

Analysis of Synaptic Particle Movement under the Thermal Gradient at the Presynaptic Terminal.

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A mathematical method for analyzing the movements of bimolecular particle under the thermal gradient was applied to the synaptic particle movement at the presynaptic terminal of neural system. The basic method was originally proposed by Meyyappan (1984, 86). The potential field was expressed by the differential equation for the temperature field in spherical coordinates. The continuity and moment equations for the particle could be solved by analytic method to obtain the three dimensional velocities and pressure around the particle. These solutions consisted of series of products of hyperbolic function and Legendre spherical functions. By complex mathematical treatment the unknown coefficients were determined under the rational physical boundary condition. The present method will be available for evaluating the synaptic particle movement.

Presynaptic terminal, Bimolecular particle, Thermal gradient, Moment equation, Legendre function

シナプス前頭部におけるシナプス顆粒の温度勾配下における粒子運動解析

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温度勾配が存在する場合の生体分子運動の数学的解析法を神経系におけるシナプス前頭部におけるシナプス顆粒運動の解析に応用する方法を紹介する。系の基礎方程式はMeyyappan (1984, 86)によって最初に提唱されたが生体組織に対しては応用されてはいなかった。粒子場は温度場と粒子運動場の2つの方程式系で記述した。温度場は球状座標系を用いて記述した。粒子運動系は連続式とモメント式で記述した。両系は解析的に解を得ることが可能であり、双曲線関数とレジェンドレの球関数の積の級数展開形式で記述できた。未定係数は物理的境界条件を設定することで複雑な数学的過程を経ることで決定した。本方法はシナプス前頭部が脱分極し、電気化学ポテンシャルが変化した場合の温度変化が生じた場合のシナプス顆粒の温度勾配下での運動解析に有用と考えられる。

シナプス前頭部, 生体分子運動, 温度勾配, モメント式, レジェンドレ球関数, 生体膜

1. Introduction

Biochemical information is transmitted by neuro transmitter stored in presynaptic terminal. The neural impulse evoking electro chemical potential gradient change in the terminal membrane will alter the thermal gradient. The transmitters stored in terminal will be affected by such potential disturbance. The present paper gives a method for tangential stress acting $\tau \xi \eta (\xi = \beta)$ on the transmitter particle.

2. Mathematical method.

2-1. Thermal equations.

In the Fig, w the unit vector in the direction of thermal gradient is parallel to the plane surface and is aligned along the x direction. The scaled **undisturbed thermal gradient** is selected as $w = ix$ -----[4-56]

The **temperature field** satisfies

$$\nabla^2 T = 0 \quad \text{-----[4-57]}$$

Since $\nabla T (\rho \rightarrow \infty) = ix$, the temperature away from the molecular surface is given by

$$T(\rho \rightarrow \infty) = T_{\infty} = x = \rho \cos \phi \quad \text{-----[5-58]}$$

The temperature field is written as the sum of two fields $T_{\infty}(x)$ and T^* (the perturbation in temperature arising from the presence of the molecular surface).

$$T = T(x) + T^* \quad \text{-----[4-59]}$$

Substitution of (4-58) and (4-59) in (4-57)

$$\nabla^2 T^* = 0 \quad \text{-----[4-60]}$$

The formal **solution** of {4-60} in a **bipolar coordinates** which vanishes at infinity is given by

$$T^* = (\cosh \xi - \mu)^{1/2} \sin \eta \cos \phi \sum [A_n \cosh((n+1/2) \xi) + B_n \sinh((n+1/2) \xi)] P_n(\mu) \quad (n \geq 1) \quad \text{-----[4-61]}$$

where $P_n(\mu)$ is the Legendre polynomial of order n .

' denotes differentiation with respect to μ .

The **boundary condition** at the solid plane surface satisfied by the temperature field under the condition that the gradient w is oriented arbitrary is

$$T(\xi = 0) = x = \rho \cos \phi \quad \text{-----[4-62]}$$

From the (4-58, 59, 62), we have

$$T^*(\xi = 0) = 0 \quad \text{-----[4-63]}$$

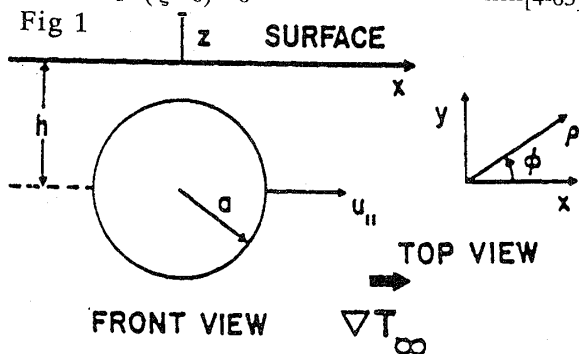
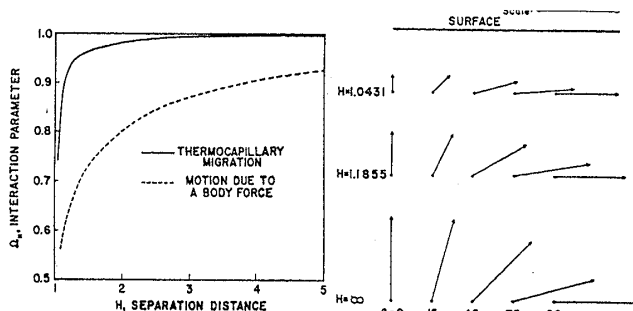


Fig 2 Data by Meyyappan 1984.



2-2. Velocity Field

The **continuity and moment equations** are

$$\nabla \cdot v = 0, \quad \nabla^2 v = \nabla p \quad \text{-----[4-68, 69]}$$

The **components of v** in a cylindrical coordinate are v_ρ, v_ϕ, v_z . Now we set

$$p = p_1 \cos \phi$$

$$v_\rho = [\rho p_1 + (x_1 + x_1)] \cos \phi / 2$$

$$v_\phi = (v_1 - x_1) \sin \phi / 2$$

$$v_z = (z p_1 + 2 w_1) \cos \phi / 2 \quad \text{-----[4-70]}$$

Substituting [4-70] into [4-69]

$$L_0^2 x_1 = L_2^2 v_1 = L_1^2 p_1 = L_1^2 w_1 = 0 \quad \text{-----[4-71]}$$

$$\text{where } L_m^2 = \partial^2 / \partial \rho^2 + 1/\rho \partial / \partial \rho + \partial^2 / \partial z^2 - m^2 / \rho^2 \quad \text{-----[4-72]}$$

The **solutions** are of the form of

$$p_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum [a_n \cosh((n+1/2) \xi) + a_n \sinh((n+1/2) \xi)] P_n(\mu) \quad (n \geq 1)$$

$$v_1 = (\cosh \xi - \mu)^{1/2} \sin \eta^2 \sum [e_n \cosh((n+1/2) \xi) + f_n \sinh((n+1/2) \xi)] P_n(\mu) \quad (n \geq 2)$$

$$x_1 = (\cosh \xi - \mu)^{1/2} \sum [g_n \cosh((n+1/2) \xi) + h_n \sinh((n+1/2) \xi)] P_n(\mu) \quad (n \geq 0)$$

$$w_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum [c_n \cosh((n+1/2) \xi) + d_n \sinh((n+1/2) \xi)] P_n(\mu) \quad (n \geq 1) \quad \text{-----[4-76]}$$

The **no slip condition** at the solid plane surface is

$$v_\rho = v_\phi = v_z = 0 \text{ at } \xi = 0 \quad \text{-----[4-77]}$$

These are expressed in another form by

$$v_1 = x_1 = -\rho p_1 / 2 \text{ and } w_1 = 0 \text{ at } \xi = 0 \quad \text{[4-78, 79]}$$

[4-79] requires

$$c_n = 0 \text{ for all } n \geq 1 \quad \text{-----[4-80] also}$$

$$p_1(\xi = 0) = -\text{Limit} (2w_1 / z) (\xi = 0) \quad \text{[4-81]}$$

Using the solutions for p_1 and w_1 in 4-72 and 4-76 and applying the L'Hospitals rule, we have

$$a_n = (n-1) d_{n-1} - (2n+1) d_n + (n+2) d_{n+1} \quad n \geq 1 \quad \text{[4-82]}$$

From [4-78] and [4-81], eliminate p_1

$$v_1(\xi = 0) = x_1(\xi = 0) = \text{limit} (\rho w_1 / z) (\xi = 0) \quad \text{-----[4-83]}$$

Solutions for v_1, x_1 and w_1 are in equations [4-74], [4-75] and [4-76]. Substituting these into [4-83] and evaluating the limit using the L'Hospitals rule, we have

$$e_n = (d_{n-1} - d_{n+1}) / 2 \text{ for } n \geq 2 \quad \text{-----[4-84]}$$

$$g_n = [(n+1)(n+2) d_{n+1} - (n-1) n d_{n-1}] / 2 \quad n \geq 0 \quad \text{-----[4-85]}$$

The **condition of vanishing normal velocity at the molecular surface** is expressed as

$$v_n = u_n \text{ at } \xi = \beta \quad \text{-----[4-86]}$$

v_n at the molecular surface is the velocity component in the ξ direction, v_ξ is given by

$$v_\xi = -1/(\cosh \xi - \mu) [\sinh \xi \sin \eta v_\rho - (\cosh \xi \cos \eta - 1) v_z] \quad \text{-----[4-87]}$$

Equations [4-70], [4-73, 74, 75 and 76] may be used in [4-87] to express v_ξ in terms of eight unknown coefficients.

