdot \frac{P_{c-2}^*(s)}{P_{c-1}^*(s)} \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right\}$$

$$+ f(n=c-1, c-2)$$

$$f(n=c-1, c-2) = P_{c-1}^* \left[\alpha_2^{c-1} + 2\alpha_2^{c-2} \frac{P_{c-2}^*}{P_{c-1}^*} \right]$$

----- derivation of the equation 924) $P_n(s)$.

Returning to the equation (5), we modify $P_n^*(s)$ by expanding in terms of z^n

$$P^*(z, s) = \frac{\left[z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n) z^n \cdot P_n^*(s) \right]}{-\lambda(z - \alpha_1)(z - \alpha_2)}$$

From the Leibnitz formula

$$\frac{1}{m!} \frac{d^m}{dz^m} \left[(z - \alpha_1)^{-1} (z - \alpha_2)^{-1} \right]_{z=0}$$

$$= \frac{1}{m!} \sum_{k=0}^m C_k \cdot \frac{d^k (z - \alpha_1)^{-1}}{dz^k} \cdot \frac{\partial^{(m-k)} (z - \alpha_2)^{-1}}{\partial z^{(m-k)}} \Big|_{z=0}$$

$$= \sum_{k=0}^m (\alpha_2/\alpha_1)^{k+1} \cdot (\alpha_2)^{-(m+2)}$$

$$= 1/(\alpha_2)^{m+2} \cdot (\alpha_2/\alpha_1) \sum_{k=0}^m (\alpha_2/\alpha_1)^k$$

$$= \frac{1}{\alpha_2^{m+1}} \cdot \frac{1}{\alpha_1} \left[\frac{1 - (\alpha_2/\alpha_1)^{m+1}}{1 - \alpha_2/\alpha_1} \right]$$

where $\alpha_1 \alpha_2 = c \cdot \mu \quad \therefore 1/\alpha_1 = \alpha_2/(c\mu)$

$$= \frac{\lambda}{c \cdot \mu \cdot \alpha_2^{m+2}} \left[\frac{1 - (\alpha_2/\alpha_1)^{m+1}}{1 - \alpha_2/\alpha_1} \right]$$

As a result, we obtain the equation (24)

The case $c=2$

For $c=2$, the Laplace transform of the generating function (5) is given by (25)

$$P^*(z, s) = \frac{z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n) z^n \cdot P_n^*(s)}{sz - (1-z)(c\mu - \lambda z)} = \frac{z^{i+1} - \mu(1-z)(2P_0^* + zP_1^*)}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)}$$

The condition on which the numerator becomes zero at $z = \alpha_2$ is $\alpha_2^{i+1} = \mu(1 - \alpha_2)(2P_0^* + \alpha_2 P_1^*)$

The Laplace transform of equation (1) - a is

$$s \cdot P_0^* = -\lambda \cdot P_0^* + \mu \cdot P_1^* \quad (s + \lambda) \cdot P_0^* = \mu \cdot P_1^*$$

hence P_1^*, P_0^* are

$$2P_0^* + \alpha_2 \cdot P_1^* = \alpha_2^{i+1} / [\mu(1 - \alpha_2)] = 2 \cdot P_0^* + \alpha_2 \cdot (s + \lambda) \cdot P_0^* / \mu$$

$$\therefore P_0^* = \frac{\alpha_2^{i+1}}{(1 - \alpha_2)[2\mu + \alpha_2(s + \lambda)]} \quad (26)$$

$$P_1^* = \frac{(s + \lambda)}{\mu} \frac{\alpha_2^{i+1}}{(1 - \alpha_2)[2\mu + \alpha_2(s + \lambda)]} \quad (27)$$

Substitute (26) (27) to (25)

$$P^*(z, s) = \left\{ z^{i+1} - \frac{(1-z)\alpha_2^{i+1} \cdot [2\mu + (s + \lambda)]}{(1 - \alpha_2)[2\mu + (s + \lambda)\alpha_2]} \right\} \frac{1}{[-\lambda(z - \alpha_1)(z - \alpha_2)]} \quad (32)$$

to modify (32)

$$2\mu + (s + \lambda) \cdot z = 2\mu + (s + \lambda) \cdot \alpha_2 + (s + \lambda)(z - \alpha_2)$$

$$P^*(z, s) = \frac{z^{i+1}}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)}$$

$$- \frac{(1-z) \cdot \alpha_2^{i+1} [2\mu + (s + \lambda) \cdot \alpha_2 + (s + \lambda)(z - \alpha_2)]}{(1 - \alpha_2)[2\mu + (s + \lambda) \cdot \alpha_2] \cdot [-\lambda(z - \alpha_1)(z - \alpha_2)]}$$

$$= \frac{z^{i+1}(1 - \alpha_2) - (1-z) \cdot \alpha_2^{i+1}}{-\lambda(z - \alpha_1)(z - \alpha_2)(1 - \alpha_2)} - \frac{(1-z) \cdot \alpha_2^{i+1}(s + \lambda)}{\lambda \cdot \alpha_1(1 - z/\alpha_1)[2\mu + (s + \lambda) \cdot \alpha_2] \cdot [1 - \alpha_2]} \quad (33)$$

The numerator of the first term in (33)

$$z^{i+1} \cdot (1 - \alpha_2) - (1 - z) \cdot \alpha_2^{i+1}$$

$$= z^{i+1} \cdot [1 - (\alpha_2/z)^{i+1}] - z^{i+1} \cdot \alpha_2 [1 - (\alpha_2/z)^i]$$

$$\text{since } (1 - (\alpha_2/z)^{i+1}) = (1 - \alpha_2/z) \cdot \sum_{r=0}^i (\alpha_2/z)^r$$

$$= z^{i+1} \cdot (1 - \alpha_2/z) \cdot \sum_{r=0}^i (\alpha_2/z)^r - z^{i+1} \cdot \alpha_2 \cdot (1 - \alpha_2/z) \cdot \sum_{r=0}^{i-1} (\alpha_2/z)^r$$

$$= (z - \alpha_2) (z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \alpha_2^3 \cdot z^{i-3} + \dots + \alpha_2^i)$$

$$- z \cdot \alpha_2 (z - \alpha_2) (z^{i-1} + \alpha_2 \cdot z^{i-2} + \alpha_2^2 \cdot z^{i-3} + \dots + \alpha_2^{i-1})$$

