

Method for Analyzing Mutual Molecular Interactions between the Bio-Molecular Particles at an Unsteady Potential Filed.

H, Hirayama., N, Kitagawa., Y, Okita and T, Kazui.

Department of Public Health Asahikawa medical college
The Graduate School of Shizuoka university
The First Department of Surgery Hamamatsu medical college.

We introduce a mathematical method proposed by Rushton (Appl. Sci. Res. vol 37-. 1973) for analyzing the molecular particle interactions which always exist in any biological potential fields. The basic equations to be solved are the Stokes's equation and continuity equations. They were expressed in a bipolar coordinates system under the no slipping condition on the surface of the molecular particle. A rigorous mathematical treatment revealed that the solutions are series expansion of the products of Gegenbauer function and hyperbolic functions. The unknown coefficient of the series were determined by the boundary conditions. Extension of his method affords a formula for computing the shear stress acting on the molecular surface. His method will be available for evaluating the functional interaction between the bio molecular particles in a potential field.

Bio-molecular particles. Bi-polar coordinates, Potential field. Gegenbauer function, Shear stress

非定常状態における生体分子相互干渉の解析方法

平山博史, 北川敬之, *沖田善光, **数井暉久

旭川市西神楽4-5 旭川医科大学 公衆衛生学講座
(電話0166-65-2111、内2411) E mail hirayama@asahikawa-med.ac.jp

* 静岡大学大学院電子科学研究施設

** 浜松医科大学外科学第一講座

生体分子同士が相互に干渉する状況は種々の生体反応場に存在する。本稿ではRushton (Appl. Sci. Res vol pp 37-. 1973)が提唱した分子間干渉の解法を詳細に述べる。流体力学方程式と連続方程式をBi-polar 座標系上で記述し、生体分子表面上での滑りなしの境界条件のもとで解析的に方程式を解いた。解はゲーゲンバウアー関数と双曲線関数との級数として得られた。境界条件を満たす係数は複雑な4元に代数方程式となった。これらを変形することで生体分子表面における接線方向、法線方向のずり応力を導出する式を得ることができる。Rushtonの方法は生体分子の反応場でも応用可能な数学的解法であり、遺伝子制御蛋白と遺伝子領域との反応や生化学的反応の解析にも有用と考えられる。

生体分子相互干渉. 生体反応場. 非定常. ゲーゲンバウアー関数. 双曲線関数. 境界条件

1. Introduction

We introduce a mathematical method for analyzing interaction of biomolecular particle under the unsteady state firstly proposed by Rushton (Appl. Sci. Res. vol 28, pp 37-1973). The basic equations for the molecular force potential filed consisted of

$$\partial v / \partial t + v \cdot \nabla v + \nabla p / \rho = \nu \nabla^2 v \quad (1), \quad \nabla \cdot v = 0 \quad (2).$$

The present paper affords a method to solve above equations. The numbers for equations are those in Rushton's paper.

The shear stress is obtained from Newton's law of viscosity written in bi-polar co-ordinates, (Love [17]), as

$$(\tau_{\xi\eta})_i = \mu_i \frac{(\cosh \xi - \mu)}{c} \left\{ \frac{\partial v_{\xi i}}{\partial \eta} + \frac{\partial v_{\eta i}}{\partial \xi} + \frac{v_{\eta i} \sinh \xi + v_{\xi i} \sin \eta}{(\cosh \xi - \mu)} \right\} \quad (12)$$

$$(\tau_{\eta\theta})_i = (\tau_{\xi\theta})_i = 0 \text{ (axi-symmetric).}$$

written in terms of the stream function ψ_i as follows

$$(\tau_{\xi\eta})_i = \frac{\mu_i (\cosh \xi - \mu)^2}{c^3 (1 - \mu^2)^{3/2}} \left\{ (1 - \mu^2) \left[(\cosh \xi - \mu) \frac{\partial^2 \psi_i}{\partial \mu^2} - 3 \frac{\partial \psi_i}{\partial \mu} \right] - \left[(\cosh \xi - \mu) \frac{\partial^2 \psi_i}{\partial \xi^2} + 3 \sinh \xi \frac{\partial \psi_i}{\partial \xi} \right] \right\} \quad (17)$$

The value of the shear stress at an interface $\xi = \xi_j$ ($j = 1, 2$) may be expressed as (see [19])

$$\begin{aligned} ((\tau_{\xi\eta})_i)_{\xi=\xi_i} = & -\frac{\mu_i}{c^3} (1 - \mu^2)^{3/2} (\cosh \xi_j - \mu)^4 \sum_{n=0}^{\infty} \left\{ \frac{\partial^2 U_{n i}}{\partial \xi^2} (\xi_j) + \right. \\ & \left. + \frac{1}{2} (2n+3)(2n-1) U_{n i}(\xi_j) - \right. \\ & \left. - \frac{U_j c^2 \sqrt{2}}{4} n(n+1)(2n+1) (e^{\pi(n-1)\xi_j} - e^{\pi(n+1)\xi_j}) \right\} \frac{C_{n+1}^{-1/2}(\mu)}{(1 - \mu^2)} \quad (18) \end{aligned}$$

APPENDIX

1. Derivation of (15.1) (15.2)

For $\xi_1 = \alpha$, $\xi_2 = \beta$

$\varphi_j = R^2 U_j / 2$ holds, Hence multiply

$f^{3/2}$ on the both sides: $\varphi_j \cdot f^{3/2} = R^2 U_j / 2 \cdot f^{3/2}$
On the other hand

$$\begin{aligned} f^{3/2} \cdot R^2 &= \sqrt{2} \sum n(n+1) \cdot \left\{ \frac{e^{-(n-1/2)\xi}}{2n-1} - \frac{e^{-(n+3/2)\xi}}{2n+3} \right\} C_{n+1}^{-1/2} \\ \varphi_j &= \sum \varphi_{nj} = \sum U_{nj} \frac{C_{n+1}^{-1/2}}{f^{3/2}} \quad \text{Thus} \\ \sum U_{nj} \frac{C_{n+1}^{-1/2}}{f^{3/2}} \cdot f^{3/2} &= \frac{\sqrt{2}}{2} \sum n(n+1) \cdot \left\{ \frac{e^{-(n-1/2)\xi}}{2n-1} - \frac{e^{-(n+3/2)\xi}}{2n+3} \right\} C_{n+1}^{-1/2} \\ \therefore U_{nj} &= \frac{1}{\sqrt{2}} \sum n(n+1) \cdot \left\{ \frac{e^{-(n-1/2)\xi}}{2n-1} - \frac{e^{-(n+3/2)\xi}}{2n+3} \right\} C_{n+1}^{-1/2} \end{aligned}$$

2. Derivation of (17)

$$(12) \rightarrow (17)$$

$$(\tau_{\xi\eta})_i = \mu_i \cdot \frac{f}{c} \cdot \left\{ \frac{\partial v_{\xi i}}{\partial \eta} + \frac{\partial v_{\eta i}}{\partial \xi} + \frac{v_{\eta i} \sinh \xi}{f} + \frac{v_{\xi i} \sin \eta}{f} \right\}$$