Balance of **tangential stress at the molecule surface**

$$n \tau_s = \nabla T_s$$

This is equivalent to [4-91, 92]

$$\tau_\xi \phi(\xi = \beta) = (\cosh \beta - \mu) / \sin \eta \partial T / \partial \phi (\xi = \beta)$$

$$\tau_\xi \eta(\xi = \beta) = -(\cosh \beta - \mu) * \sin \eta \partial T / \partial \mu (\xi = \beta)$$

$$= \partial [(\cosh \xi - \mu) * v_\eta] / \partial \xi + \partial [(\cosh \xi - \mu) * v_\xi] / \partial \eta \quad \text{-----[4-99]}$$

$$\text{where } v_\eta = [v_\rho (\cosh \xi \cos \eta - 1) - v_z \sinh \xi \sin \eta] / (\cosh \xi - \mu) \quad \text{-----[4-100]}$$

In the previous our work, we have shown the method for obtaining $\tau_\xi \phi$. This paper presents the method for $\tau_\xi \eta$.

3. Reference

1. Meyyappan, M. PHD. 1984. Clarkson Univ.
2. H, Hirayama. IEICE. Tech. Rep. vol99-41. pp25-32. 1999

while

$$\begin{aligned}\cos \eta \cdot P'_n &= \frac{(n+1)P'_{n-1} + n \cdot P'_{n+1}}{(2n+1)} \quad \text{and} \\ \cos \eta \cdot P'_{n-2} &= \frac{(n-1)P'_{n-3} + (n-2)P'_{n-1}}{(2n-3)} \\ \cos \eta \cdot P'_{n+2} &= \frac{(n+3)P'_{n+1} + (n+2)P'_{n+3}}{(2n+5)}\end{aligned}$$

Hence,

$$\begin{aligned}&= -\frac{n(n+1)}{(2n+1)(2n-1)} \cdot \left[\frac{(n-1)P'_{n-3} + (n-2)P'_{n-1}}{(2n-3)} \right] \\ &- \frac{n(n+1)}{(2n+1)(2n+3)} \cdot \left[\frac{(n+3)P'_{n+1} + (n+2)P'_{n+3}}{(2n+5)} \right] \\ &+ \frac{2n(n+1)}{(2n-1)(2n+3)} \cdot \left[\frac{(n+1)P'_{n-1} + n \cdot P'_{n+1}}{(2n+1)} \right]\end{aligned}$$

Therefore

$$\begin{aligned}\sin^2 \eta \cdot \cos \eta \cdot P'_n(\mu) &= -\frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} P'_{n-3} + \frac{n^2(n+1)}{(2n+1)(2n-3)(2n+3)} P'_{n-1} \\ &+ \frac{n(n+1)^2}{(2n+1)(2n+5)(2n-1)} P'_{n+1} - \frac{n(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} P'_{n+3} \\ &***** \\ -\cos^2 \eta \cdot P_n &= (\sin^2 \eta - 1) P_n \\ &= -\frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} P'_{n+3} + \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n+3)} P'_{n-3} \\ &+ P'_{n-1} \left[\frac{(3n^2+5n-4)}{(2n+1)(2n+5)(2n-1)} - \frac{1}{2n+1} \right] \\ &- P'_{n-1} \left[\frac{(3n^2+n-6)}{(2n+1)(2n+3)(2n-3)} - \frac{1}{2n+1} \right] \\ &= -\frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} P'_{n+3} + \frac{(n-1)n}{(2n+1)(2n-1)(2n+3)} P'_{n-3} \\ &- P'_{n+1} \frac{(n^2+3n-1)}{(2n+1)(2n+5)(2n-1)} + P'_{n-1} \frac{(n^2-n-3)}{(2n+1)(2n+3)(2n-3)}\end{aligned}$$

$$\sin^4 \eta \cdot P'_n(\mu)$$

$$\begin{aligned}&= P'_{n-3} \left[-\frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n-1)(2n-3)} \right] + P'_{n+3} \left[\frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \right] \\ &+ P'_{n-1} \left[\frac{3 \cdot (n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n-3)(2n+3)} \right] + P'_{n+1} \left[\frac{-3 \cdot (n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n-1)(2n+5)} \right]\end{aligned}$$

$$\cos \eta \cdot P'_{n-1} \cdot \sin^2 \eta$$

$$= \frac{(n-1) \cdot n \cdot (n+1)}{(2n-1)(2n-3)} \cdot P'_{n-3} + \frac{3 \cdot (n-2)(n+1)}{(2n-3)(2n+1)} \cdot P'_{n-1} - \frac{(n-2)(n-1)}{(2n-1)(2n+1)} \cdot n \cdot P'_{n+1}$$

$$\cos \eta \cdot P'_{n+1} \cdot \sin^2 \eta$$

$$= \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+1)} \cdot P'_{n-1} + \frac{3 \cdot n(n+3)}{(2n+1)(2n+5)} \cdot P'_{n+1} - \frac{n(n+1)(n+2)}{(2n+3)(2n+5)} \cdot P'_{n+3}$$

$$\sin^2 \eta \cdot P'_n = \frac{(n+1)(n+2) \cdot P'_{n-1} - n(n-1) \cdot P'_{n+1}}{2n+1}$$

$$\sin^2 \eta \cdot P'_{n-1} = \frac{n \cdot (n+1) \cdot P'_{n-3} - (n-1)(n-2) \cdot P'_{n-1}}{2n-1}$$

$$\sin^2 \eta \cdot P'_{n+1} = \frac{(n+2)(n+3) \cdot P'_{n+1} - (n+1) \cdot n \cdot P'_{n+3}}{2n+3}$$

$$\begin{aligned}\cos \eta \cdot \sin^2 \eta \cdot P'_n &= \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} P'_{n-3} + \frac{3 \cdot (n-1)(n+2)}{(2n-1)(2n+3)} P'_{n-1} \\ &- \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} P'_{n+2}\end{aligned}$$

while

$$\begin{aligned}\cos \eta \cdot P_n &= \frac{(n+1) \cdot P'_{n+2}}{(2n+1)(2n+3)} + \frac{P'_n}{(2n+3)(2n-1)} - \frac{n \cdot P'_{n-2}}{(2n+1)(2n-1)} \\ P_n &= \frac{P'_{n+1} - P'_{n-1}}{2n+1}\end{aligned}$$

$$\begin{aligned}\sin^2 \eta \cdot P_n &= -\frac{(n+1)(n+2) \cdot P'_{n+3}}{(2n+1)(2n+3)(2n+5)} + \frac{(3n^2+5n-4) \cdot P'_{n+1}}{(2n+1)(2n+5)(2n-1)} \\ &- \frac{(3n^2+n-6) \cdot P'_{n-1}}{(2n+1)(2n+3)(2n-3)} + \frac{(n-1)n \cdot P'_{n-3}}{(2n+1)(2n-1)(2n-3)}\end{aligned}$$

$$\cos \eta \cdot \sin^2 \eta \cdot P'_n(\mu)$$

$$\begin{aligned}&= -\frac{n(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} P'_{n+3} + \frac{n(n+1)^2}{(2n+1)(2n+5)(2n-1)} P'_{n+1} \\ &- \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} P'_{n-3} + \frac{n^2(n+1)}{(2n+1)(2n-3)(2n+3)} P'_{n-1} \\ &*****\end{aligned}$$

$$\sin^2 \eta \cdot P'_n(\mu) = -\frac{n(n+1) \cdot P'_{n-2}}{(2n+1)(2n-1)} + 2 \frac{n(n+1) \cdot P'_n}{(2n-1)(2n+3)} - \frac{n(n+1) \cdot P'_{n+2}}{(2n+1)(2n+3)}$$

$$\cos^3 \eta \cdot P'_n = \cos^2 \eta \cdot P'_n \cdot \cos \eta$$

$$= (1 - \sin^2 \eta) \cdot P'_n \cdot \cos \eta$$

$$= \cos \eta \cdot P'_n - \sin^2 \eta \cdot \cos \eta \cdot P'_n$$

$$= \frac{(n+1) \cdot P'_{n-1} + n \cdot P'_{n+1}}{(2n+1)}$$

$$+ \frac{n(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} P'_{n+3} - \frac{n(n+1)^2}{(2n+1)(2n+5)(2n-1)} P'_{n+1}$$

$$+ \frac{(n-1)n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} P'_{n-3} - \frac{n^2(n+1)}{(2n+1)(2n-3)(2n+3)} P'_{n-1}$$