Therefore the first term is

$$= \frac{(z - \alpha_2)(z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \dots + \alpha_2^i)}{\lambda \alpha_1 (1 - z/\alpha_1)(z - \alpha_2)(1 - \alpha_2)}$$

$$= \frac{z \cdot \alpha_2 \cdot (z - \alpha_2)(z^{i-1} + \alpha_2 \cdot z^{i-2} + \dots + \alpha_2^{i-1})}{\lambda \alpha_1 (1 - z/\alpha_1)(z - \alpha_2)(1 - \alpha_2)}$$

$$= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \cdot \frac{1}{(1 - z/\alpha_1)}$$

$$= \frac{z \cdot \alpha_2 \cdot (z^{i-1} + \alpha_2 \cdot z^{i-2} + \dots + \alpha_2^{i-2} \cdot z + \alpha_2^{i-1})}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \cdot \frac{1}{(1 - z/\alpha_1)}$$

$$= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m$$

$$= \frac{\alpha_2 \cdot (z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^{i-2} \cdot z^2 + z \cdot \alpha_2^{i-1})}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m$$

$$= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m \cdot [1 - \alpha_2] + \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m$$

$$= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1} \sum_{m=0}^{\infty} (z/\alpha_1)^m + \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m \quad (34)$$

The second term in (34) is

$$\frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} (z/\alpha_1)^m = \sum_{m=0}^{\infty} \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2) \alpha_1^{m+1}} \cdot z^m$$

Then $\alpha_2^{i+1} / [\lambda \cdot (1 - \alpha_2) \cdot \alpha_1^{m+1}]$ can be expanded as

$$\frac{\alpha_2^{i+1}}{\lambda \cdot (1 - \alpha_2) \cdot \alpha_1^{m+1}} = \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1^{m+1}} \sum_{n=0}^{\infty} \alpha_2^n$$

since $\alpha_1 \cdot \alpha_2 = 2\mu/\lambda$ we have $\alpha_2 = 2\mu/(\lambda \alpha_1)$

$$= \frac{1}{\lambda} \left(\frac{2\mu}{\lambda \alpha_1} \right)^{i+1} \frac{1}{\alpha_1^{m+1}} \sum_{n=0}^{\infty} \left(\frac{2\mu}{\lambda \alpha_1} \right)^n = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{2\mu}{\lambda} \right)^{n+i+1} \frac{1}{\alpha_1^{n+i+1+m+1}}$$

The series of α_1 starts from $i+m+2$

$$= \frac{1}{\lambda} \sum_{k=i+m+2}^{\infty} \left(\frac{1}{\alpha_1^k} \right) \left(\frac{2\mu}{\lambda} \right)^k \cdot \left(\frac{2\mu}{\lambda} \right)^{-(m+1)} = \frac{1}{\lambda} \left(\frac{\lambda}{2\mu} \right)^{m+1} \sum_{k=i+m+2}^{\infty} \frac{1}{\alpha_1^k} \cdot \left(\frac{2\mu}{\lambda} \right)^k$$

Therefore

$$(34) \text{ second} = \sum_{m=0}^{\infty} \frac{1}{\lambda} \left(\frac{\lambda}{2\mu} \right)^{m+1} \sum_{k=i+m+2}^{\infty} \frac{1}{\alpha_1^k} \cdot \left(\frac{2\mu}{\lambda} \right)^k \cdot z^m$$

The first term in the (34) is

$$= 1/(\lambda \alpha_1) \left[\sum_{m=0}^i z^m \cdot \sum_{l=0}^i + \sum_{m=i+1}^{\infty} z^m \cdot \sum_{l=0}^{\infty} \right] (\alpha_2^{i-l} / \alpha_1^{m-l})$$

$$= 1/(\lambda \alpha_1) \left[\sum_{m=0}^i z^m \cdot \sum_{l=0}^{i-m} \right] (\alpha_2^{i-l} / \alpha_1^{m-l})$$

The second term in (33) is

$$= \frac{(1-z)\alpha_2^{i+1}(s + \lambda)}{\lambda \cdot \alpha_1 \cdot (1 - z/\alpha_1)(1 - \alpha_2)[2\mu + (s + \lambda)\alpha_2]}$$

$$= \frac{\alpha_2^{i+1}(s + \lambda)(1-z)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)[2\mu + (s + \lambda)\alpha_2]} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m$$

$1-z$ in numerator is separated to 1 and $1-z$

$$-\frac{\alpha_2^{i+1} \cdot (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \cdot \alpha_2]} \left[\frac{z}{\alpha_1} \cdot \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m - \frac{1}{\alpha_1} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m \right]$$

The coefficient of z^m is

$$\sum_{n=0}^m \left(\frac{z}{\alpha_1} \right)^{m+1} - \sum_{n=0}^m \frac{z^m}{\alpha_1^{m+1}} = \sum_{n=0}^m \left(\frac{1}{\alpha_1^n} - \frac{1}{\alpha_1^{m+1}} \right) z^m - 1$$

Therefore expand the second term in (3.3) by z^m

$$= \frac{\alpha_2^{i+1} (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \cdot \alpha_2]} \left[\sum_{m=0}^{\infty} \left(\frac{1}{\alpha_1^m} - \frac{1}{\alpha_1^{m+1}} \right) z^m - 1 \right]$$

Hence

$$\frac{\alpha_2^{i+1} (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \cdot \alpha_2]} \left(\frac{1}{\alpha_1^m} - \frac{1}{\alpha_1^{m+1}} \right) = -\frac{\alpha_2^{i+1} (s+\lambda) (1-\alpha_1)}{\lambda (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} \frac{1}{\alpha_1^{m+1}} + \frac{1}{4\mu} \sum_{i=0}^m C_m \cdot (-1)^i \left(\frac{\lambda}{2\mu} \right)^m \frac{1}{2^i} \frac{(m+k+i+2) (2\sqrt{2\lambda\mu})^{m+k+i+2}}{(2\lambda)^{m+k+i+2}} I_{m+k+i+2} (2\sqrt{2\lambda\mu} \cdot t)$$