From (13)

$$\frac{\partial v_{\xi i}}{\partial \eta} = -\frac{1}{c^2} \frac{\partial}{\partial \eta} \left(f^2 \frac{\partial \varphi_i}{\partial \mu} \right)$$

from (10), since φ_i is a function of μ and η

$$= -\frac{1}{c^2} \left[2f \cdot \sin \eta \cdot \frac{\partial \varphi_i}{\partial \mu} + f^2 \cdot \frac{\partial^2 \varphi_i}{\partial \mu^2} (-\sin \eta) \right]$$

$$\begin{aligned} \frac{\partial v_{\eta i}}{\partial \xi} &= -\frac{1}{c^2 \sin \eta} \frac{\partial}{\partial \xi} \left(f^2 \frac{\partial \varphi_i}{\partial \xi} \right) \\ &= -\frac{1}{c^2 \sin \eta} \cdot \left(2f \cdot \sinh \xi \cdot \frac{\partial \varphi_i}{\partial \xi} + f^2 \frac{\partial^2 \varphi_i}{\partial \xi^2} \right) \end{aligned}$$

Therefore

$$\begin{aligned} (\tau_{\xi\eta})_i &= \mu_i \cdot \frac{f}{c} \cdot \left\{ -\frac{2f \sin \eta}{c^2} \frac{\partial \varphi_i}{\partial \xi} + \frac{f^2 \sin \eta}{c^2} \frac{\partial^2 \varphi_i}{\partial \mu^2} \right. \\ &\quad \left. - \frac{2f \sinh \xi}{c^2 \sin \eta} \frac{\partial \varphi_i}{\partial \xi} - \frac{f^2}{c^2 \sin \eta} \frac{\partial^2 \varphi_i}{\partial \xi^2} + \frac{\sinh \xi}{f} \left(-\frac{f^2}{c^2 \sin \eta} \right) \frac{\partial \varphi_i}{\partial \xi} + \frac{\sin \eta}{f} \left(-\frac{f^2}{c^2} \right) \frac{\partial \varphi_i}{\partial \mu} \right\} \\ &= \mu_i \cdot \frac{f^2}{c^3} \cdot \left\{ -2 \sin \eta \frac{\partial \varphi_i}{\partial \xi} + f \sin \eta \frac{\partial^2 \varphi_i}{\partial \mu^2} - \frac{2 \sinh \xi}{\sin \eta} \frac{\partial \varphi_i}{\partial \xi} - \frac{f}{\sin \eta} \frac{\partial^2 \varphi_i}{\partial \xi^2} \right. \end{aligned}$$

$$\left. - \frac{\sinh \xi}{\sin \eta} \frac{\partial \varphi_i}{\partial \xi} - \sin \eta \frac{\partial \varphi_i}{\partial \mu} \right\}$$

$$\begin{aligned} \text{while } \sin \eta &= \sqrt{1 - \cos^2 \eta} = \sqrt{1 - \mu^2} \\ &= \mu_i \cdot \frac{f^2}{c^3} \cdot \frac{1}{(1 - \mu^2)^{3/2}} \cdot \left\{ (1 - \mu^2) \left[f \cdot \frac{\partial^2 \varphi_i}{\partial \mu^2} - 3 \frac{\partial \varphi_i}{\partial \mu} \right] - \left(f \frac{\partial^2 \varphi_i}{\partial \xi^2} + 3 \sinh \xi \frac{\partial \varphi_i}{\partial \xi} \right) \right\} \end{aligned}$$

3. Transformation from (17) \rightarrow (18)

Putting $f = \cosh \xi - s$ and putting

$$\chi_i = (\cosh \xi - s)^{3/2} \cdot \varphi_i \quad \chi_i = f^{3/2} \cdot \varphi_i$$

The 1st term of (17) is

$$\begin{aligned} [] &= f \cdot \frac{\partial^2 \varphi_i}{\partial s^2} - 3 \frac{\partial \varphi_i}{\partial s} = f \cdot \frac{\partial^2 (\chi_i \cdot f^{-3/2})}{\partial s^2} - 3 \frac{\partial (\chi_i \cdot f^{-3/2})}{\partial s} \\ &= f \cdot \left[f^{-3/2} \cdot \frac{\partial^2 \chi_i}{\partial s^2} + 2 \frac{\partial \chi_i}{\partial s} \cdot \frac{\partial f^{-3/2}}{\partial s} + \chi_i \cdot \frac{\partial^2 f^{-3/2}}{\partial s^2} \right] \\ &\quad - 3 \cdot \left[f^{-3/2} \cdot \frac{\partial \chi_i}{\partial s} + \chi_i \cdot \frac{\partial f^{-3/2}}{\partial s} \right] \\ [] &= f \cdot \left[f^{-3/2} \cdot \frac{\partial^2 \chi_i}{\partial s^2} + \chi_i \cdot \frac{15}{4} \cdot f^{-7/2} \right] - 3 \chi_i \cdot \frac{3}{2} \cdot f^{-3/2} \\ &= f^{-1/2} \cdot \frac{\partial^2 \chi_i}{\partial s^2} - \frac{3}{4} \cdot \chi_i \cdot f^{-3/2} \end{aligned}$$

As a result

$$\begin{aligned} (\cosh \xi - \mu) \frac{\partial^2 \varphi_i}{\partial \mu^2} - 3 \frac{\partial \varphi_i}{\partial \mu} &= f^{-1/2} \frac{\partial^2 \chi_i}{\partial s^2} - \frac{3}{4} f^{-3/2} \cdot \chi_i \\ &= f^{-1/2} \cdot \frac{\partial^2 \chi_i}{\partial s^2} - \frac{3}{4} \cdot f^{-1} \cdot \varphi_i \end{aligned}$$

About the second groups

$$\begin{aligned} [] &= (\cosh \xi - \mu) \cdot \frac{\partial^2 \varphi_i}{\partial \xi^2} + 3 \cdot \sinh \xi \cdot \frac{\partial \varphi_i}{\partial \xi} \\ &= f \cdot \left[f^{-3/2} \cdot \frac{\partial^2 \chi_i}{\partial \xi^2} + 2 \frac{\partial \chi_i}{\partial \xi} \cdot \frac{\partial f^{-3/2}}{\partial \xi} + \chi_i \cdot \frac{\partial^2 f^{-3/2}}{\partial \xi^2} \right] \\ &\quad + 3 \cdot \sinh \xi \cdot \left[f^{-3/2} \cdot \frac{\partial \chi_i}{\partial \xi} + \chi_i \cdot \frac{\partial f^{-3/2}}{\partial \xi} \right] \\ \frac{\partial f^{-3/2}}{\partial \xi} &= \frac{\partial f^{-3/2}}{\partial f} \cdot \frac{\partial f}{\partial \xi} = -\frac{3}{2} \cdot f^{-5/2} \cdot \sinh \xi \\ \frac{\partial^2 f^{-3/2}}{\partial \xi^2} &= \frac{15}{4} \cdot f^{-7/2} \cdot \sinh^2 \xi - \frac{3}{2} \cdot f^{-5/2} \cdot \cosh \xi \end{aligned}$$

Sum of the 2nd and 4th terms is

$$\begin{aligned} 2 \cdot f \cdot \frac{\partial \varphi_i}{\partial \xi} \cdot \left(-\frac{3}{2} \right) \cdot f^{-5/2} \cdot \sinh \xi + 3 \cdot \sinh \xi \cdot f^{-3/2} \cdot \frac{\partial \chi_i}{\partial \xi} \\ = \frac{\partial \chi_i}{\partial \xi} \cdot \sinh \xi \cdot \left[-3 \cdot f^{-3/2} + 3 \cdot f^{-3/2} \right] = 0 \end{aligned}$$