$$*****$$

$$\cos^2 \eta \cdot \sin^2 \eta \cdot P'_n = \cos \eta [\cos \eta \cdot \sin^2 \eta \cdot P'_n]$$

$$= \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} \cos \eta \cdot P'_{n-2} + \frac{3(n-1)(n+2)}{(2n-1)(2n+3)} \cos \eta \cdot P'_n$$

$$- \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} \cos \eta \cdot P'_{n+2}$$

while

$$\cos \eta \cdot P'_n = \frac{(n+1) \cdot P'_{n-1} + n \cdot P'_{n+1}}{2n+1}$$

$$\cos \eta \cdot P'_{n-2} = \frac{(n-1) \cdot P'_{n-3} + (n-2) \cdot P'_{n-1}}{2n-3}$$

$$\cos \eta \cdot P'_{n+2} = \frac{(n+3) \cdot P'_{n+1} + (n+2) \cdot P'_{n+3}}{2n+5}$$

$$= \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} \left[\frac{(n-1)}{(2n-3)} P'_{n-3} + \frac{(n-2)}{(2n-3)} P'_{n-1} \right]$$

$$+ \frac{3(n-1)(n+2)}{(2n-1)(2n+3)} \left[\frac{(n+1)}{(2n+1)} P'_{n-1} + \frac{n}{(2n+1)} P'_{n+1} \right]$$

$$- \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} \left[\frac{(n+3)}{(2n+5)} P'_{n+1} + \frac{(n+2)}{(2n+5)} P'_{n+3} \right]$$

$$+++++$$

Tangential Stress

$$\tau_{\eta\beta} \Big|_{\eta=\beta} = -(\cosh \beta - \mu) \cdot \sin \eta \cdot \frac{\partial T}{\partial \mu} (\mu = \beta)$$

$$= \frac{\partial}{\partial \xi} [(\cosh \xi - \mu) \cdot v_\eta] + \frac{\partial}{\partial \eta} [(\cosh \xi - \mu) \cdot v_\xi]$$

Here we set $f = (\cosh \xi - \cos \eta)$. Because

$$\begin{aligned}v_\xi &= -\frac{\cos \phi}{2 \cdot f} [\sinh \xi \cdot \sin \eta \cdot (\rho p_1 + v_1 + x_1) \\ &+ (\cosh \xi \cdot \cos \eta - 1) \cdot (z p_1 + 2w_1)]\end{aligned}$$

$$*****$$

1.The first term

Since, we have

$$\begin{aligned}v_\eta &= \frac{1}{f} [v_\rho (\cosh \xi \cdot \cos \eta - 1) - v_z \cdot \sinh \xi \cdot \sin \eta] \\ &= \frac{1}{f} \left[\frac{\cos \phi}{2} (\rho p_1 + v_1 + x_1) \cdot (\cosh \xi \cdot \cos \eta - 1) \right.\end{aligned}$$

$$\begin{aligned}
& -\frac{\cos \phi}{2}(\rho p_1 + 2w_1) \cdot (\sinh \xi \cdot \sin \eta - 1) \\
& = \frac{\cos \phi}{2 \cdot f} [(\rho p_1 + v_1 + x_1)(\cosh \xi \cdot \cos \eta - 1) \\
& \quad - (z p_1 + 2w_1) \sinh \xi \cdot \sin \eta]
\end{aligned}$$

Then, the first term is by using $f = (\cosh \xi - \cos \eta)$

$$\begin{aligned}
(\cosh \xi - \mu) \cdot v_\eta & = \frac{\cos \phi}{2} [(\rho p_1 + v_1 + x_1) \cdot (\cosh \xi \cdot \cos \eta - 1) \\
& \quad - (z p_1 + 2w_1) \sinh \xi \cdot \sin \eta]
\end{aligned}$$

In v_η and v_ξ , $\cos \phi/2$ is common. Since ξ , η are independent from the differential operation, we factor out.

We modify the first term as

$$\begin{aligned}
& (\rho p_1 + v_1 + x_1) \cdot (\cosh \xi \cdot \cos \eta - 1) - (z p_1 + 2w_1) \cdot \sinh \xi \cdot \sin \eta \\
& = p_1 [\rho (\cosh \xi \cdot \cos \eta - 1) - z \cdot \sinh \xi \cdot \sin \eta] \\
& + v_1 \cdot (\cosh \xi \cdot \cos \eta - 1) + x_1 (\cosh \xi \cdot \cos \eta - 1) + w_1 (-2 \cdot \sinh \xi \cdot \sin \eta)
\end{aligned}$$

about the coefficient of P_1 , since

$$\begin{aligned}
\rho & = \frac{\sin \eta}{f} \quad z = \frac{\sinh \xi}{f} \quad \text{then,} \\
\frac{\sin \eta}{f} \cdot (\cosh \xi \cdot \cos \eta - 1) - \frac{\sinh \xi}{f} \cdot \sinh \xi \cdot \sin \eta \\
& = \frac{\sin \eta}{f} \cdot (\cosh \xi \cdot \cos \eta - 1 - \sinh^2 \xi) \quad * \cosh^2 \xi - \sinh^2 = 1 \\
& = \frac{\sin \eta}{f} \cdot (\cosh \xi \cdot \cos \eta - \cosh^2 \xi) \\
& = \sin \eta \cdot \cosh \xi \cdot \frac{(\cos \eta - \cosh \xi)}{f} = -\sin \eta \cdot \cosh \xi
\end{aligned}$$

Thus

$$\frac{\partial}{\partial \xi} [-\sin \eta \cdot \cosh \xi \cdot p_1 + (\cosh \xi \cdot \cos \eta - 1) \cdot v_1 + (-2 \sinh \xi \cdot \sin \eta) w_1]$$

2. The second term

$$\begin{aligned}
(\cosh \xi - \mu) \cdot v_\xi & = -\frac{\cos \phi}{2} [\sinh \xi \cdot \sin \eta \cdot (\rho p_1 + v_1 + x_1) \\
& + (\cosh \xi \cdot \cos \eta - 1) \cdot (z p_1 + 2w_1)]
\end{aligned}$$

We modify above as

$$\begin{aligned}
& \sinh \xi \cdot \sin \eta \cdot (\rho p_1 + v_1 + x_1) + (\cosh \xi \cdot \cos \eta - 1)(z p_1 + 2w_1) \\
& = p_1 (\sinh \xi \cdot \sin \eta \cdot \rho + (\cosh \xi \cdot \cos \eta - 1) \cdot z) \\
& + v_1 \cdot \sinh \xi \cdot \sin \eta \\
& + x_1 \cdot \sinh \xi \cdot \sin \eta \\
& + w_1 \cdot (2(\cosh \xi \cdot \cos \eta - 1))
\end{aligned}$$

About the coefficients of P_1

$$\begin{aligned}
& \sinh \xi \cdot \sin \eta \cdot \frac{\sin \eta}{f} + (\cosh \xi \cdot \cos \eta - 1) \cdot \frac{\sinh \xi}{f} \\
& = \frac{\sinh \xi}{f} \cdot (\sin^2 \eta + \cosh \xi \cdot \cos \eta - 1) \\
& = \frac{\sinh \xi}{f} \cdot (1 - \cos^2 \eta + \cosh \xi \cdot \cos \eta - 1) \\
& = \sinh \xi \cdot \frac{\cos \eta}{f} \cdot (\cosh \xi - \cos \eta) = \sinh \xi \cdot \cos \eta
\end{aligned}$$

Hence, the operation of the second term is

$$\frac{\partial}{\partial \eta} [\sinh \xi \cdot \cos \eta \cdot p_1 + \sinh \xi \cdot \sin \eta \cdot v_1 - \frac{\partial}{\partial \eta} [\sinh \xi \cdot \sin \eta \cdot x_1 + w_1 [2 \cdot (\cosh \xi \cdot \cos \eta - 1)]]]$$

.....