Thus the second term in (3.3) is

$$= -P_0^* \frac{(s+\lambda)(1-\alpha_1)}{\lambda} \sum_{m=0}^{\infty} \frac{z^m}{\alpha_1^{m+1}} \quad (3.3)$$

$$\text{The 1st term} = \frac{1}{\lambda \alpha_1} \left[\sum_{m=0}^{\infty} z^m \cdot \sum_{n=0}^m + \sum_{m=1}^{\infty} z^m \cdot \sum_{n=0}^{m-1} \right] \left(\alpha_2^{i-n} / \alpha_1^{m-n} \right) + \frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{\lambda}{2\mu} \right)^m \sum_{k=m+2}^{\infty} \frac{1}{\alpha_1^k} \left(\frac{2\mu}{\lambda} \right)^k \cdot z^m$$

Then, we determine $P_0(t)$ from (2.6). Since

$$\left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k = \left(\frac{\alpha_2}{2} \right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m$$

Hence

$$\left[1 - \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right) \right]^{-1} = \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k = \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} \right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m$$

To expand $P_0^* 2\mu + (s+\lambda) \cdot \alpha_2$ in the denominator of (2.6)

P_0^* , we replace $s+\lambda$ by the eigen roots. Since

$$s = -\lambda \cdot (1-\alpha_1) \cdot (1-\alpha_2) \quad \alpha_1 = 2\mu / (\lambda \alpha_2) \quad \text{Thus}$$

$$s + \lambda = \lambda / \alpha_2 [2\mu + \lambda \alpha_2^2 - 2\mu \alpha_2] \quad \text{Hence, from (2.6)}$$

$$P_0^* = \frac{\alpha_2^{i+1}}{(1-\alpha_2) [2\mu + 2\mu + \lambda \alpha_2^2 - 2\mu \alpha_2]} = \frac{\alpha_2^{i+1}}{(1-\alpha_2) (\lambda \alpha_2^2 - 2\mu \alpha_2 + 4\mu)} = \frac{\alpha_2^{i+1}}{(\lambda + 2\mu) \left[\frac{1}{(1-\alpha_2)} + \frac{\lambda \cdot \alpha_2 + (\lambda - 2\mu)}{(\lambda \alpha_2^2 - 2\mu \alpha_2 + 4\mu)} \right]}$$

The denominator of the second term is

$$= 4\mu \left[1 - \left(\alpha_2/2 - \lambda \alpha_2^2/4\mu \right) \right] \quad \text{Hence,}$$

$$P_0^* = \frac{\alpha_2^{i+1}}{(\lambda + 2\mu)} \left[\frac{1}{1-\alpha_2} + \frac{(\lambda \alpha_2 + \lambda - 2\mu)}{4\mu \left[1 - \left(\alpha_2/2 - \lambda \alpha_2^2/4\mu \right) \right]} \right] = \frac{\alpha_2^{i+1}}{(\lambda + 2\mu)} \left[\sum_{k=0}^{\infty} (\alpha_2)^k + \frac{(\lambda \alpha_2 + \lambda - 2\mu)}{4\mu} \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k \right] = \frac{1}{(\lambda + 2\mu)} \left[\sum_{k=0}^{\infty} \alpha_2^{i+k+1} + \frac{\lambda}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{1}{2k} \right) \left(\frac{\lambda}{2\mu} \right)^m \alpha_2^{i+2+k+m} + \frac{(\lambda - 2\mu)}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k C_m \cdot (-1)^m \frac{1}{2k} \left(\frac{\lambda}{2\mu} \right)^m \alpha_2^{i+1+k+m} \right]$$

The Laplace transform is

$$L^{-1}[\alpha_2^i] = L^{-1} \left[\lambda + 2\mu + s - \left[(\lambda + 2\mu + s)^2 - 4 \cdot 2\mu \lambda \right]^{1/2} \right] / (2\lambda)^n$$

$$L^{-1} \left[\left(s - (s^2 - 8\mu \lambda)^{1/2} \right)^n \right] = \frac{n \cdot (2\sqrt{2\lambda\mu})^n}{t} \cdot I_n(2\sqrt{2\lambda\mu} \cdot t)$$

$$L^{-1}[\alpha_2^{k+i+1}] = \frac{(k+i+1) (2\sqrt{2\lambda\mu})^{k+i+1}}{t (2\lambda)^{k+i+1}} \cdot I_{k+i+1}(2\sqrt{2\lambda\mu} \cdot t)$$

$$L^{-1}[\alpha_2^{m+k+i+2}] = \frac{(m+k+i+2) (2\sqrt{2\lambda\mu})^{m+k+i+2}}{t (2\lambda)^{m+k+i+2}} \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu} \cdot t)$$

$$L^{-1}[\alpha_2^{m+k+i+1}] = \frac{(m+k+i+1) (2\sqrt{2\lambda\mu})^{m+k+i+1}}{t (2\lambda)^{m+k+i+1}} \cdot I_{m+k+i+1}(2\sqrt{2\lambda\mu} \cdot t)$$

Thus, the invert Laplace transform of $P_0^*(s)$ is

$$P_0(t) = \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu) \cdot t} \left[\sum_{k=0}^{\infty} \frac{(k+i+1) \cdot (2\sqrt{2\lambda\mu})^{k+i+1}}{(2\lambda)^{k+i+1}} \cdot I_{k+i+1}(2\sqrt{2\lambda\mu} \cdot t) \right]$$

$$+ \frac{1}{4\mu} \sum_{i=0}^{\infty} \sum_{m=0}^i C_m \cdot (-1)^m \left(\frac{\lambda}{2\mu} \right)^m \frac{1}{2^i} \frac{(m+k+i+2) (2\sqrt{2\lambda\mu})^{m+k+i+2}}{(2\lambda)^{m+k+i+2}} \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu} \cdot t) + \frac{(\lambda-2\mu)}{4\mu} \sum_{i=0}^{\infty} \sum_{m=0}^i C_m \cdot (-1)^m \left(\frac{\lambda}{2\mu} \right)^m \frac{1}{2^i} \frac{(m+k+i+1) (2\sqrt{2\lambda\mu})^{m+k+i+1}}{(2\lambda)^{m+k+i+1}} \cdot I_{m+k+i+1}(2\sqrt{2\lambda\mu} \cdot t)$$