Hence,

$$\begin{aligned} [] &= f \cdot \left[f^{-3/2} \cdot \frac{\partial^2 \chi_i}{\partial \xi^2} + \chi_i \cdot \left(\frac{15}{4} f^{-7/2} \cdot \sinh^2 \xi - \frac{3}{2} f^{-5/2} \cdot \cosh \xi \right) \right] \\ &\quad + 3 \cdot \sinh \xi \cdot \chi_i \cdot \left(-\frac{3}{2} \right) \cdot f^{-5/2} \cdot \sinh \xi \\ &\quad \text{since } \chi_i = f^{3/2} \cdot \varphi_i \\ &= f^{-1/2} \cdot \frac{\partial^2 \chi_i}{\partial \xi^2} - \varphi_i \cdot \frac{3}{4} \cdot \left[\sinh^2 \xi \cdot f^{-1} + 2 \cdot \cosh \xi \right] \end{aligned}$$

Therefore

$$\begin{aligned} (\cosh \xi - s) \cdot \frac{\partial^2 \varphi_i}{\partial \xi^2} + 3 \cdot \sinh \xi \cdot \frac{\partial \varphi_i}{\partial \xi} \\ = f^{-1/2} \cdot \frac{\partial^2 \chi_i}{\partial \xi^2} - \frac{3}{4} \cdot \varphi_i \cdot \left[2 \cosh \xi + \sinh^2 \xi \cdot f^{-1} \right] \end{aligned}$$

By these, (17) can be

$$\begin{aligned} \tau_{\xi\eta} &= \frac{\mu_i \cdot f^2}{c^3 \cdot (1 - s^2)^{3/2}} \cdot \left\{ (1 - \mu^2) \cdot \left[f^{-1/2} \frac{\partial^2 \chi_i}{\partial s^2} - \frac{3}{4} f^{-1} \cdot \varphi_i \right] \right. \\ &\quad \left. - \left[f^{-1/2} \cdot \frac{\partial^2 \chi_i}{\partial \xi^2} - \frac{3}{4} \varphi_i \cdot (2 \cosh \xi + \sinh^2 \xi \cdot f^{-1}) \right] \right\} \end{aligned}$$

$$= \frac{\mu_i}{c^3 \cdot (1-s^2)^{1/2}} \cdot \left\{ f^{3/2} \cdot \left[(1-\mu^2) \cdot \frac{\partial^2 \chi_i}{\partial s^2} - \frac{\partial^2 \chi_i}{\partial \xi^2} \right] - \frac{3}{4} \cdot f \cdot \varphi_i \cdot \left[(1-s^2) - 2 \cdot f \cdot \cosh \xi - \sinh^2 \xi \right] \right\}$$

since

$$1 - S^2 - 2 \cdot f \cdot \cosh \xi - \sinh^2 \xi = -(\cosh \xi - s)^2 - 2 \cdot \sinh^2 \xi$$

Thus above equation is

$$= \frac{\mu_i}{c^3 \cdot (1-s^2)^{1/2}} \cdot \left\{ f^{3/2} \cdot \left[(1-\mu^2) \cdot \frac{\partial^2 \chi_i}{\partial s^2} - \frac{\partial^2 \chi_i}{\partial \xi^2} \right] + \frac{3}{4} \cdot f \cdot \varphi_i \cdot \left[(\cosh \xi - s)^2 + 2 \cdot \sinh^2 \xi \right] \right\} \quad A. 2, 3, 2$$

$$\text{since } \varphi_{ni} = \frac{U_{ni}(\xi) \cdot C_{n+1}^{-1/2}(s)}{(\cosh \xi - s)^{1/2}}$$

$$\chi_i = (\cosh \xi - s)^{3/2} \cdot \varphi_i = U_{ni}(\xi) \cdot C_{n+1}^{-1/2}(s)$$

We express χ_i by $U(\xi)$ and $C_{n+1}^{-1/2}(s)$

$$\frac{\partial^2 \chi_i}{\partial \xi^2} = \sum_{n=0}^{\infty} C_{n+1}^{-1/2}(s) \cdot \frac{\partial^2 U_i(\xi)}{\partial \xi^2} \quad \frac{\partial^2 \chi_i}{\partial s^2} = \sum U_i(\xi) \cdot \frac{\partial^2 C_{n+1}^{-1/2}(s)}{\partial s^2}$$

Since $C_n^\lambda(s)$ satisfies the equation

$$(1-s^2) \cdot \frac{\partial^2 y}{\partial s^2} - (2\lambda+1) \cdot s \cdot \frac{\partial y}{\partial s} + n(n+2\lambda) \cdot y = 0$$

Putting $\lambda = -1/2$ and replacing $n \rightarrow n+1$

$$(1-s^2) \cdot \frac{\partial^2 C_{n+1}^{-1/2}(s)}{\partial s^2} + (n+1) \cdot (n+1-1) \cdot C_{n+1}^{-1/2}(s) = 0$$

Thus,

$$(1-s^2) \cdot \frac{\partial^2 C_{n+1}^{-1/2}(s)}{\partial s^2} = -(n+1) \cdot n \cdot C_{n+1}^{-1/2}(s)$$

$$(1-s^2) \cdot \frac{\partial^2 \chi_i}{\partial s^2} = \sum U_i(\xi) \cdot (1-s^2) \cdot \frac{\partial^2 C_{n+1}^{-1/2}(s)}{\partial s^2} = \sum -U_i(\xi) \cdot (n+1) \cdot n \cdot C_{n+1}^{-1/2}(s)$$

As a result

$$\begin{aligned} \tau_{\eta} &= \frac{\mu_i}{c^3 \cdot (1-s^2)^{1/2}} \cdot \left\{ f^{3/2} \sum \left[-U(\xi) \cdot (n+1) \cdot n \cdot C_{n+1}^{-1/2}(s) - C_{n+1}^{-1/2}(s) \cdot \frac{\partial^2 U(\xi)}{\partial \xi^2} \right] \right. \\ &\quad \left. + \frac{3}{4} \cdot f \cdot \varphi_i \cdot [f^2 + 2 \cdot \sinh^2 \xi] \right\} \\ &= -\frac{\mu_i}{c^3} \cdot f^{3/2} \cdot (1-s^2)^{1/2} \sum \left[U_i(\xi) \cdot n(n+1) + \frac{\partial^2 U(\xi)}{\partial \xi^2} \right] \cdot \frac{C_{n+1}^{-1/2}}{(1-s^2)} \\ &\quad + \frac{\mu_i}{c^3} \cdot \frac{3}{4} \cdot \frac{\varphi_i}{(1-s^2)^{1/2}} \cdot [f^3 + 2 \cdot f \cdot \sinh^2 \xi] \end{aligned}$$

About the second term

$$\varphi_i = \frac{R^2 \cdot U_i}{2} = \frac{1}{2} \left(\frac{c \cdot \sin \eta}{\cosh \xi - s} \right)^2 \cdot U_i = \frac{U_i \cdot C^2 \cdot \sin^2 \eta}{(\cosh \xi - s)^2} = \frac{U_i \cdot C^2 \cdot (1-s^2)}{f^2} \quad \text{Hence}$$

$$\frac{\mu_i}{c^3} \cdot \frac{3}{4} \cdot \frac{U_i}{(1-s^2)^{1/2}} \cdot \frac{C^2 \cdot (1-s^2)}{f^2} \cdot [f^3 + 2 \cdot f \cdot \sinh^2 \xi]$$

$$= \frac{3}{8} \cdot \frac{\mu_i}{c} \cdot U_i \cdot (1-s^2)^{1/2} \cdot f^{3/2} \cdot [f^{-1/2} + 2 \cdot f^{-3/2} \cdot \sinh^2 \xi]$$