3. Operation of $\tau \xi \eta$ for the terms related to

$v_1 = f^{\frac{1}{2}} \cdot \sin^2 \eta \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot p_n''(\mu)$. We have

$$\frac{\partial}{\partial \xi} [(\cosh \xi \cdot \cos \eta - 1) \cdot v_1] - \frac{\partial}{\partial \eta} [\sinh \xi \cdot \sin \eta \cdot v_1]$$

$$= \sinh \xi \cdot \cos \eta \cdot v_1 + (\cosh \xi \cdot \cos \eta - 1) \frac{\partial v_1}{\partial \xi}$$

$$= \sinh \xi \cdot \cos \eta \cdot v_1 - \sinh \xi \cdot \sin \eta \cdot \frac{\partial v_1}{\partial \eta}$$

$$\frac{\partial v_1}{\partial \xi} = \frac{1}{2} f^{-\frac{1}{2}} \cdot \sinh \xi \cdot \sin^2 \eta \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot p_n''(\mu)$$

$$+ f^{-\frac{1}{2}} \sin^2 \eta \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot p_n''(\mu)$$

$$\frac{\partial v_1}{\partial \eta} = \frac{\partial}{\partial \eta} \left[f^{\frac{1}{2}} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot (\sin^2 \eta \cdot p_n''(\mu)) \right]$$

$$= \frac{1}{2} f^{-\frac{1}{2}} \cdot \sin \eta \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''$$

$$+ f^{\frac{1}{2}} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{\partial}{\partial \eta} \left[\frac{(n+1)(n+2) p_{n-1}'}{2n+1} - \frac{n(n-1) p_{n+1}'}{2n+1} \right]$$

$$= \frac{f^{-\frac{1}{2}}}{2} \cdot \sin \eta \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''
+ f^{\frac{1}{2}} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \left[-\frac{(n+1)(n+2)}{(2n+1)} p_{n-1}'' \cdot \sin \eta + \frac{n(n-1)}{(2n+1)} p_{n+1}'' \cdot \sin \eta \right]$$

$$= (\cosh \xi \cdot \cos \eta - 1) \left[\frac{f^{-\frac{1}{2}}}{2} \cdot \sinh \xi \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''(\mu) + f^{\frac{1}{2}} \cdot \sum f_n (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''(\mu) \right]$$

$$= \sinh \xi \cdot \sin \eta \cdot \left[\frac{f^{-\frac{1}{2}}}{2} \cdot \sinh \eta \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n'' \right.$$

$$\left. + f^{\frac{1}{2}} \cdot \sum f_n \sinh(n + \frac{1}{2}) \xi \cdot \left\{ -\frac{(n+1)(n+2)}{(2n+1)} p_{n-1}'' \cdot \sin \eta + \frac{n(n-1)}{(2n+1)} p_{n+1}'' \cdot \sin \eta \right\} \right]$$

Multiply $\frac{1}{2}$ on both sides

$$\Rightarrow \cosh \xi \cdot \cos \eta \cdot \frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''(\mu)$$

$$- \frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''(\mu)$$

$$+ \cosh \xi \cdot \cos \eta \cdot (\cosh \xi - \cos \eta) \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''$$

$$- (\cosh \xi - \cos \eta) \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''$$

$$- \sinh \xi \cdot \sin \eta \cdot \frac{\sin \eta}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''$$

$$+ \sinh \xi \cdot \sin \eta \cdot (\cosh \xi - \cos \eta) \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)(n+2)}{(2n+1)} \cdot p_{n-1}'' \cdot \sin \eta$$

$$- \sinh \xi \cdot \sin \eta \cdot (\cosh \xi - \cos \eta) \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{n(n-1)}{(2n+1)} \cdot p_{n+1}'' \cdot \sin \eta$$

$$= \frac{\cosh \xi \cdot \sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \cos \eta \cdot \sin^2 \eta \cdot p_n''(\mu)$$

$$- \frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''(\mu)$$

$$+ \cosh^2 \xi \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \cos \eta \cdot \sin^2 \eta \cdot p_n''$$

$$- \cosh \xi \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \cos^2 \eta \cdot \sin^2 \eta \cdot p_n''$$

$$- \cosh \xi \sum f_n (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \sin^2 \eta \cdot p_n''$$

$$+ \sum f_n (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \cos \eta \cdot \sin^2 \eta \cdot p_n''$$

$$\begin{aligned}
& -\frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \sin^4 \eta \cdot p_n'' \\
& + \sinh \xi \cdot \cosh \xi \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)(n+2)}{(2n+1)} \cdot \sin^2 \eta \cdot p_{n-1}'' \\
& - \sinh \xi \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)(n+2)}{(2n+1)} \cos \eta \cdot \sin^2 \eta \cdot p_{n-1}'' \\
& - \sinh \xi \cdot \cosh \xi \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{n \cdot (n-1)}{(2n+1)} \sin^2 \eta \cdot p_{n+1}'' \\
& + \sinh \xi \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{n \cdot (n-1)}{(2n+1)} \cos \eta \cdot \sin^2 \eta \cdot p_{n+1}'' \quad \textcircled{1} \\
& = \frac{\cosh \xi \cdot \sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \\
& \left[\frac{n(n+1)(n+2)}{(2n+1)(2n-1)} p_{n-2}' + \frac{3(n-1)(n+2)}{(2n-1)(2n+3)} p_{n-1}' - \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} p_{n+2}' \right] \\
& - \frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \left[\frac{(n+1)(n+2)}{2n+1} p_{n-1}' - \frac{n(n-1)}{2n+1} p_{n+1}' \right] \\
& - \frac{\sinh \xi}{2} \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \\
& \left[-p_{n-3}' \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n-3)(2n+3)} + p_{n+3}' \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \right. \\
& \left. + p_{n-1}' \frac{3 \cdot (n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n-3)(2n+3)} - p_{n+1}' \frac{3 \cdot (n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n-1)(2n+5)} \right] \\
& + \sinh \xi \cdot \cosh \xi \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)(n+2)}{(2n+1)} \cdot \\
& \left[\frac{n(n+1) \cdot p_{n-2}' - (n-1)(n-2) \cdot p_{n+2}'}{2n-1} \right] \\
& - \sinh \xi \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)(n+2)}{(2n+1)} \cdot \\
& \left[\frac{(n-1) \cdot n \cdot (n+1)}{(2n-1)(2n-3)} p_{n-2}' + \frac{3 \cdot (n-2)(n+1)}{(2n-3)(2n+1)} p_{n-1}' - \frac{(n-2)(n-1)n}{(2n-1)(2n+1)} p_{n+1}' \right] \\
& - \sinh \xi \cdot \cosh \xi \cdot \sum f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \frac{n(n-1)}{(2n+1)} \cdot \\
& \left[\frac{(n+2)(n+3)p_{n-1}' - (n+1) \cdot n \cdot p_{n+2}'}{2n+3} \right] \\
& + \sinh \xi \cdot \sum f_n \sinh(n + \frac{1}{2}) \xi \cdot \frac{n(n-1)}{(2n+1)} \cdot \\
& \left[\frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+1)} p_{n-1}' + \frac{3n(n+3)}{(2n+1)(2n+5)} p_{n+1}' - \frac{n(n+1)(n+2)}{(2n+3)(2n+5)} p_{n+3}' \right] \\
& + \cosh^2 \xi \cdot \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \\
& \left[\frac{n(n+1)(n+2)}{(2n+1)(2n-1)} p_{n-2}' + \frac{3 \cdot (n-1)(n+2)}{(2n-1)(2n+3)} p_{n-1}' - \frac{(n-1)n(n+1)}{(2n+1)(2n+3)} p_{n+2}' \right] \\
& - \cosh \xi \cdot \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \\
& \left[\frac{n(n+1)(n+2)(n-1)}{(2n+1)(2n-1)(2n-3)} p_{n-3}' - \frac{(n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n+3)(2n+1)} p_{n+3}' \right. \\
& \left. + \left\{ \frac{n(n+1)(n+2)(n-2)}{(2n+1)(2n-1)(2n-3)} + \frac{3(n-1)(n+2)(n+1)}{(2n-1)(2n+3)(2n+1)} \right\} p_{n-1}' \right. \\
& \left. + \left\{ -\frac{(n-1) \cdot n \cdot (n+1)(n+3)}{(2n+1)(2n+3)(2n+5)} + \frac{3(n-1)(n+2) \cdot n}{(2n-1)(2n+3)(2n+1)} \right\} p_{n+1}' \right] \\
& - \cosh \xi \cdot \sum f_n \cdot (n + \frac{1}{2}) \cosh(n + \frac{1}{2}) \xi \left[\frac{(n+1)(n+2)}{2n+1} p_{n-1}' - \frac{n(n-1)}{(2n+1)} p_{n+1}' \right] \\
& + \sum f_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \\
& \left[\frac{n(n+1)(n+2)}{(2n+1)(2n-1)} p_{n-2}' + \frac{3(n-1)(n+2)}{(2n-1)(2n+3)} p_{n-1}' - \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} p_{n+2}' \right]
\end{aligned}$$

A. Rearrange about p_{n-3} in the 7th and the 9th terms

$$\begin{aligned}
& -\frac{\sinh \xi}{2} \cdot \left[\frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n-1)(2n-3)} \right] \\
& - \sinh \xi \cdot \frac{(n+1)(n+2)(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} \\
& = -\sinh \xi \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n-1)(2n-3)} \left(n + \frac{1}{2} \right) \cdot \\
& \text{Substitute } n \text{ instead of } n-3 \\
& = -\sinh \xi \frac{(n+5)(n+4)(n+3)(n+2)}{(2n+7)(2n+5)(2n+3)} \left(n + \frac{7}{2} \right) f_{n+3} \sinh(n + \frac{7}{2}) \xi \\
& \text{About the term of } e_n, \text{ substitute } \cosh(n+7/2) \xi \text{ instead of } \sinh(n+7/2) \xi \\
& = -\sinh \xi \frac{(n+5)(n+4)(n+3)(n+2)}{(2n+7)(2n+5)(2n+3)} \left(n + \frac{7}{2} \right) \cdot e_{n+3} \cdot \cosh(n + \frac{7}{2}) \xi
\end{aligned}$$