Thereby, we replace $\lambda/2\mu$ by $\rho = \lambda/2\mu$

$$P_0(t) = \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu) \cdot t} \left[\sum_{k=0}^{\infty} (\rho)^{-(k+i+1)/2} \cdot (k+i+1) I_{k+i+1}(2\sqrt{2\lambda\mu} \cdot t) + \frac{1}{4} \cdot \frac{\lambda}{\mu} \cdot \sum_{i=0}^{\infty} \sum_{m=0}^i C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^i (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu} \cdot t) + \frac{1}{4} \cdot (\lambda-2\mu) \sum_{i=0}^{\infty} \sum_{m=0}^i C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^i (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot I_{m+k+i+1}(2\sqrt{2\lambda\mu} \cdot t) \right]$$

$$P_1^* = \frac{(s+\lambda) \alpha_2^{i+1}}{\mu \cdot (1-\alpha_2) [2\mu + (s+\lambda) \cdot \alpha_2]} = \frac{(s+\lambda)}{\mu} P_0^* = \frac{1}{\mu} \cdot [s \cdot P_0^* + \lambda \cdot P_0^*]$$

While

$$L^{-1}[s \cdot \alpha_1^{-n}] = \frac{\partial}{\partial t} L^{-1}[\alpha_1^{-n}] = (2\lambda)^n \frac{\partial}{\partial t} \left[L^{-1} \left[(8\mu \lambda)^{-n} \cdot (s - (s^2 - 8\mu \lambda)^{1/2})^n \right] \cdot e^{-(\lambda+2\mu)t} \right] = L^{-1} \left[\left(s - (s^2 - 8\mu \lambda)^{1/2} \right)^n \right] = \frac{n \cdot (2\sqrt{2\lambda\mu})^n}{t} \cdot I_n(2\sqrt{2\lambda\mu} \cdot t) = (2\lambda)^n (2\sqrt{2\lambda\mu})^{-n} \cdot \frac{\partial}{\partial t} \left[\frac{n \cdot I_n(2\sqrt{2\lambda\mu} \cdot t)}{t} \cdot e^{-(\lambda+2\mu)t} \right]$$

The differentiation of the n order modified Bessel function

$$\frac{\partial}{\partial t} \left[\frac{I_n(a \cdot t)}{t} \right] = [a \cdot t^{-1} \cdot I_{n-1}(at) - (n+1) \cdot t^{-2} \cdot I_n(at)]$$

Putting $a = 2\sqrt{2\lambda\mu}$

$$L^{-1}[s \cdot \alpha_1^{-n}] = (2\lambda)^n (2\sqrt{2\lambda\mu})^{-n} \cdot \left[n \left[2\sqrt{2\lambda\mu} t^{-1} \cdot I_{n-1}(2\sqrt{2\lambda\mu} \cdot t) - (n+1) \cdot t^{-2} \cdot I_n(2\sqrt{2\lambda\mu} \cdot t) \right] \cdot e^{-(\lambda+2\mu)t} + n \cdot I_n(2\sqrt{2\lambda\mu} \cdot t) \cdot (-\lambda - 2\mu) e^{-(\lambda+2\mu)t} \right]$$

For the practical use, we set

$$P_1(t) = L^{-1} \left[\frac{(s \cdot P_0^* + \lambda P_0^*)}{\mu} \right] = \frac{1}{\mu} L^{-1}[s \cdot P_0^*] + \frac{\lambda}{\mu} L^{-1}[P_0^*]$$

Therefore we simply add the differential of $P_0(t)$

$$\frac{\partial I_n(at)}{\partial t} = a \cdot I_{n-1}(at) - n/t \cdot I_n(at)$$

By setting $n = k+i+1, m+k+i+2, m+k+i+1$

$$1. \frac{\partial I_{k+i+1}(2\sqrt{2\lambda\mu} \cdot t)}{\partial t} = 2\sqrt{2\lambda\mu} \cdot I_{k+i} \cdot (2\sqrt{2\lambda\mu} \cdot t) - (k+i+1)/t \cdot I_{k+i+1}(2\sqrt{2\lambda\mu} \cdot t); F_1(t) = f_1(t)$$

$$2. \frac{\partial I_{m+k+i+2}(2\sqrt{2\lambda\mu} \cdot t)}{\partial t} = 2\sqrt{2\lambda\mu} \cdot I_{m+k+i+1} \cdot (2\sqrt{2\lambda\mu} \cdot t) - (m+k+i+2)/t \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu} \cdot t); F_2(t) = f_2(t)$$

$$3. \frac{\partial I_{m+k+i+1}(2\sqrt{2\lambda\mu} \cdot t)}{\partial t} = 2\sqrt{2\lambda\mu} \cdot I_{m+k+i} \cdot (2\sqrt{2\lambda\mu} \cdot t) - (m+k+i+1)/t \cdot I_{m+k+i+1}(2\sqrt{2\lambda\mu} \cdot t); F_3(t) = f_3(t)$$

$$\begin{aligned}
P_0(t) &= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu)t} \left[\sum_{i=0}^{\infty} (\sqrt{\rho})^{-(k+i+1)} \cdot (k+i+1) \cdot F_1(t) \right. \\
&+ \frac{1}{4} \cdot \left(\frac{\lambda}{\mu} \right) \sum_{i=0}^{\infty} \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot F_2(t) \\
&+ \left. \frac{1}{4\mu} (\lambda-2\mu) \sum_{i=0}^{\infty} \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot F_3(t) \right] \\
\frac{\partial}{\partial t} \left[\frac{e^{-(\lambda+2\mu)t}}{t} \right] &= -e^{-(\lambda+2\mu)t} \left[(\lambda+2\mu) \cdot t^{-1} + t^{-2} \right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial P_0(t)}{\partial t} &= \frac{-(\lambda+2\mu)}{(\lambda+2\mu)} e^{-(\lambda+2\mu)t} \cdot \left[(\lambda+2\mu) \cdot t^{-1} \right] \cdot P_0(t) \\
&+ \frac{1}{(\lambda+2\mu)} \frac{e^{-(\lambda+2\mu)t}}{t} \cdot \frac{\partial}{\partial t} \left[\frac{1}{t} \right] \\
&= -\left[(\lambda+2\mu) \cdot t^{-1} \right] P_0(t) + \frac{1}{(\lambda+2\mu)} \frac{e^{-(\lambda+2\mu)t}}{t} \cdot \\
&\left[\sum_{i=0}^{\infty} (\sqrt{\rho})^{-(k+i+1)} \cdot (k+i+1) \cdot f_1(t) \right. \\
&+ \frac{1}{4} \cdot \left(\frac{\lambda}{\mu} \right) \sum_{i=0}^{\infty} \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot f_2(t) \\
&+ \left. \frac{1}{4\mu} \cdot (\lambda-2\mu) \sum_{i=0}^{\infty} \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot f_3(t) \right]
\end{aligned}$$