Therefore

$$\begin{aligned} \tau_{\eta} &= -\frac{\mu_i}{c^3} \cdot f^{3/2} \cdot (1-s^2)^{1/2} \sum \left[U_i(\xi) \cdot n(n+1) + \frac{\partial^2 U(\xi)}{\partial \xi^2} \right] \cdot \frac{C_{n+1}^{-1/2}}{(1-s^2)} \\ &\quad + \frac{3}{8} \cdot \frac{\mu_i}{c} \cdot U_i \cdot (1-s^2)^{1/2} \cdot f^{3/2} \cdot [f^{-1/2} + 2 \cdot f^{-3/2} \cdot \sinh^2 \xi] \end{aligned}$$

→ A. 2, 3, 3

.....

In the next, we expand $(\cosh \xi - s)^m$ by potential function.

1. It stands that

$$(\cosh \xi - s)^{-1/2} = \sqrt{2} \sum e^{\mp(n+1/2)\xi} \cdot P_n(s)$$

Differentiate both sides with s

$$-\frac{1}{2} \cdot (\cosh \xi - s)^{-3/2} \cdot (-1) = \sqrt{2} \sum e^{\mp(n+1/2)\xi} \cdot \frac{\partial P_n(s)}{\partial s}$$

$$\therefore (\cosh \xi - s)^{-3/2} = 2\sqrt{2} \sum e^{\mp(n+1/2)\xi} \cdot \frac{\partial P_n(s)}{\partial s}$$

2. To expand $(\cosh \xi - s)^{-3/2}$. Since

$$(\cosh \xi - s)^{-1/2} = \frac{1}{\sqrt{2}} \sum \frac{\partial P_n}{\partial s} \left\{ \frac{e^{\mp(n-1/2)\xi}}{n-1/2} - \frac{e^{\mp(n+3/2)\xi}}{n+3/2} \right\}$$

$$(1-s^2) \cdot (\cosh \xi - s)^{-1/2} = \frac{\sqrt{2}}{2} \sum (1-s^2) \frac{\partial P_n}{\partial s} \left\{ \frac{e^{\mp(n-1/2)\xi}}{n-1/2} - \frac{e^{\mp(n+3/2)\xi}}{n+3/2} \right\}$$

while

$$(1-s^2) \cdot \frac{\partial P_n}{\partial s} = n(n+1) \cdot C_{n+1}^{-1/2}(s)$$

$$= \sqrt{2} \cdot \sum n(n+1) \cdot C_{n+1}^{-1/2}(s) \cdot \left\{ \frac{e^{\mp(n-1/2)\xi}}{2n-1} - \frac{e^{\mp(n+3/2)\xi}}{2n+3} \right\}$$

here putting

$$f_n(\xi) = n(n+1) \cdot \left\{ \frac{e^{\mp(n-1/2)\xi}}{2n-1} - \frac{e^{\mp(n+3/2)\xi}}{2n+3} \right\}$$

then,

$$(1-s^2) \cdot (\cosh \xi - s)^{-1/2} = \sqrt{2} \sum f_n(\xi) \cdot C_{n+1}^{-1/2}(s)$$

Differentiate both sides by ξ

$$(1-s^2) \cdot \left(-\frac{1}{2} \right) \cdot (\cosh \xi - s)^{-3/2} \cdot \sinh \xi = \sqrt{2} \sum \frac{\partial f_n(\xi)}{\partial \xi} \cdot C_{n+1}^{-1/2}(s)$$

eliminate $\sinh \xi$

$$(1-s^2) \cdot (\cosh \xi - s)^{-3/2} = -2\sqrt{2} \sum \frac{\partial f_n(\xi)}{\partial \xi} \cdot \frac{C_{n+1}^{-1/2}(s)}{\sinh \xi}$$

Again differentiate by ξ

$$(1-s^2) \cdot \left(-\frac{3}{2} \right) \cdot (\cosh \xi - s)^{-5/2} \cdot \sinh \xi$$

$$= -2\sqrt{2} \cdot \sum \left[\frac{\partial^2 f_n(\xi)}{\partial \xi^2} \cdot \sinh \xi - \cosh \xi \cdot \frac{\partial f_n(\xi)}{\partial \xi} \right] \cdot \frac{C_{n+1}^{-1/2}(s)}{\sinh^2 \xi}$$

$$= -2\sqrt{2} \cdot \sum \left[\frac{1}{\sinh \xi} \cdot \frac{\partial^2 f_n(\xi)}{\partial \xi^2} - \frac{1}{\sinh \xi} \cdot \coth \xi \cdot \frac{\partial f_n(\xi)}{\partial \xi} \right] \cdot C_{n+1}^{-1/2}(s)$$

Multiply $\sinh \xi$ on the both sides

$$\frac{3}{4} (1-s^2) \cdot (\cosh \xi - s)^{-5/2} \cdot \sinh^2 \xi$$

$$= \sqrt{2} \cdot \sum \left[\frac{\partial^2 f_n(\xi)}{\partial \xi^2} - \coth \xi \cdot \frac{\partial f_n(\xi)}{\partial \xi} \right] \cdot C_{n+1}^{-1/2}(s)$$

on the while

$$\frac{\partial f_n}{\partial \xi} = \frac{n(n+1)}{2} \left\{ \mp e^{\mp(n-1/2)\xi} \pm e^{\mp(n+3/2)\xi} \right\}$$

$$\frac{\partial^2 f_n}{\partial \xi^2} = \frac{n(n+1)}{2} \left\{ (n-1/2) e^{\mp(n-1/2)\xi} - (n+3/2) e^{\mp(n+3/2)\xi} \right\}$$

$$\begin{aligned} &\frac{f_n(\xi)}{\partial \xi^2} - \coth \xi \cdot \frac{\partial f_n}{\partial \xi} \\ &= \frac{n(n+1)}{2} \left\{ (n-1/2) \cdot e^{\mp(n-1/2)\xi} - (n+3/2) \cdot e^{\mp(n+3/2)\xi} \right. \\ &\quad \left. - \coth \xi \cdot (\mp e^{\mp(n-1/2)\xi} \pm e^{\mp(n+3/2)\xi}) \right\} \end{aligned}$$

On the while

$$\coth \xi = \frac{\cosh \xi}{\sinh \xi} = \frac{e^{2\xi} + 1}{e^{2\xi} - 1} = 1 + \frac{2}{e^{2\xi} - 1}$$