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B. Rearrange for p_{n+3} in the 7th and the 11th terms,

$$\begin{aligned}
& -\frac{\sinh \xi}{2} \left[\frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \right] \\
& + \sinh \xi \cdot \left[\frac{n \cdot (n-1)}{(2n+1)} \right] \cdot \left(-\frac{n(n+1)(n+2)}{(2n+3)(2n+5)} \right) \\
& = -\sinh \xi \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \cdot \left(n + \frac{1}{2} \right) \\
& \text{Substitute } n+3 \text{ to } n \\
& = -\sinh \xi \frac{(n-1)(n-2)(n-3)(n-4)}{(2n-5)(2n-3)(2n-1)} \cdot \left(n - 3 + \frac{1}{2} \right) f_{n-3} \cdot \\
& * \sinh(n-5/2) \xi \\
& \text{About the term of } e_n, \text{ substitute } \cosh(n-5/2) \xi \text{ instead of } \sinh(n-5/2) \xi \\
& = -\sinh \xi \frac{(n-1)(n-2)(n-3)(n-4)}{(2n-5)(2n-3)(2n-1)} \left(n - \frac{5}{2} \right) e_{n-3} \cdot \cosh(n - \frac{5}{2}) \xi
\end{aligned}$$

The contribution from $\cosh \xi$ is

$$\begin{aligned}
& -\cosh \xi \cdot \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \cos^2 \eta \cdot \sin^2 \eta \cdot p_n'' \\
& = -\cosh \xi \cdot \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh(n + \frac{1}{2}) \xi \\
& \left[\frac{n(n+1)(n+2)(n-1)}{(2n+1)(2n-1)(2n-3)} p_{n-3}' - \frac{(n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} p_{n+3}' \right]
\end{aligned}$$

Summing the terms from $\sinh(n+1/2) \xi$, we have

$$\begin{aligned}
& -p_{n-3}' \left[\sinh \xi \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n-1)(2n-3)} \left(n + \frac{1}{2} \right) \cdot \sinh(n + \frac{1}{2}) \xi \cdot f_n \right. \\
& \left. + \cosh \xi \frac{n(n+1)(n+2)(n-1)}{(2n+1)(2n-1)(2n-3)} \left(n + \frac{1}{2} \right) \cdot \cosh(n + \frac{1}{2}) \xi \cdot f_n \right] \\
& -p_{n+3}' \left[\sinh \xi \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \left(n + \frac{1}{2} \right) \cdot \sinh(n + \frac{1}{2}) \xi \cdot f_n \right. \\
& \left. - \cosh \xi \frac{(n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} \left(n + \frac{1}{2} \right) \cdot \cosh(n + \frac{1}{2}) \xi \cdot f_n \right] \\
& = -p_{n-3}' \frac{(n+2)(n+2) \cdot n \cdot (n-1)}{(2n+1)(2n-1)(2n-3)} f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \\
& \left[\left(n + \frac{1}{2} \right) \cdot \sinh \xi + \left(n + \frac{1}{2} \right) \cdot \cosh \xi \cdot \coth(n + \frac{1}{2}) \xi \right] \\
& - p_{n+3}' \frac{(n+2)(n+1) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} f_n \cdot \sinh(n + \frac{1}{2}) \xi \cdot \\
& \left[\left(n + \frac{1}{2} \right) \cdot \sinh \xi - \left(n + \frac{1}{2} \right) \cdot \cosh \xi \cdot \coth(n + \frac{1}{2}) \xi \right]
\end{aligned}$$

By converting in the 1st term $n-3 \rightarrow n$ andin the 2nd term as $n+3 \rightarrow n$

$$= -p_{n-3}' \frac{(n+5)(n+4)(n+3)(n+2)}{(2n+7)(2n+5)(2n+3)} \cdot f_{n+3} \cdot \sinh(n + \frac{7}{2}) \xi \cdot \cosh \xi$$

$$\cdot \left[\left(n + \frac{7}{2} \right) \cdot \frac{\sinh \xi}{\cosh \xi} + \left(n + \frac{7}{2} \right) \cdot \coth \left(n + \frac{7}{2} \right) \xi \right]$$

$$+ p'_n \frac{(n-1)(n-2)(n-3)(n-4)}{(2n-5)(2n-3)(2n-1)} f_{n-3} \cdot \sinh \left(n - \frac{5}{2} \right) \xi \cdot \cosh \xi$$

$$\cdot \left[\left(n - \frac{5}{2} \right) \cdot \frac{\sinh \xi}{\cosh \xi} - \left(n - \frac{5}{2} \right) \cdot \coth \left(n - \frac{5}{2} \right) \xi \right]$$

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C. Rearrange for p'_{n+2} in the 1st and the 8th terms

$$\frac{\cosh \xi \cdot \sinh \xi}{2} \left[\frac{n(n+1)(n+2)}{(2n+1)(2n-1)} + 2 \frac{(n+1)n(n+1)(n+2)}{(2n-1)(2n+1)} \right]$$

$$= \frac{\cosh \xi \cdot \sinh \xi}{2} \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} [1 + 2(n+1)]$$

$$= \cosh \xi \cdot \sinh \xi \cdot \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} \left(n + \frac{3}{2} \right)$$

Converting $n-2$ to n

$$\Rightarrow \cosh \xi \cdot \sinh \xi \cdot \frac{(n+2)(n+3)(n+4)}{(2n+5)(2n+3)} \left(n + \frac{7}{2} \right) f_{n+2}$$

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D. Rearrange for p'_{n+2} in the 1st and the 10th terms

$$\frac{\cosh \xi \cdot \sinh \xi}{2} \left[- \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} - 2 \frac{n(n-1)}{(2n+1)} (-1) \frac{(n+1)n}{(2n+3)} \right]$$

$$= \frac{\cosh \xi \cdot \sinh \xi}{2} \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} [-1 + 2n]$$

Converting $n+2$ to n

$$= \cosh \xi \cdot \sinh \xi \cdot \frac{(n-3)(n-2)(n-1)}{(2n-3)(2n-1)} \left(n - \frac{5}{2} \right) f_{n-2}$$

Contribution from the terms related to $\cosh \xi$ are

$$\cosh^2 \xi \cdot \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot (\cos \eta \cdot \sin^2 \eta \cdot p'_n)$$

$$+ \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot (\cos \eta \cdot \sin^2 \eta \cdot p'_n)$$

$$= (1 + \cosh^2 \xi) \cdot \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot (\cos \eta \cdot \sin^2 \eta \cdot p'_n)$$

$$= (1 + \cosh^2 \xi) \cdot \sum f_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi$$

$$\cdot \left[\frac{n(n+1)(n+2)}{(2n+1)(2n-1)} p'_{n-2} - \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} p'_{n+2} \right]$$

Summing the terms from $\sinh \xi$

$$p'_{n-2} \left[\cosh \xi \cdot \sinh \xi \cdot \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} \left(n + \frac{3}{2} \right) \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot f_n \right.$$

$$\left. + (1 + \cosh^2 \xi) \cdot \frac{n \cdot (n+1)(n+2)}{(2n+1)(2n-1)} \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot f_n \right]$$

$$+ p'_{n+2} \left[\cosh \xi \cdot \sinh \xi \cdot \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} \left(n - \frac{1}{2} \right) \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot f_n \right.$$

$$\left. - (1 + \cosh^2 \xi) \cdot \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot f_n \right]$$

$$= p'_{n-2} \cdot \frac{n \cdot (n+1)(n+2)}{(2n+1)(2n-1)} \cdot f_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi$$

$$\cdot \left[\cosh \xi \cdot \sinh \xi \cdot \left(n + \frac{3}{2} \right) + (1 + \cosh^2 \xi) \cdot \left(n + \frac{1}{2} \right) \coth \left(n + \frac{1}{2} \right) \xi \right]$$

$$+ p'_{n+2} \cdot \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} f_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi$$

$$\cdot \left[\cosh \xi \cdot \sinh \xi \cdot \left(n - \frac{1}{2} \right) - (1 + \cosh^2 \xi) \cdot \left(n + \frac{1}{2} \right) \cdot \coth \left(n + \frac{1}{2} \right) \xi \right]$$