The exact form of $P_0(t)$ can be obtained by inverting the Laplace form of $P_0(s)^*$ (equation (33)).

$$\begin{aligned}
P_0^* &= \frac{\alpha_2^{i+1}}{\lambda+2\mu} \left[\frac{1}{1-\alpha_2} + \frac{\lambda\alpha_2+\lambda-2\mu}{4\mu-2\mu\alpha_2+\lambda\alpha_2^2} \right] \\
&= \frac{1}{\lambda+2\mu} \left[\sum_{k=0}^{\infty} \alpha_2^{k+i+1} + \alpha_2^{i+1} \frac{(\lambda\alpha_2+\lambda-2\mu)}{4\mu} \sum_{k=0}^{\infty} \left(\frac{1}{2} \alpha_2 - \frac{\lambda}{4\mu} \alpha_2^2 \right)^k \right] \\
&= \frac{1}{\lambda+2\mu} \left[\sum_{k=0}^{\infty} \alpha_2^{k+i+1} + \alpha_2^{i+1} \frac{(\lambda\alpha_2+\lambda-2\mu)}{4\mu} \sum_{k=0}^{\infty} \left(\frac{1}{2} \alpha_2 \right)^k \cdot \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \alpha_2 \right)^m \right] \\
&= \frac{1}{\lambda+2\mu} \left[\sum_{k=0}^{\infty} \alpha_2^{k+i+1} + \frac{\lambda}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{1}{2} \alpha_2 \right)^{m+k+i+2} \right. \\
&\quad \left. + \frac{\lambda-2\mu}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{1}{2} \alpha_2 \right)^{m+k+i+1} \right] \\
F_0(t) &= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu)t} \left[\sum_{k=0}^{\infty} \left(\frac{2\mu}{\lambda} \right)^{k+i+1} (2\sqrt{2\lambda\mu})^{-(k+i+1)} (k+i+1) I_{k+i+1}(2\sqrt{2\lambda\mu}t) \right. \\
&+ \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{2\mu}{\lambda} \right)^{m+k+i+2} \\
&\quad \cdot (2\sqrt{2\lambda\mu})^{-(m+k+i+2)} (m+k+i+2) I_{m+k+i+2}(2\sqrt{2\lambda\mu}t) \\
&+ \frac{\lambda-2\mu}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{2\mu}{\lambda} \right)^{m+k+i+1} \\
&\quad \cdot (2\sqrt{2\lambda\mu})^{-(m+k+i+1)} (m+k+i+1) I_{m+k+i+1}(2\sqrt{2\lambda\mu}t) \left. \right] \\
&= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu)t} \left[\sum_{k=0}^{\infty} \left(\frac{2\mu}{\lambda} \right)^{k+i+1} \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^{k+i+1} (k+i+1) I_{k+i+1}(2\sqrt{2\lambda\mu}t) \right. \\
&+ \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{2\mu}{\lambda} \right)^{m+k+i+1} \\
&\quad \cdot \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^{m+k+i+2} (m+k+i+2) I_{m+k+i+2}(2\sqrt{2\lambda\mu}t) \\
&+ \frac{\lambda-2\mu}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{2\mu}{\lambda} \right)^{m+k+i+1} \\
&\quad \cdot \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^{m+k+i+1} (m+k+i+1) I_{m+k+i+1}(2\sqrt{2\lambda\mu}t) \left. \right] \\
&= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu)t} \left[\sum_{k=0}^{\infty} (\sqrt{\rho})^{-(k+i+1)} (k+i+1) I_{k+i+1}(2\sqrt{2\lambda\mu}t) \right. \\
&+ \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-\rho)^m \left(\frac{1}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+2)} (m+k+i+2) \\
&\quad \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu}t) + \left(\frac{1}{4} \frac{\lambda}{\mu} - \frac{1}{2} \right) \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m (-\rho)^m \left(\frac{1}{2} \right)^k \\
&\quad \cdot (\sqrt{\rho})^{-(m+k+i+1)} (m+k+i+1) I_{m+k+i+1}(2\sqrt{2\lambda\mu}t) \left. \right].
\end{aligned} \tag{37}$$

If we use L^{-1} to denote the inverse Laplace transform, we have

$$\begin{aligned}
L^{-1}(s\alpha_1^{-n}) &= (2\sqrt{2\lambda\mu})^{-n} [2\sqrt{2\lambda\mu}nt^{-1} I_{n-1}(2\sqrt{2\lambda\mu}t) - n(n+1)t^{-1} I_n(2\sqrt{2\lambda\mu}t)], \\
L^{-1}(\alpha_1^{-n}) &= (2\sqrt{2\lambda\mu})^{-n} nt^{-1} I_n(2\sqrt{2\lambda\mu}t).
\end{aligned}$$

Then:

$$\begin{aligned}
P_1(t) &= \frac{e^{-(\lambda+2\mu)t}}{\mu(\lambda+2\mu)} \left[\sum_{k=0}^{\infty} \rho^{-(k+i+1)} L^{-1}[(s+\lambda)\alpha_1^{-(k+i+1)}] \right] \\
&+ \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m \left(\frac{1}{2} \right)^k \rho^{-(m+k+i+2)} L^{-1}[(s+\lambda)\alpha_1^{-(m+k+i+2)}] \\
&+ \frac{1}{4} \frac{\lambda-2\mu}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m \left(\frac{1}{2} \right)^k \rho^{-(m+k+i+1)} L^{-1}[(s+\lambda)\alpha_1^{-(m+k+i+1)}].
\end{aligned} \tag{38}$$