Thus, operation only for $(-)$

$$\{ \} = (n-1/2) e^{-(n-1/2)\xi} - (n+3/2) e^{-(n+3/2)\xi}$$

$$- \left(1 + \frac{2}{e^{2\xi} - 1} \right) \cdot \left(-e^{-(n-1/2)\xi} + e^{-(n+3/2)\xi} \right)$$

$$\begin{aligned} &= (n-1/2) e^{-(n-1/2)\xi} + e^{-(n-1/2)\xi} - (n+3/2) e^{-(n+3/2)\xi} - e^{-(n+3/2)\xi} \\ &\quad + \frac{2}{(e^{2\xi} - 1)} \cdot (e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}) \end{aligned}$$

$$= (n+1/2) \cdot e^{-(n-1/2)\xi} - (n+1/2) \cdot e^{-(n+3/2)\xi} - 2 \cdot e^{-(n+3/2)\xi} \\ + 2 \cdot \frac{[e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}]}{(e^{2\xi} - 1)}$$

Sum of 3rd and 4th terms are

$$= 2 \cdot \frac{[e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi} - e^{-(n+3/2)\xi} + e^{-(n+3/2)\xi}]}{(e^{2\xi} - 1)} = 2 \cdot 0$$

hence,

$$\{ \} = (n+1/2) [e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}]$$

Therefore

$$\frac{3}{4}(1-s^2) \cdot (\cosh \xi - s)^{-3/2} \cdot \sinh^2 \xi \\ = \frac{\sqrt{2}}{4} \sum n(n+1)(2n+1) \cdot [e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}] C_{n+1}^{-1/2}(s) \\ \rightarrow A, 2, 3, 6$$

Substitute

$$(1-s^2) \cdot (\cosh \xi - s)^{-1/2} = \sqrt{2} \sum f_n(\xi) \cdot C_{n+1}^{-1/2}(s) \\ \text{into } A, 2, 3, 3 \\ \tau_{\xi\eta} = -\frac{\mu_i}{c^3} \cdot (1-s^2)^{1/2} \cdot (\cosh \xi - s)^{3/2} \cdot \sum \left[n(n+1)U_i(\xi) + \frac{\partial^2 U_i}{\partial \xi^2} \right] \cdot \frac{C_{n+1}^{-1/2}}{(1-s^2)} \\ + \mu_i \left\{ U_j \left[\frac{\sqrt{2} \sum n(n+1)(2n+1)}{4} [e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}] C_{n+1}^{-1/2}(s) \cdot \frac{1}{(1-s^2)} \right] \right. \\ \left. + \frac{3}{8} \sqrt{2} \cdot \sum \frac{f_n(\xi) \cdot C_{n+1}^{-1/2}(s)}{(1-s^2)} \right\} (1-s^2)^{1/2} \cdot (\cosh \xi - s)^{3/2} \\ = -\frac{\mu_i}{c^3} \cdot (1-s^2)^{1/2} \cdot (\cosh \xi - s)^{3/2} \cdot \frac{C_{n+1}^{-1/2}}{(1-s^2)} \cdot \sum_{n=0}^{\infty} \\ \left[\frac{\partial^2 U_i}{\partial \xi^2} + n(n+1)U_i - \frac{3}{8} \sqrt{2} f_n(\xi) - \frac{\sqrt{2}}{4} n(n+1)(2n+1) \cdot U_j \cdot c^2 \right. \\ \left. \cdot [e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}] \right]$$

where

$$f_n(\xi) = \frac{\sqrt{2}}{U_j c^2} U_n(\xi_j) \quad \text{we have} \\ n(n+1)U_i(\xi) - \frac{3}{8} \sqrt{2} \cdot \sqrt{2} \cdot U_n(\xi) = [n^2 + n - 3/4] U_n(\xi) \\ = \frac{1}{4} [4n^2 + 4n - 3] U_n(\xi) = \frac{(2n+3)(2n-1) \cdot U_n(\xi)}{4}$$

Thus, we finally have

$$\tau_{\xi\eta} = -\frac{\mu_i}{c^3} \cdot (1-s^2)^{1/2} \cdot (\cosh \xi - s)^{3/2} \cdot \frac{C_{n+1}^{-1/2}}{(1-s^2)} \\ \sum \left[\frac{\partial^2 U_i}{\partial \xi^2} + \frac{(2n+3)(2n-1)}{4} U_n - \frac{\sqrt{2}}{4} U_j \cdot \sqrt{2} (n+1)(2n+1) \right. \\ \left. \cdot [e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}] \right]$$

APPENDIX

1. Derivation of (23.1), (23.3)

$$1. \tilde{V}_n = A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi$$

From the first relation of (21.1) and its right side

$$A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi \\ = \frac{U_i}{4} \left\{ (2n+3)e^{-(n-1/2)\xi} - (2n-1)e^{-(n+3/2)\xi} \right\}$$

2. From the first equation of (21.1)

$$A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi \\ = A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi$$

3. From the relation of (21.2)

$$\frac{\partial \tilde{V}_{n1}}{\partial \xi}(\alpha) = \frac{\partial \tilde{V}_{n3}}{\partial \xi}(\alpha) \quad (21.2)$$

After differentiation

$$(n-1/2) \{ A_n \sinh(n-1/2)\xi + B_n \cosh(n-1/2)\xi \} \\ + (n+3/2) \{ C_n \sinh(n+3/2)\xi + D_n \cosh(n+3/2)\xi \} \\ = (n-1/2) \{ A_n \sinh(n-1/2)\xi + B_n \cosh(n-1/2)\xi \} \\ + (n+3/2) \{ C_n \sinh(n+3/2)\xi + D_n \cosh(n+3/2)\xi \}$$

since

$$\sinh(n-1/2)\xi = \sinh(n+1/2)\xi \cdot \cosh \alpha - \cosh(n+1/2)\xi \cdot \sinh \alpha \\ \sinh(n+3/2)\xi = \sinh(n+1/2)\xi \cdot \cosh \alpha + \cosh(n+1/2)\xi \cdot \sinh \alpha \\ A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi \\ = \frac{U_i}{4} \left\{ (2n+3)e^{-(n-1/2)\xi} - (2n-1)e^{-(n+3/2)\xi} \right\}$$

can be expanded by the terms including $\cosh, \sinh(n+1/2)$. The left side of above equation is

$$= A_n [\cosh(n+1/2)\xi \cdot \cosh \alpha - \sinh(n+1/2)\xi \cdot \sinh \alpha] \\ + B_n [\sinh(n+1/2)\xi \cdot \cosh \alpha - \cosh(n+1/2)\xi \cdot \sinh \alpha] \\ + C_n [\cosh(n+1/2)\xi \cdot \cosh \alpha - \sinh(n+1/2)\xi \cdot \sinh \alpha] \\ + D_n [\sinh(n+1/2)\xi \cdot \cosh \alpha - \cosh(n+1/2)\xi \cdot \sinh \alpha]$$

Putting

$$X = (n+1/2)\xi, \quad A_{n1}^* = A_{n3} + C_{n3}, \quad A_{n2}^* = B_{n3} - D_{n3} \\ A_{n2}^* = B_{n3} + D_{n3}, \quad A_{n4}^* = A_{n3} - C_{n3} \\ = \cosh X \cdot \cosh \alpha \cdot (A_{n3} + C_{n3}) - \sinh X \cdot \sinh \alpha \cdot (A_{n3} - C_{n3}) \\ + \sinh X \cdot \cosh \alpha \cdot (B_{n3} + D_{n3}) - \cosh X \cdot \sinh \alpha \cdot (B_{n3} - D_{n3}) \\ = \cosh X \cdot \cosh \alpha \cdot A_{n1}^* - \sinh X \cdot \sinh \alpha \cdot A_{n2}^* \\ + \sinh X \cdot \cosh \alpha \cdot A_{n2}^* - \cosh X \cdot \sinh \alpha \cdot A_{n3}^* \\ = \cosh \alpha [\cosh X \cdot A_{n1}^* + \sinh X \cdot A_{n2}^*] - \sinh \alpha [\sinh X \cdot A_{n1}^* + \cosh X \cdot A_{n3}^*]$$