In the 1st term, converting $n-2 \rightarrow n$ and in the 2nd term $n+2 \rightarrow n$. Then, we have

$$= p'_n \cdot \frac{(n+2)(n+3)(n+4)}{(2n+5)(2n+3)} f_{n+2} \cdot \sinh \left(n + \frac{5}{2} \right) \xi$$

$$\cdot \left[\cosh \xi \cdot \sinh \xi \cdot \left(n + \frac{7}{2} \right) + (1 + \cosh^2 \xi) \cdot \left(n + \frac{5}{2} \right) \cdot \coth \left(n + \frac{5}{2} \right) \xi \right]$$

$$- p'_n \frac{(n-3)(n-2)(n-1)}{(2n-3)(2n-1)} f_{n-2} \cdot \sinh \left(n - \frac{3}{2} \right) \xi$$

$$\cdot \left[- \cosh \xi \cdot \sinh \xi \cdot \left(n - \frac{5}{2} \right) + (1 + \cosh^2 \xi) \cdot \left(n - \frac{3}{2} \right) \cdot \coth \left(n - \frac{3}{2} \right) \xi \right]$$

About the terms in e_n , we change $\sinh(n+m/2) \xi$ to $\cosh(n+m/2) \xi$

$$\cosh \xi \cdot \sinh \xi \cdot \frac{(n+2)(n+3)(n+4)}{(2n+5)(2n+3)} \frac{(n+7/2)}{(n+5/2)} \cosh \left(n + 5/2 \right) \xi \cdot e_{n+2}$$

$$\cosh \xi \cdot \sinh \xi \cdot \frac{(n-3)(n-2)(n-1)}{(2n-3)(2n-1)} \frac{(n-5/2)}{(n-3/2)} \cosh \left(n - 3/2 \right) \xi \cdot e_{n-2}$$

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We next operate for the term of x_1

$$x_1 = f^{\frac{1}{2}} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n(\mu)$$

$$\frac{\partial}{\partial \xi} \left[(\cosh \xi \cdot \cos \eta - 1) \cdot x_1 \right] - \frac{\partial}{\partial \eta} \left[\sinh \xi \cdot \sin \eta \cdot x_1 \right]$$

$$= \sinh \xi \cdot \cos \eta \cdot x_1 + (\cosh \xi \cdot \cos \eta - 1) \frac{\partial x_1}{\partial \xi} - \sinh \xi \cdot \cos \eta \cdot x_1 - \sinh \xi \cdot \sin \eta \cdot \frac{\partial x_1}{\partial \eta}$$

$$\text{since}$$

$$\frac{\partial x_1}{\partial \xi} = \frac{f^{-\frac{1}{2}}}{2} \cdot \sinh \xi \cdot \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n + f^{\frac{1}{2}} \sum h_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot p_n$$

$$\frac{\partial x_1}{\partial \eta} = \frac{f^{-\frac{1}{2}}}{2} \cdot \sin \eta \cdot \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n + f^{\frac{1}{2}} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p'_n(\mu) (-\sin \eta)$$

$$= (\cosh \xi \cdot \cos \eta - 1) \left[\frac{f^{-\frac{1}{2}}}{2} \sinh \xi \cdot \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n \right.$$

$$\left. + f^{\frac{1}{2}} \sum h_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot p_n \right]$$

$$- \sinh \xi \cdot \sin \eta \cdot \left[\frac{f^{-\frac{1}{2}}}{2} \sin \eta \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n \right.$$

$$\left. - f^{\frac{1}{2}} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p'_n(\mu) \cdot \sin \eta \right]$$

Multiply $f^{1/2}$ on both sides

$$\Rightarrow \cosh \xi \cdot \cos \eta \cdot \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n$$

$$- \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n$$

$$+ \cosh \xi \cdot \cos \eta \cdot (\cosh \xi - \cos \eta) \sum h_n \cdot \left(n + \frac{1}{2} \right) \cosh \left(n + \frac{1}{2} \right) \xi \cdot p_n$$

$$- \sinh \xi \cdot \sin \eta \cdot \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n$$

$$+ \sinh \xi \cdot \sin \eta \cdot (\cosh \xi - \cos \eta) \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p'_n(\mu) \cdot \sin \eta$$

$$= \frac{\cosh \xi \cdot \sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot \cos \eta \cdot p_n \quad (1)$$

$$- \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot p_n \quad (2)$$

$$+ \cosh^2 \xi \sum h_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot \cos \eta \cdot p_n \quad (3)$$

$$- \cosh \xi \sum h_n \cdot \left(n + \frac{1}{2} \right) \cdot \cosh \left(n + \frac{1}{2} \right) \xi \cdot \cos^2 \eta \cdot p_n \quad (4)$$

$$- \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot \sin^2 \eta \cdot p_n \quad (5)$$

$$+ \sinh \xi \cdot \cosh \xi \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot \sin^2 \eta \cdot p'_n(\mu) \quad (6)$$

$$- \sinh \xi \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot \cos \eta \cdot \sin^2 \eta \cdot p'_n(\mu) \quad (7)$$

$$= \frac{\cosh \xi \cdot \sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \quad (1)$$

$$\cdot \left[\frac{(n+1)}{(2n+1)(2n+3)} p'_{n+2} + \frac{p'_n}{(2n+3)(2n-1)} - \frac{n}{(2n+1)(2n-1)} p'_{n-2} \right]$$

$$- \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot \left[\frac{p'_{n+1}}{2n+1} - \frac{p'_{n-1}}{2n+1} \right] \quad (2)$$

$$- \frac{\sinh \xi}{2} \sum h_n \cdot \sinh \left(n + \frac{1}{2} \right) \xi \cdot (5)$$

$$\begin{aligned}
& \left[\frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} p'_{n-3} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} p'_{n+3} \right. \\
& \left. - \frac{(3n^2+n-6)}{(2n+1)(2n+3)(2n-3)} p'_{n-1} + \frac{(3n^2+5n-4)}{(2n+1)(2n+5)(2n-1)} p'_{n+1} \right] \\
& + \sinh \xi \cdot \cosh \xi \cdot \sum h_n \cdot \sinh(n + \frac{1}{2}) \xi \quad \textcircled{6} \\
& \left[-\frac{n(n+1)}{(2n+1)(2n-1)} p'_{n-2} + \frac{2 \cdot n \cdot (n+1)}{(2n-1)(2n+3)} p'_n - \frac{n \cdot (n+1)^2}{(2n+1)(2n+3)} p'_{n+2} \right] \\
& - \sinh \xi \cdot \sum h_n \cdot \sinh(n + \frac{1}{2}) \xi \quad (7) \\
& \left[-\frac{n(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} p'_{n+3} - \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} p'_{n-3} \right. \\
& \left. + \frac{n \cdot (n+1)^2}{(2n+1)(2n+5)(2n-1)} p'_{n+1} + \frac{n^2(n+1)}{(2n+1)(2n-3)(2n+3)} p'_{n-1} \right] \\
& + \cosh^2 \xi \cdot \sum h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \quad \textcircled{3} \\
& \left[\frac{(n+1)}{(2n+1)(2n+3)} p'_{n+2} + \frac{p'_n}{(2n+3)(2n-1)} - \frac{n}{(2n+1)(2n-1)} p'_{n-2} \right] \\
& - \cosh \xi \cdot \sum h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \quad \textcircled{4} \\
& \left[\frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} p'_{n+3} - \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n+3)} p'_{n-3} \right. \\
& \left. + \frac{(n^2+3n-1)}{(2n+1)(2n+5)(2n-1)} p'_{n+1} - \frac{(n^2-n-3)}{(2n+1)(2n+3)(2n-3)} p'_{n-1} \right]
\end{aligned}$$

A.. Terms related to $p_{n,3}$ are involved in ⑤ and ⑦th

$$\begin{aligned}
& \frac{\sinh \xi}{2} \cdot \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} - \sinh \xi \left[-\frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n-1)(2n-3)} \right] \\
& = \sinh \xi \cdot \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} (n + \frac{1}{2})
\end{aligned}$$

Convert $n \rightarrow n+3$

$$= \sinh \xi \cdot \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} \cdot (n + \frac{7}{2}) \cdot h_{n+3} \cdot \sinh(n + \frac{7}{2}) \xi$$

About g_n exchange $\sinh(n+7/2) \xi$ to $\cosh(n+7/2) \xi$

$$= \sinh \xi \cdot \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} (n + \frac{7}{2}) \cdot g_{n+3} \cdot \cosh(n + \frac{7}{2}) \xi$$

Contribution from $\cosh \xi$ are

$$- \cosh \xi \cdot \sum h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \cos^2 \eta \cdot p_n$$

Contribution from this term to $p'_{n,3}$ are

$$\cosh \xi \cdot \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n+3)} \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi$$

Thus, summin with the terms timed by $\sinh \xi$, we have

$$\begin{aligned}
& \left[\sinh \xi \cdot \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} \cdot (n + \frac{1}{2}) \cdot \sinh(n + \frac{1}{2}) \xi \right. \\
& \left. + \cosh \xi \cdot \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \right] \cdot p'_{n-3} \cdot h_n
\end{aligned}$$