If we write the second component of the Laplace transform of $P_n(t)$ in the form

$$[(s+\lambda)/\lambda\alpha_1^{-n} - (s+\lambda)/\lambda\alpha_1^{-(n+1)}] P_0^*(s),$$

we have:

$$\begin{aligned}
P_n(t) &= e^{-(\lambda+2\mu)t} \left[(\sqrt{\rho})^{n-i} I_{n-i}(2\sqrt{2\lambda\mu}t) + (\sqrt{\rho})^{n-i-1} I_{n+i+1}(2\sqrt{2\lambda\mu}t) \right. \\
&+ (1-\rho)\rho^n \sum_{k=n+i+2}^{\infty} (\sqrt{\rho})^{-k} I_k(2\sqrt{2\lambda\mu}t) \left. \right] + \frac{e^{-(\lambda+2\mu)t}}{\lambda(\lambda+2\mu)} \\
&\cdot \left\{ \sum_{k=0}^{\infty} (\rho)^{-(k+i+1)} L^{-1}[(s+\lambda)(\alpha_1^{-(n+k+i+1)} - \alpha_1^{-(n+k+i+2)})] \right. \\
&+ \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m (\rho)^m \left(\frac{1}{2} \right)^k \rho^{-(m+k+i+2)} \\
&\cdot L^{-1}[(s+\lambda)(\alpha_1^{-(n+m+k+i+2)} - \alpha_1^{-(n+m+k+i+3)})] + \frac{\lambda-2\mu}{4\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \\
&\cdot \left(\frac{1}{2} \frac{\lambda}{\mu} \right)^m \left(\frac{1}{2} \right)^k \rho^{-(m+k+i+1)} L^{-1}[(s+\lambda)(\alpha_1^{-(n+m+k+i+1)} - \alpha_1^{-(n+m+k+i+2)})] \left. \right\}.
\end{aligned} \tag{39}$$

$$\begin{aligned}
L^{-1}[P_n(s)^*] &= 1/\lambda (\lambda/2\mu)^{n+1} \sum_{k=i+n+2}^{\infty} (2\mu/\lambda)^k L^{-1}[\alpha_1^{-k}] \\
&+ 1/(\lambda\alpha_1) L^{-1}[\sum_{k=0}^n \alpha_2^{i-k} / \alpha_1^{n-k} \text{ (for } n \leq i) + \\
&+ \sum_{k=0}^i \alpha_2^{i-k} / \alpha_1^{n-k} \text{ (for } i+1 \leq n)] \\
&- 1/\lambda L^{-1}[(s+\lambda)(1-\alpha_1)P_0(s)^* \alpha_1^{-n-1}]
\end{aligned}$$

putting

$$\begin{aligned}
\rho &= \lambda/(2\mu) \\
&= \rho^{n+1}/\lambda \sum_{k=i+n+2}^{\infty} \rho^{-k} L^{-1}[\alpha_1^{-k}] \\
&+ 1/\lambda L^{-1}[\sum_{k=0}^n \alpha_2^{i-k} / \alpha_1^{n-k+1} \text{ (for } n \leq i) + \\
&+ \sum_{k=0}^i \alpha_2^{i-k} / \alpha_1^{n-k+1} \text{ (for } i+1 \leq n)] \\
&- 1/\lambda L^{-1}[s(1-\alpha_1)P_0(s)^* \alpha_1^{-n-1} + \lambda(1-\alpha_1)P_0(s)^* \alpha_1^{-n-1}]
\end{aligned}$$

We express the first and the second terms through

$$\alpha_2/\alpha_1 = 2\mu/\lambda, \quad \alpha_1^{-1} = (2\mu/(\lambda\alpha_2))^{-1} = \rho/\alpha_2.$$

Hence, the first term

$$\begin{aligned}
&= \rho^{n+1}/\lambda \sum_{k=i+n+2}^{\infty} \rho^{-k} L^{-1}[\rho\alpha_2^k] \\
&= \rho^{n+1}/\lambda \sum_{k=i+n+2}^{\infty} \rho^{-k/2} k I_k(2t\sqrt{2\mu\lambda})/t \exp(-(2\mu+\lambda)t)
\end{aligned}$$

The second term =

$$\begin{aligned}
&1/\lambda L^{-1}[\sum_{k=0}^n \alpha_2^{i-k} / (\rho\alpha_2)^{n-k+1} \text{ (for } n \leq i) \\
&+ 1/\lambda L^{-1}[\sum_{k=0}^i \alpha_2^{i-k} / (\rho\alpha_2)^{n-k+1} \text{ (for } i+1 \leq n)] \\
&= \rho^{n+1}/(t\lambda \exp(-(2\mu+\lambda)t)) \left[\sum_{k=0}^n \sum_{k=0}^i \rho^{-(i+1+n)/2} * \right. \\
&\quad * (i+1+n-2k) I_{i+1+n-2k}(2t\sqrt{2\mu\lambda}) \\
&\quad \text{Hence, summing with the first term} \\
&\quad \rho^{n+1}/(t\lambda \exp(-(2\mu+\lambda)t)) \left[\sum_{k=i+n+2}^{\infty} \sum_{k=0}^i \rho^{-(i+1+n)/2} * \right. \\
&\quad * (i+1+n-2k) I_{i+1+n-2k}(2t\sqrt{2\mu\lambda}) \\
&\quad \left. + \sum_{k=0}^n \sum_{k=0}^i \rho^{-(i+1+n)/2} * (i+1+n-2k) I_{i+1+n-2k}(2t\sqrt{2\mu\lambda}) \right]
\end{aligned}$$

The third term

$$= s (\alpha_1^{-n-1} P_0^*(s) - \alpha_1^{-n} P_0^*(s)) + \lambda (\alpha_1^{-n-1} P_0^*(s) - \alpha_1^{-n} P_0^*(s)) \\ = s (\rho^{n+1} \alpha_2^{n+1} P_0^*(s) - \rho^n \alpha_2^n P_0^*(s)) \\ + \lambda (\rho^{n+1} \alpha_2^{n+1} P_0^*(s) - \rho^n \alpha_2^n P_0^*(s))$$