The right side is transformed to

$$= \frac{U_i}{4} [(2n+3)e^{-(n+1/2)\xi} - (2n-1)e^{-(n+1/2)\xi}] \\ = \frac{U_i}{4} [(2n+1+2)e^{-(n+1/2)\xi} \cdot e^\alpha - (2n+1-2)e^{-(n+1/2)\xi} \cdot e^{-\alpha}] \\ = \frac{U_i}{4} [(2n+1+2)e^{-X} \cdot e^\alpha - (2n+1-2)e^{-X} \cdot e^{-\alpha}] \\ = \frac{U_i e^{-X}}{4} [(2n+1) \cdot (e^\alpha - e^{-\alpha}) + 2e^\alpha + 2e^{-\alpha}] \\ = \frac{U_i e^{-X}}{4} [(2n+1) \cdot 2 \cdot \sinh \alpha + 2 \cdot 2 \cdot \cosh \alpha] \\ = \frac{U_i e^{-X}}{2} [(2n+1) \cdot \sinh \alpha + 2 \cdot \cosh \alpha] \\ \rightarrow (23, 1)$$

Similar procedure leads to (23.2).

2. Derivation of (23.3)

2-1. Modification of (21.5) and (21.6)

$$V_n^* = K(n)U_n^* \quad \text{and from} \quad (19, 3)$$

$$= K(n) \cdot \left\{ 4 \frac{\alpha^2 U_n}{\alpha \xi^2} + (2n+3)(2n-1) \cdot U_n \right\} \\ = K(n) \left\{ 4(n-1/2)^2 a_n \cosh(n-1/2)\xi + 4(n-1/2)^2 b_n \sinh(n-1/2)\xi \right. \\ \left. + 4(n+3/2)^2 c_n \cosh(n+3/2)\xi + 4(n+3/2)^2 d_n \sinh(n+3/2)\xi \right. \\ \left. + (2n+3)(2n-1) [A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi \right. \\ \left. + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi] \right\}$$

From (20) such that $A_n = K(n) \cdot a_n$

$$= 4A_n (n-1/2)^2 \cosh(n-1/2)\xi + 4B_n (n-1/2)^2 \sinh(n-1/2)\xi \\ + 4C_n (n+3/2)^2 \cosh(n+3/2)\xi + 4D_n (n+3/2)^2 \sinh(n+3/2)\xi \\ + (2n+3)(2n-1) [A_n \cosh(n-1/2)\xi + B_n \sinh(n-1/2)\xi \\ + C_n \cosh(n+3/2)\xi + D_n \sinh(n+3/2)\xi] \\ + \frac{U_i}{2} \cdot (2n+1)(2n-1)(2n+3) (e^{-(n-1/2)\xi} - e^{-(n+3/2)\xi}) (\mu_i - \mu_j)$$

By using

$$\begin{aligned} e^{-(n-1/2)\alpha} - e^{-(n+3/2)\alpha} &= e^{-(n+1/2-1)\alpha} - e^{-(n+1/2+1)\alpha} \\ &= e^{-(n+1/2)\alpha} [e^\alpha - e^{-\alpha}] \\ &= e^{-X} \cdot \sinh \alpha \cdot 2 \end{aligned}$$

The right sides of (21, 5), (21, 6) are $\frac{U_1}{2} \cdot (2n+1)(2n-1)(2n+3) \cdot e^{-X} \cdot 2 \cdot \sinh \alpha \cdot (\mu_1 - \mu_3)$

The left side of (21, 5) regarding only to A_n, C_n

$$\begin{aligned} &\mu_1 V_{n1}^*(\alpha) - \mu_3 V_{n3}^*(\alpha) \\ &= \mu_1 [4A_{n1} \cdot (n-1/2)^2 \cosh(n-1/2)\alpha + 4C_{n1} \cdot (n+3/2)^2 \cosh(n+3/2)\alpha \\ &\quad + (2n+3)(2n-1)(A_{n1} \cdot \cosh(n-1/2)\alpha + C_{n1} \cdot \cosh(n+3/2)\alpha)] \\ &\quad - \mu_3 [4A_{n3} \cdot (n-1/2)^2 \cosh(n-1/2)\alpha + 4C_{n3} \cdot (n+3/2)^2 \cosh(n+3/2)\alpha \\ &\quad + (2n+3)(2n-1)(A_{n3} \cdot \cosh(n-1/2)\alpha + C_{n3} \cdot \cosh(n+3/2)\alpha)] \end{aligned}$$

A). For the first, rearrange only with respect to μ_3 . Since $4 \cdot (n-1/2)^2 = (2n-1)^2 = [(2n+1)-2]^2 = (2n+1)^2 - 4 \cdot (2n+1) + 4$
 $4 \cdot (n+3/2)^2 = (2n+3)^2 = [(2n+1)+2]^2 = (2n+1)^2 + 4 \cdot (2n+1) + 4$
 $(2n+3)(2n-1) = [(2n+1)+2][(2n+1)-2] = (2n+1)^2 - 4$

Hence terms regarding to μ_3 are

$$\begin{aligned} &-\mu_3 [A_{n3} \cdot \cosh(n-1/2)\alpha \cdot [(2n+1)^2 - 4 \cdot (2n+1) + 4] \\ &\quad + C_{n3} \cdot \cosh(n+3/2)\alpha \cdot [(2n+1)^2 + 4 \cdot (2n+1) + 4] \\ &\quad + A_{n3} \cdot \cosh(n-1/2)\alpha \cdot [(2n+1)^2 - 4] \\ &\quad + C_{n3} \cdot \cosh(n+3/2)\alpha \cdot [(2n+1)^2 - 4]] \\ &= -\mu_3 [2 \cdot (2n+1)^2 [A_{n3} \cdot \cosh(n-1/2)\alpha + C_{n3} \cdot \cosh(n+3/2)\alpha] \\ &\quad - 4 \cdot (2n+1) [A_{n3} \cdot \cosh(n-1/2)\alpha - C_{n3} \cdot \cosh(n+3/2)\alpha]] \\ &= -\mu_3 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [A_{n3} \cdot \cosh(n-1/2)\alpha + C_{n3} \cdot \cosh(n+3/2)\alpha] \\ &\quad - 2[A_{n3} \cdot \cosh(n-1/2)\alpha - C_{n3} \cdot \cosh(n+3/2)\alpha]] \end{aligned}$$