Converting $n-3 \rightarrow n$

$$\begin{aligned}
& \left[\sinh \xi \cdot \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} (n + \frac{7}{2}) \cdot \sinh(n + \frac{7}{2}) \xi \right. \\
& \left. + \cosh \xi \cdot \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} (n + \frac{7}{2}) \cdot \sinh(n + \frac{7}{2}) \xi \right] \cdot p'_n \cdot h_{n+3}
\end{aligned}$$

$$= \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} \sinh(n + \frac{7}{2}) \xi \cdot \cosh \xi \cdot \left[\frac{\sinh \xi}{\cosh \xi} \cdot (n + \frac{7}{2}) + \cosh(n + \frac{7}{2}) \cdot \coth(n + \frac{7}{2}) \xi \right] \cdot p'_n \cdot h_{n+3}$$

$$\rightarrow \frac{(n+2)(n+3)}{(2n+7)(2n+5)(2n+3)} \sinh(n + \frac{7}{2}) \xi \cdot \cosh \xi \cdot [g_{n+3}] h_{n+3} \cdot p'_n$$

~~~~~

B.. Terms related to  $p_{n,3}$  are ⑤th and ⑦th terms

$$\begin{aligned}
& -\frac{\sinh \xi}{2} (-1) \frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} - \sinh \xi \cdot (-1) \frac{n(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} \\
& = \frac{\sinh \xi}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} [1+2n]
\end{aligned}$$

~~~~~

C.. Terms related to p'_{n-2} are ①st and ⑥th ones

$$\begin{aligned}
& \frac{\cosh \xi \cdot \sinh \xi}{2} \cdot \frac{(-n)}{(2n+1)(2n-1)} + \sinh \xi \cdot \cosh \xi \cdot \frac{(-1)n(n+1)}{(2n+1)(2n-1)} \\
& = -\cosh \xi \cdot \sinh \xi \cdot \frac{n}{(2n+1)(2n-1)} \left[n + \frac{3}{2} \right]
\end{aligned}$$

Convert $n-2$ to n

$$\Rightarrow -\cosh \xi \cdot \sinh \xi \cdot \frac{(n+2)}{(2n+5)(2n+3)} \frac{(n+7/2)}{(n+5/2)} \sinh(n+2 + \frac{1}{2}) \xi \cdot h_n$$

For g_{n+2} , change $\sinh(n+5/2) \xi$ to $\cosh(n+5/2) \xi$

$$- \cosh \xi \cdot \sinh \xi \cdot \frac{(n+2)}{(2n+5)(2n+3)} \frac{(n+7/2)}{(n+5/2)} \cdot \cosh(n + \frac{5}{2}) \xi \cdot g_{n+2}$$

Contribution to p_{n-2} from terms timed by $\cosh \xi$

$\cosh^2 \xi \cdot \sum h_n (n+1/2) \cdot \cosh(n+1/2) \xi$ since

$$\cosh^2 \xi \cdot h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \left(-\frac{n}{(2n+1)(2n-1)} \right) p'_{n-2}$$

In p'_{n-2} , with $\sinh \xi \cdot \sinh \xi / 2$

$$\left[-\cosh \xi \cdot \sinh \xi \cdot \frac{n}{(2n+1)(2n-1)} \cdot (n + \frac{3}{2}) \cdot h_n \cdot \sinh(n + \frac{1}{2}) \xi \right.$$

$$\left. - \cosh^2 \xi \cdot \frac{n}{(2n+1)(2n-1)} \cdot (n + \frac{1}{2}) \cdot h_n \cdot \cosh(n + \frac{1}{2}) \xi \right] \cdot p'_{n-2}$$

$$= \frac{-n}{(2n+1)(2n-1)} h_n \cdot \left[\cosh \xi \cdot \sinh \xi \cdot (n + \frac{3}{2}) \cdot \sinh(n + \frac{1}{2}) \xi \right.$$

$$\left. + \cosh^2 \xi \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \right] \cdot p'_{n-2}$$

~~~~~

D.. Terms related to  $p'_{n+2}$  are ①st and ⑥th ones

$$\begin{aligned}
& \frac{\cosh \xi \cdot \sinh \xi}{2} \cdot \frac{(n+1)}{(2n+1)(2n+3)} + \sinh \xi \cdot \cosh \xi \cdot \frac{(-1)n(n+1)}{(2n+1)(2n+3)} \\
& = \frac{\cosh \xi \cdot \sinh \xi}{2} \cdot \frac{(n+1)}{(2n+1)(2n+3)} (n - \frac{1}{2})
\end{aligned}$$

Convert  $n+2 \rightarrow n$

$$= -\cosh \xi \cdot \sinh \xi \cdot \frac{(n-1)}{(2n-3)(2n-1)} \frac{(n-5/2)}{(n-3/2)} \sinh(n-2 + \frac{1}{2}) \xi \cdot h_{n-2}$$

For  $g_{n-2}$ , change  $\sinh(n-3/2) \xi$  to  $\cosh(n-3/2) \xi$

$$- \cosh \xi \cdot \sinh \xi \cdot \frac{(n-1)}{(2n-3)(2n-1)} \frac{(n-5/2)}{(n-3/2)} \cosh(n-3/2) \xi \cdot g_{n-2}$$

Contribution to  $p'_{n+2}$  from  $\cosh \xi$  are only

$\cosh^2 \xi \cdot \sum h_n (n+1/2) \cdot \cosh(n+1/2) \xi$  and since

$$\cosh^2 \xi \cdot h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)}{(2n+1)(2n+3)} \cdot p'_{n+2}$$

with the terms of  $\cosh \xi \cdot \sinh \xi / 2$

$$\left[ -\cosh \xi \cdot \sinh \xi \cdot \frac{(n+1)}{(2n+1)(2n+3)} (n - \frac{1}{2}) \cdot h_n \cdot \sinh(n + \frac{1}{2}) \xi \right.$$

$$\left. + \cosh^2 \xi \cdot h_n \cdot (n + \frac{1}{2}) \cdot \cosh(n + \frac{1}{2}) \xi \cdot \frac{(n+1)}{(2n+1)(2n+3)} \right] \cdot p'_{n+2}$$

Convert  $n+2$  to  $n$

$$\rightarrow \frac{(n-1) \cdot h_{n-2}}{(2n-3)(2n-1)} \cdot \sinh(n - \frac{3}{2}) \xi \cdot \left[ -\cosh \xi \cdot \sinh \xi \cdot (n - \frac{5}{2}) \right.$$

$$\left. + \cosh^2 \xi \cdot (n - \frac{3}{2}) \cdot \coth(n - \frac{3}{2}) \xi \right]$$

E.. Terms related to  $p_n$  are ①st and ⑥th terms

$$\frac{\cosh \xi \cdot \sinh \xi}{2} \cdot \frac{1}{(2n+3)(2n-1)} + \sinh \xi \cdot \cosh \xi \cdot \frac{2n(n+1)}{(2n-1)(2n+3)}$$

~~~~~

F.. Terms related to p'_{n-1} are ②⑤⑥⑦

$$-\frac{\sinh \xi}{2} \cdot \left(-\frac{p'_{n-1}}{2n+1} \right) - \frac{\sinh \xi}{2} \cdot \left(-\frac{(3n^2+n-6)}{(2n+1)(2n+3)(2n-3)} \right)$$

$$+ \sinh \xi \cdot \cosh \xi \cdot \frac{n(n+1)}{(2n+1)} - \sinh \xi \cdot \frac{n^2(n+1)}{(2n+1)(2n-3)(2n+3)}$$

$$= \frac{1}{2} \frac{\sinh \xi}{(2n+1)} \cdot \left[1 + \frac{(3n^2+n-6)}{(2n+3)(2n-3)} - \frac{2n^2(n+1)}{(2n-3)(2n+3)} \right]$$

APPENDIX

Associate about $P'_n(\mu)$ (1) $P''_n \rightarrow P'_n$

$$\sin^2 \eta \cdot P''_n = \frac{(n+1)(n+2) \cdot P'_{n-1} - n(n-1) \cdot P'_{n+1}}{2n+1}$$

$$\sin^4 \eta \cdot P''_n = \frac{1}{(2n+1)} \left[(n+1)(n+2) \cdot \sin^2 \eta \cdot P'_{n-1} - n(n-1) \cdot \sin^2 \eta \cdot P'_{n+1} \right]$$

while

$$\sin^2 \eta \cdot P'_n = \frac{n(n+1)}{(2n+1)} (P_{n-1} - P_{n+1}) \quad \text{and}$$

$$P_n = \frac{P'_{n+1} - P'_{n-1}}{(2n+1)} \quad P_{n-1} = \frac{P'_n - P'_{n-2}}{(2n-1)} \quad \text{Thus}$$

$$\begin{aligned} \sin^2 \eta \cdot P'_n &= \frac{n(n+1)}{(2n+1)} \left[\frac{P'_n - P'_{n-2}}{2n-1} - \frac{P'_{n+2} - P'_n}{2n+3} \right] \\ &= \frac{n(n+1)}{(2n+1)} \left[-\frac{P'_{n-2}}{2n-1} - \frac{P'_{n+2}}{2n+3} + \frac{2(2n+1)P'_n}{(2n-1)(2n+3)} \right] \\ &= -\frac{n(n+1)}{(2n+1)(2n-1)} \cdot P'_{n-2} + \frac{2n(n+1)}{(2n-1)(2n+3)} \cdot P'_n - \frac{n(n+1)}{(2n+1)(2n+3)} \cdot P'_{n+2} \end{aligned}$$