Then, we have

$$L^{-1} [\alpha_2^n P_0^*(s)]$$

$$= L^{-1} [\alpha_2^n / (2\mu + \lambda)] \sum_{k=0}^{\infty} \alpha_2^{i+1+k}$$

$$+ \lambda / (4\mu) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-1)^m (\lambda / (2\mu))^m 1/2^k \alpha_2^{m+k+i+2}$$

$$+ (\lambda - 2\mu) / (4\mu) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-1)^m (\lambda / (2\mu))^m 1/2^k \alpha_2^{m+k+i+1}$$

$$= \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[\sum_{k=0}^{\infty} (\sqrt{\rho})^{-(i+1+n+k)} \right.$$

$$(n+k+i+1) I_{n+k+i+1}(2t\sqrt{2\mu}\lambda)$$

$$+ \lambda / (4\mu) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k (\sqrt{\rho})^{-(m+n+k+i+2)}$$

$$* (m+n+k+i+2) I_{m+n+k+i+2}(2t\sqrt{2\mu}\lambda)$$

$$+ (\lambda - 2\mu) / (4\mu) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k (\sqrt{\rho})^{-(m+n+k+i+1)}$$

$$* (m+n+k+i+1) I_{m+n+k+i+1}(2t\sqrt{2\mu}\lambda)$$

Multiplying s on them indicates differentiation with respect to t . Generally, we have

$$\partial I_i(at) / \partial t = a I_{i-1}(at) - L I_i(at)$$

Putting $L1 = n+k+i+1$, $L2 = n+m+k+i+2$,

$L3 = n+m+k+i+1$. Then we have

$$\partial I_{n+k+i+1}(2t\sqrt{2\mu}\lambda) / \partial t = 2\sqrt{2\mu}\lambda I_{n+k+i}(2t\sqrt{2\mu}\lambda) \\ - (n+k+i+1) / t I_{n+k+i+1}(2t\sqrt{2\mu}\lambda) \quad \text{set } F4'(t) = f4(t) \\ \partial I_{n+m+k+i+2}(2t\sqrt{2\mu}\lambda) / \partial t = 2\sqrt{2\mu}\lambda I_{n+m+k+i+1}(2t\sqrt{2\mu}\lambda) \\ - (n+m+k+i+2) / t I_{n+m+k+i+2}(2t\sqrt{2\mu}\lambda) \quad \text{set } F5'(t) = f5(t) \\ \partial I_{n+m+k+i+1}(2t\sqrt{2\mu}\lambda) / \partial t = 2\sqrt{2\mu}\lambda I_{n+m+k+i}(2t\sqrt{2\mu}\lambda) \\ - (n+m+k+i+1) / t I_{n+m+k+i+1}(2t\sqrt{2\mu}\lambda) \quad \text{set } F6'(t) = f6(t)$$

Then putting other serial coefficients in $L^{-1} [\alpha_2^n P_0^*(s)]$

$$\sum_{k=0}^{\infty} (\rho)^{-(i+1+n+k)/2} (n+k+i+1) = g1(k)$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k \rho^{-(m+n+k+i+2)/2} (m+n+k+i+2) = g2(k)$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k \rho^{-(m+n+k+i+1)/2} (m+n+k+i+1) = g3(k)$$

and putting other serial coefficients in $L^{-1} [\alpha_2^{n+2} P_0^*(s)]$

$$\sum_{k=0}^{\infty} (\rho)^{-(i+1+n+1+k)/2} (n+1+k+i+1) = g4(k)$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k \rho^{-(m+n+1+k+i+2)/2} (m+n+1+k+i+2) = g5(k)$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_k (-\rho)^m 1/2^k \rho^{-(m+n+1+k+i+1)/2} (m+n+1+k+i+1) = g6(k)$$

$$L^{-1} [\alpha_2^n P_0^*(s)]$$

$$= \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[g1(k) F4(t) + \lambda / (4\mu) g2(k) F5(t) \right. \\ \left. + (\lambda - 2\mu) / (4\mu) g3(k) F6(t) \right]$$

$$L^{-1} [s \alpha_2^n P_0^*(s)]$$

$$= 1 / (2\mu + \lambda) \partial \exp(- (2\mu + \lambda)t) / \partial t \left[g1(k) F4(t) + \lambda / (4\mu) g2(k) F5(t) \right. \\ \left. + (\lambda - 2\mu) / (4\mu) g3(k) F6(t) \right] / \partial t$$

$$= \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[g1(k) f4(t) \right. \\ \left. + \lambda / (4\mu) g2(k) f5(t) + (\lambda - 2\mu) / (4\mu) g3(k) f6(t) \right]$$

$$- \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[(2\mu + \lambda) + 1/t \right] * \\ \left[g1(k) F4(t) + \lambda / (4\mu) g2(k) F5(t) \right. \\ \left. + (\lambda - 2\mu) / (4\mu) g3(k) F6(t) \right]$$

$$= - (2\mu + \lambda) + 1/t * L^{-1} [\alpha_2^n P_0^*(s)]$$

$$+ \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[g1(k) f4(t) \right. \\ \left. + \lambda / (4\mu) g2(k) f5(t) + (\lambda - 2\mu) / (4\mu) g3(k) f6(t) \right]$$

Similarly

$$L^{-1} [\alpha_2^{n+1} P_0^*(s)]$$

$$= \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[g4(k) F7(t) \right. \\ \left. + \lambda / (4\mu) g5(k) F8(t) + (\lambda - 2\mu) / (4\mu) g6(k) F9(t) \right]$$

where

$$F7(t) = I_{n+k+i+2}(2t\sqrt{2\mu}\lambda)$$

$$F7'(t) = f7 = 2\sqrt{2\mu}\lambda I_{n+k+i+1}(2t\sqrt{2\mu}\lambda) \\ - (n+k+i+2) / t I_{n+k+i+2}(2t\sqrt{2\mu}\lambda)$$

$$F8(t) = I_{n+m+k+i+3}(2t\sqrt{2\mu}\lambda)$$

$$F8'(t) = f8 = 2\sqrt{2\mu}\lambda I_{n+m+k+i+2}(2t\sqrt{2\mu}\lambda) \\ - (n+m+k+i+3) / t I_{n+m+k+i+3}(2t\sqrt{2\mu}\lambda)$$