On the other hand,

$$\begin{aligned} &A_{n3} \cosh(n-1/2)\alpha + C_{n3} \cosh(n+3/2)\alpha \\ &= A_{n3} [\cosh(n+1/2)\alpha \cdot \cosh \alpha - \sinh(n+1/2)\alpha \cdot \sinh \alpha] \\ &\quad + C_{n3} [\cosh(n+1/2)\alpha \cdot \cosh \alpha - \sinh(n+1/2)\alpha \cdot \sinh \alpha] \\ &= \cosh(n+1/2)\alpha \cdot \cosh \alpha \cdot (A_{n3} + C_{n3}) \\ &\quad - \sinh(n+1/2)\alpha \cdot \sinh \alpha \cdot (A_{n3} - C_{n3}) \\ &= \cosh X \cdot \cosh \alpha \cdot A_{n4}^* - \sinh X \cdot \sinh \alpha \cdot A_{n4}^* \end{aligned}$$

and

$$\begin{aligned} &A_{n3} \cosh(n-1/2)\alpha - C_{n3} \cosh(n+3/2)\alpha \\ &= A_{n3} [\cosh(n+1/2)\alpha \cdot \cosh \alpha - \sinh(n+1/2)\alpha \cdot \sinh \alpha] \\ &\quad - C_{n3} [\cosh(n+1/2)\alpha \cdot \cosh \alpha - \sinh(n+1/2)\alpha \cdot \sinh \alpha] \\ &= \cosh X \cdot \cosh \alpha \cdot A_{n4}^* - \sinh X \cdot \sinh \alpha \cdot A_{n4}^* \end{aligned}$$

Hence

$$\begin{aligned} &= -\mu_3 \cdot 2 \cdot (2n+1) [(2n+1) \cdot (\cosh X \cdot \cosh \alpha \cdot A_{n4}^* - \sinh X \cdot \sinh \alpha \cdot A_{n4}^*) \\ &\quad - 2(\cosh X \cdot \cosh \alpha \cdot A_{n4}^* - \sinh X \cdot \sinh \alpha \cdot A_{n4}^*)] \\ &= -2 \cdot (2n+1) \cdot A_{n4}^* [(2n+1) \mu_3 \cdot \cosh X \cdot \cosh \alpha + 2 \sinh X \cdot \sinh \alpha \cdot \mu_3] \\ &\quad + 2 \cdot (2n+1) \cdot A_{n4}^* [(2n+1) \mu_3 \cdot \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha \cdot \mu_3] \end{aligned}$$

B). For the second, rearrange about μ_1 . The process is the same until expanding $(n-1/2) \cdot (n+3/2)$ with those of μ_3

$$\begin{aligned} &\mu_1 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [A_{n1} \cdot \cosh(n-1/2)\alpha + C_{n1} \cdot \cosh(n+3/2)\alpha] \\ &\quad - 2[A_{n1} \cdot \cosh(n-1/2)\alpha - C_{n1} \cdot \cosh(n+3/2)\alpha]] \end{aligned}$$

from (21, 2)

$$\begin{aligned} &(2n-1) \cdot A_{n1} \cdot \sinh(n-1/2)\alpha + (2n+3) \cdot C_{n1} \cdot \sinh(n+3/2)\alpha \\ &= (2n-1) \cdot A_{n3} \cdot \sinh(n-1/2)\alpha + (2n+3) \cdot C_{n3} \cdot \sinh(n+3/2)\alpha \end{aligned}$$

and from (21, 1)

$$\begin{aligned} &A_{n1} \cdot \cosh(n-1/2)\alpha + C_{n1} \cdot \cosh(n+3/2)\alpha \\ &= A_{n3} \cdot \cosh(n-1/2)\alpha + C_{n3} \cdot \cosh(n+3/2)\alpha \end{aligned}$$

Hence

$$\begin{aligned} &A_{n1} \cdot \cosh(n-1/2)\alpha + C_{n1} \cdot \cosh(n+3/2)\alpha \\ &= A_{n3} \cdot \sinh(n-1/2)\alpha + C_{n3} \cdot \sinh(n+3/2)\alpha \\ &A_{n1} \cdot \cosh(n-1/2)\alpha - C_{n1} \cdot \cosh(n+3/2)\alpha \\ &= A_{n3} \cdot \sinh(n-1/2)\alpha - C_{n3} \cdot \sinh(n+3/2)\alpha \end{aligned}$$

Therefore rearranging about μ_1

$$\begin{aligned} &\mu_1 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [A_{n3} \cdot \sinh(n-1/2)\alpha + C_{n3} \cdot \sinh(n+3/2)\alpha] \\ &\quad - 2 \cdot [A_{n3} \cdot \sinh(n-1/2)\alpha - C_{n3} \cdot \sinh(n+3/2)\alpha]] \\ &= \mu_1 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [A_{n3} (\sinh(n+1/2)\alpha \cdot \cosh \alpha - \cosh(n+1/2)\alpha \cdot \sinh \alpha) \\ &\quad + C_{n3} (\sinh(n+1/2)\alpha \cdot \cosh \alpha + \cosh(n+1/2)\alpha \cdot \sinh \alpha)] \\ &\quad - 2 \cdot [A_{n3} (\sinh(n+1/2)\alpha \cdot \cosh \alpha - \cosh(n+1/2)\alpha \cdot \sinh \alpha) \\ &\quad - C_{n3} (\sinh(n+1/2)\alpha \cdot \cosh \alpha + \cosh(n+1/2)\alpha \cdot \sinh \alpha)]] \end{aligned}$$

Putting $(n+1/2)\alpha = X$

$$\begin{aligned} &= \mu_1 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [A_{n3} (\sinh X \cdot \cosh \alpha - \cosh X \cdot \sinh \alpha) \\ &\quad + C_{n3} (\sinh X \cdot \cosh \alpha + \cosh X \cdot \sinh \alpha)] \\ &\quad - 2 \cdot [A_{n3} (\sinh X \cdot \cosh \alpha - \cosh X \cdot \sinh \alpha) \\ &\quad - C_{n3} (\sinh X \cdot \cosh \alpha + \cosh X \cdot \sinh \alpha)]] \\ &= \mu_1 \cdot 2 \cdot (2n+1) [(2n+1) \cdot [\sinh X \cdot \cosh \alpha \cdot A_{n4}^* - \cosh X \cdot \sinh \alpha \cdot A_{n4}^*] \\ &\quad - 2[\sinh X \cdot \cosh \alpha \cdot A_{n4}^* - \cosh X \cdot \sinh \alpha \cdot A_{n4}^*]] \\ &= 2 \cdot (2n+1) \cdot A_{n4}^* [(2n+1) \mu_1 \cdot \sinh X \cdot \cosh \alpha + 2 \cosh X \cdot \sinh \alpha \cdot \mu_1] \\ &\quad - 2 \cdot (2n+1) \cdot A_{n4}^* [(2n+1) \mu_1 \cdot \cosh X \cdot \sinh \alpha + 2 \sinh X \cdot \cosh \alpha \cdot \mu_1] \end{aligned}$$

Associate the terms of μ_1 and μ_3 , (21, 5) is

$$\begin{aligned} &\mu_1 V_{n1}^*(\alpha) - \mu_3 V_{n3}^*(\alpha) \\ &= 2 \cdot (2n+1) \{A_{n4}^* [(2n+1) \mu_1 \sinh X \cdot \cosh \alpha + 2 \cosh X \cdot \sinh \alpha \cdot \mu_1 \\ &\quad + (2n+1) \mu_3 \cosh X \cdot \cosh \alpha + 2 \sinh X \cdot \sinh \alpha \cdot \mu_3] \\ &\quad - A_{n4}^* [(2n+1) \mu_1 \cosh X \cdot \sinh \alpha + 2 \sinh X \cdot \cosh \alpha \cdot \mu_1 \\ &\quad + (2n+1) \mu_3 \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha \cdot \mu_3]\} \end{aligned}$$

→ These are the 1st and 4th terms of (23, 3)

By the same process, we have have (23, 4)