Hence

$$\sin^2 \eta \cdot P'_{n-1} = -\frac{(n-1)n}{(2n-1)(2n-3)} \cdot P'_{n-3} + \frac{2(n-1)n}{(2n-3)(2n+1)} \cdot P'_{n-1}$$

$$-\frac{(n-1)n}{(2n-1)(2n+1)} \cdot P'_{n+1}$$

$$\sin^2 \eta \cdot P'_{n+1} = -\frac{(n+1)(n+2)}{(2n+3)(2n+1)} \cdot P'_{n+3}$$

$$+\frac{2(n+1)(n+2)}{(2n+1)(2n+5)} \cdot P'_{n+1} - \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cdot P'_{n+3}$$

Substitute them to the right side of $\sin^2 \eta \cdot P'_n$ About P'_{n-1}

$$\begin{aligned} (n+1)(n+2) \frac{2(n-1)n}{(2n-3)(2n+1)} + n(n-1) \frac{(n+1)(n+2)}{(2n+1)(2n+3)} \\ = \frac{(n-1) \cdot n \cdot (n+1)(n+2)}{(2n+1)} \left(\frac{2}{2n-3} + \frac{1}{2n+3} \right) \\ = \frac{3 \cdot (n-1)n(n+1)(n+2)}{(2n-3)(2n+3)} \end{aligned}$$

About P'_{n+1}

$$\begin{aligned} \frac{(n+1)(n+2)(n-1) \cdot n}{(2n-1)(2n+1)} - \frac{n(n-1) \cdot 2 \cdot (n+1)(n+2)}{(2n+1)(2n+5)} \\ = \frac{(n-1) \cdot n(n+1)(n+2)}{(2n+1)} \left[\frac{1}{2n-1} + \frac{2}{2n+5} \right] \\ = -\frac{3(n-1)n(n+1)(n+2)}{(2n-1)(2n+5)} \end{aligned}$$

Therefore,

$$\sin^4 \eta \cdot P''_n$$

$$= P'_{n-3} \left[-\frac{(n+1)(n+2) \cdot n \cdot (n-1)}{(2n+1) \cdot (2n-1)(2n-3)} \right]$$

$$+ P'_{n-1} \frac{1}{(2n+1)} \left[\frac{3(n-1) \cdot n \cdot (n+1)(n+2)}{(2n-3)(2n+3)} \right]$$

$$+ P'_{n+1} \frac{-1}{(2n+1)} \left[\frac{3(n-1) \cdot n \cdot (n+1)(n+2)}{(2n-1)(2n+5)} \right]$$

$$+ P'_{n+3} \frac{1}{(2n+1)} \left[\frac{(n+1)(n+2) \cdot n \cdot (n-1)}{(2n+1)(2n+3)(2n+5)} \right]$$

$$\cos \eta \cdot \sin^2 \eta \cdot P'_n$$

$$= \frac{1}{(2n+1)} \left[(n+1)(n+2) \cos \eta \cdot P'_{n-1} - n(n-1) \cos \eta \cdot P'_{n+1} \right]$$

whereas

$$\cos \eta \cdot P'_n = \frac{(n+1)P'_{n-1} + n \cdot P'_{n+1}}{(2n+1)}$$

$$\cos \eta \cdot P'_{n-1} = \frac{n \cdot P'_{n-2} + (n-1)P'_n}{(2n-1)}$$

$$\cos \eta \cdot P'_{n+1} = \frac{(n+2)P'_n + (n+1)P'_{n+2}}{(2n+3)}$$

$$= \frac{1}{(2n+1)} \left[\frac{(n+1)(n+2)}{(2n-1)} \left[n \cdot P'_{n-2} + (n-1)P'_n \right] \right.$$

$$\left. - \frac{n(n-1)}{(2n+3)} \left[(n+2)P'_n + (n+1)P'_{n+2} \right] \right]$$

Hence

$$= \frac{n(n+1)(n+2)}{(2n+1)(2n-1)} P'_{n-2} + \frac{3 \cdot (n-1)(n+2)}{(2n-1)(2n+3)} P'_n - \frac{(n-1) \cdot n \cdot (n+1)}{(2n+1)(2n+3)} P'_{n+2}$$

$$\sin^2 \eta \cdot P_n = \sin^2 \eta \cdot \frac{[P'_{n+1} - P'_{n-1}]}{2n+1}$$

While

$$\sin^2 \eta \cdot P'_n = \frac{n(n+1)}{2n+1} \cdot (P_{n-1} - P_{n+1})$$

Hence,

$$\sin^2 \eta \cdot P'_{n+1} = \frac{(n+1)(n+2)}{(2n+3)} (P_n - P_{n+2})$$

$$\sin^2 \eta \cdot P'_{n-1} = \frac{(n-1)n}{(2n-1)} (P_{n-2} - P_n)$$

$$= \frac{1}{(2n+1)} \left[\frac{(n+1)(n+2)}{(2n+3)} (P_n - P_{n+2}) - \frac{(n-1)n}{(2n-1)} (P_{n-2} - P_n) \right]$$

$$= -\frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{2 \cdot (n^2 + n - 1)P_n}{(2n+3)(2n-1)} - \frac{(n-1)n}{(2n+1)(2n-1)} P_{n-2}$$

While

$$P_n = \frac{P'_{n+1} - P'_{n-1}}{2n+1} \quad P_{n+2} = \frac{P'_{n+3} - P'_{n+1}}{2n+5} \quad P_{n-2} = \frac{P'_{n-1} - P'_{n-3}}{2n-3}$$

Thus,

$$\sin^2 \eta \cdot P_n$$

$$= -\frac{(n+1)(n+2)}{(2n+1)(2n+3)} \left[\frac{P'_{n+3}}{2n+5} - \frac{P'_{n+1}}{2n+5} \right]$$

$$+ \frac{2 \cdot (n^2 + n - 1)}{(2n+3)(2n-1)} \left[\frac{P'_{n+1}}{2n+1} - \frac{P'_{n-1}}{2n+1} \right]$$

$$- \frac{(n-1)n}{(2n+1)(2n-1)} \left[\frac{P'_{n-1}}{2n-3} - \frac{P'_{n-3}}{2n-3} \right]$$

Associating them

$$\sin^2 \eta \cdot P_n$$

$$= -\frac{(n+1)(n+2) \cdot P'_{n+3}}{(2n+1)(2n+3)(2n+5)} + \frac{(3n^2 + 5n - 4)}{(2n+1)(2n+5)(2n-1)} P'_{n+1}$$

$$- \frac{(3n^2 + n - 6)}{(2n+1)(2n+3)(2n-3)} P'_{n-1} + \frac{(n-1) \cdot n}{(2n+1)(2n-1)(2n-3)} P'_{n-3}$$

$$\cos \eta \cdot P_n$$

$$= \cos \eta \cdot \frac{(P'_{n+1} - P'_{n-1})}{(2n+1)}$$

$$\text{since } \cos \eta \cdot P'_n = \frac{(n+1)P'_{n-1} + nP'_{n+1}}{(2n+1)}$$

$$= \frac{(n+1)}{(2n+1)(2n+3)} P'_{n+2} + \frac{P'_n}{(2n+3)(2n-1)} - \frac{n}{(2n+1)(2n-1)} P'_{n-2}$$

$$\sin^2 \eta \cdot P'_n(\mu)$$

$$= \frac{n(n+1)}{(2n+1)} (P_{n-1} - P_{n+1})$$

$$P_n = \frac{(P'_{n+1} - P'_{n-1})}{(2n+1)} \quad P_{n-1} = \frac{(P'_n - P'_{n-2})}{(2n-1)}$$

$$P_{n+1} = \frac{(P'_{n+2} - P'_n)}{(2n+3)}$$

$$= \frac{n(n+1)}{(2n+1)} \left[\frac{P'_n}{2n-1} - \frac{P'_{n-2}}{2n-1} - \frac{P'_{n+2}}{2n+3} + \frac{P'_n}{2n+3} \right]$$

$$= -\frac{n(n+1)}{(2n+1)(2n-1)} P'_{n-2} - \frac{n(n+1)}{(2n+1)(2n+3)} P'_{n+2} + \frac{n(n+1) \cdot 2}{(2n-1)(2n+3)} P'_n$$

$$\sin^2 \eta \cdot \cos \eta \cdot P'_n(\mu)$$

$$= -\frac{n(n+1) \cos \eta \cdot P'_{n-2}}{(2n+1)(2n-1)} - \frac{n(n+1) \cos \eta \cdot P'_{n+2}}{(2n+1)(2n+3)} - \frac{2n(n+1) \cos \eta \cdot P'_n}{(2n-1)(2n+3)}$$