$$F9(t) = I_{n+m+k+i+2}(2t\sqrt{2\mu}\lambda)$$

$$F9'(t) = f9 = 2\sqrt{2\mu}\lambda I_{n+m+k+i+1}(2t\sqrt{2\mu}\lambda) \\ - (n+m+k+i+2) / t I_{n+m+k+i+2}(2t\sqrt{2\mu}\lambda)$$

$$L^{-1} [s \alpha_2^{n+1} P_0^*(s)]$$

$$= 1 / (2\mu + \lambda) \partial \exp(- (2\mu + \lambda)t) / \partial t \left[g4(k) F7(t) \right. \\ \left. + \lambda / (4\mu) g5(k) F8(t) + (\lambda - 2\mu) / (4\mu) g6(k) F9(t) \right] / \partial t$$

$$= - (2\mu + \lambda) + 1/t * L^{-1} [\alpha_2^{n+1} P_0^*(s)]$$

$$+ \exp(- (2\mu + \lambda)t) / (2\mu + \lambda) \left[g4(k) f7(t) \right. \\ \left. + \lambda / (4\mu) g5(k) f8(t) + (\lambda - 2\mu) / (4\mu) g6(k) f9(t) \right]$$

The distribution of a busy period.

The length of a busy period is the length of time between the arrival of a unit at the empty queue and the first subsequent moment at which the queue is again empty. We consider the distribution of the interval during which at least one processing neuron is busy. To find the distribution of a busy period, we compute $dP_0(t)/dt$ from the system equation (1) with $c=2$ with an absorbing barrier at the origin. We modify the equation (1)

$$dP_0(t)/dt = \mu P_1(t), dP_1(t)/dt = -(\lambda + \mu) P_1(t) + 2\mu P_2(t).$$

Remaining equations are the same to (1). Instead of $P_0(t)$ we compute $\mu P_1(t)$. The generating function is as before except add $\lambda(1-z)P_0$ to it. We have

$$P^*(z, s) = \frac{z^{i+1} - \mu(1-z)[2P_0^* + zP_1^*] + \lambda(1-z)P_0^*}{-\lambda(z - \alpha_1)(z - \alpha_2)}, \quad (40)$$

which, together with the Laplace transform of the first equation of system $P_0'(t) = \mu P_1(t)$, gives

$$(2\mu - \lambda) P_0^* + \mu \alpha_2 P_1^* = \alpha_2^{i+1} / (1 - \alpha_2), \quad s P_0^* = \mu P_1^*.$$

Substituting from the second equation into the first we have

$$\left[(2\mu - \lambda) \mu / s + \mu \alpha_2 \right] P_1^* = \alpha_2^{i+1} / (1 - \alpha_2), \\ P_1^* = \frac{s \alpha_2^{i+1}}{\mu(1 - \alpha_1)(2\mu - \lambda + s \alpha_2)} = \frac{\lambda(1 - \alpha_1) \alpha_2^{i+1}}{\mu(2\mu - \lambda + s \alpha_2)} = \frac{(\lambda/\mu)(1 - \alpha_1) \alpha_2^{i+1}}{4\mu - \lambda - (\lambda + 2\mu) \alpha_2 + \alpha_2^2}. \quad (41)$$

Arguing as before, we have

$$P_1^* = - \frac{\lambda/\mu}{4\mu - \lambda} (1 - \alpha_1) \alpha_2^{i+1} \left(1 - \frac{\lambda + 2\mu}{4\mu - \lambda} \alpha_2 + \frac{\lambda}{4\mu - \lambda} \alpha_2^2 \right)^{-1} \\ = - \frac{\lambda/\mu}{4\mu - \lambda} (1 - \alpha_1) \alpha_2^{i+1} \sum_{k=0}^{\infty} \left(\frac{\lambda + 2\mu}{4\mu - \lambda} \alpha_2 - \frac{\lambda}{4\mu - \lambda} \alpha_2^2 \right)^k \\ = - \frac{\lambda/\mu}{4\mu - \lambda} (1 - \alpha_1) \alpha_2^{i+1} \sum_{k=0}^{\infty} \left(\frac{\lambda + 2\mu}{4\mu - \lambda} \alpha_2 \right)^k \sum_{m=0}^{\infty} \binom{k}{m} (-1)^m \left(\frac{\lambda}{\lambda + 2\mu} \alpha_2 \right)^m \\ = - \frac{\lambda/\mu}{4\mu - \lambda} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k}{m} (-1)^m \left(\frac{\lambda + 2\mu}{4\mu - \lambda} \right)^k \left(\frac{\lambda}{\lambda + 2\mu} \right)^m \quad (42)$$

$$\left[\alpha_2^{i+1+k+m} - \frac{2\mu}{\lambda} \alpha_2^{i+k+m} \right] \\ = - \frac{\lambda/\mu}{4\mu - \lambda} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k}{m} (-1)^m \left(\frac{\lambda + 2\mu}{4\mu - \lambda} \right)^k \left(\frac{\lambda}{\lambda + 2\mu} \right)^m \\ \cdot \left[\left(\frac{2\mu}{\lambda} \right)^{i+k+m+1} \frac{1}{(\alpha_1^{-i-k-m} - \alpha_1^{-(i+k+m+1)})} - \alpha_1^{-(i+k+m+1)} \right].$$

We then have for the distribution of a busy period in which at least server is busy:

$$P_0'(t) = \mu P_1(t) = \frac{\lambda e^{-(\lambda+2\mu)t}}{(4\mu - \lambda)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k}{m} (-1)^m \left(\frac{\lambda + 2\mu}{4\mu - \lambda} \right)^k \left(\frac{\lambda}{\lambda + 2\mu} \right)^m \\ \left(\frac{2\mu}{\lambda} \right)^{i+k+m+1} \left[(2\sqrt{2\lambda\mu})^{-(i+k+m)} \frac{i+k+m}{t} I_{i+k+m}(2\sqrt{2\lambda\mu}t) \right. \\ \left. - (2\sqrt{2\lambda\mu})^{-(i+k+m+1)} \frac{i+k+m+1}{t} I_{i+k+m+1}(2\sqrt{2\lambda\mu}t) \right]. \quad (43)$$