3. Solving the simultaneous algebraic equations of A.1, 2, 3, 4 in equations (23-1, 23-2, 23-3, 23-4)

The entire form is simplified as

$$\begin{aligned} &A_1 h_1 + A_2 h_2 - A_3 h_3 - A_4 h_4 = f_1 \\ &A_1 l_1 + A_2 l_2 - A_3 l_3 - A_4 l_4 = f_2 \\ &A_1 (m_1 \mu_1 + n_1 \mu_3) + A_2 (m_2 \mu_1 + n_2 \mu_3) \\ &\quad - A_3 (m_3 \mu_1 + n_3 \mu_3) - A_4 (m_4 \mu_1 + n_4 \mu_3) = f_3 \end{aligned}$$

3-1. Eliminate A_4

(1) $(m_4 \mu_1 + n_4 \mu_3) - (3) \cdot h_4$ (3) can be separated to the

terms of μ_1 and μ_3 such that

$$\begin{aligned} &= [(1) \cdot m_4 - (3)(\mu_1 \text{ component}) \cdot h_4] \mu_1 \\ &\quad + [(1)n_4 - (3)(\mu_3 \text{ component}) \cdot h_4] \mu_3 \end{aligned}$$

The equations are rearranged about $\mu_1, \mu_2, \mu_3, \mu_4$ by separation.

3-1-A. For μ_1

from (23, 1), (23, 2)

$$A_{n1} \cdot \cosh \alpha \cdot \cosh X + A_{n2} \cdot \cosh \alpha \cdot \sinh X - A_{n3} \cdot \sinh \alpha \cdot \cosh X - A_{n4} \cdot \sinh \alpha \cdot \sinh X = f_1 \quad (1)$$

$$A_1 \rightarrow A_{n1}, A_2 \rightarrow A_{n2}, A_3 \rightarrow A_{n3}, A_4 \rightarrow A_{n4}$$

$$A_1 \cdot \cosh \beta \cdot \cosh Y + A_2 \cdot \cosh \beta \cdot \sinh Y - A_3 \cdot \sinh \beta \cdot \cosh Y - A_4 \cdot \sinh \beta \cdot \sinh Y = f_2 \quad (2)$$

(2 3, 3) leads

$$\begin{aligned}
& A_1 [2 \cdot \sinh \alpha \cdot \cosh X + (2n+1) \cdot \cosh \alpha \cdot \sinh X] \cdot \mu_1 \\
& + A_2 [2 \cdot \sinh \alpha \cdot \sinh X + (2n+1) \cdot \cosh \alpha \cdot \cosh X] \cdot \mu_1 \\
& - A_3 [2 \cdot \cosh \alpha \cdot \cosh X + (2n+1) \cdot \sinh \alpha \cdot \sinh X] \cdot \mu_1 \\
& - A_4 [2 \cdot \cosh \alpha \cdot \sinh X + (2n+1) \cdot \sinh \alpha \cdot \cosh X] \cdot \mu_1 = f_3
\end{aligned} \quad (3)$$

To eliminate A_4

$$A) (1) \cdot [2 \cdot \cosh \alpha \cdot \sinh X + (2n+1) \cdot \sinh \alpha \cdot \cosh X] \cdot \mu_1 - (3) \cdot \sinh \alpha \cdot \sinh X$$

The left side =

$$\begin{aligned}
& A_1 \cdot \mu_1 \cdot [\cosh \alpha \cdot \cosh X \cdot (2 \cosh \alpha \cdot \sinh X + (2n+1) \cdot \sinh \alpha \cdot \cosh X)] \\
& + A_2 \cdot \mu_1 \cdot [\cosh \alpha \cdot \sinh X \cdot (2 \cosh \alpha \cdot \sinh X + (2n+1) \cdot \sinh \alpha \cdot \cosh X)] \\
& - A_3 \cdot \mu_1 \cdot [\sinh \alpha \cdot \cosh X \cdot (2 \cosh \alpha \cdot \sinh X + (2n+1) \cdot \sinh \alpha \cdot \cosh X)]
\end{aligned}$$

$$\begin{aligned}
& - A_1 \cdot [(2 \sinh \alpha \cdot \cosh X + (2n+1) \cdot \cosh \alpha \cdot \sinh X) \cdot \sinh \alpha \cdot \sinh X] \cdot \mu_1 \\
& - A_2 \cdot [(2 \sinh \alpha \cdot \sinh X + (2n+1) \cdot \cosh \alpha \cdot \cosh X) \cdot \sinh \alpha \cdot \sinh X] \cdot \mu_1 \\
& + A_3 \cdot [(2 \cosh \alpha \cdot \cosh X + (2n+1) \cdot \sinh \alpha \cdot \sinh X) \cdot \sinh \alpha \cdot \sinh X] \cdot \mu_1
\end{aligned}$$

Factor out μ_1 after this

The left side =

$$\begin{aligned}
& A_1 \cdot [2 \cosh^2 \alpha \cdot \cosh X \cdot \sinh X - 2 \sinh^2 \alpha \cdot \sinh X \cdot \cosh X \\
& + (2n+1) \cdot (\cosh \alpha \cdot \sinh \alpha \cdot \cosh^2 X - \cosh \alpha \cdot \sinh \alpha \cdot \sinh^2 X)] \\
& + A_2 \cdot [2 \cosh^2 \alpha \cdot \sinh^2 X - 2 \sinh^2 \alpha \cdot \sinh^2 X \\
& + (2n+1) \cdot (\sinh \alpha \cdot \cosh \alpha \cdot \sinh X \cdot \cosh X - \cosh \alpha \cdot \sinh \alpha \cdot \cosh X \cdot \sinh X)] \\
& + A_3 \cdot [2 \cosh \alpha \cdot \cosh X \cdot \sinh \alpha \cdot \sinh X - 2 \sinh \alpha \cdot \cosh X \cdot \cosh \alpha \cdot \sinh X \\
& + (2n+1) \cdot (\sinh^2 \alpha \cdot \sinh^2 X - \sinh^2 \alpha \cdot \cosh^2 X)] \\
& = A_1 \cdot \left[\sinh 2X \cdot (\cosh^2 \alpha - \sinh^2 \alpha) + \frac{(2n+1)}{2} \sinh 2\alpha \cdot (\cosh^2 X - \sinh^2 X) \right] \\
& + A_2 \cdot [2 \cdot \sinh^2 X \cdot (\cosh^2 \alpha - \sinh^2 \alpha)] \\
& + A_3 \cdot [(2n+1) \cdot (-\sinh^2 \alpha) \cdot (\cosh^2 X - \sinh^2 X)] \\
& = A_1 \cdot [\sinh 2X + (n+1/2) \cdot \sinh 2\alpha] \\
& + A_2 \cdot [2 \cdot \sinh^2 X] \\
& - A_3 \cdot [(2n+1) \cdot (\sinh^2 \alpha)] \\
& = f_1 \cdot [2 \cdot \cosh \alpha \cdot \sinh X + (2n+1) \cdot \cosh X \cdot \sinh \alpha] \\
& - f_3 \cdot \sinh \alpha \cdot \sinh X = g_1 \quad (4)
\end{aligned}$$

$$B) (1) \sinh \beta \cdot \sinh Y - (2) \cdot \sinh \alpha \cdot \sinh X$$

$$\begin{aligned}
& A_1 [\cosh \alpha \cdot \cosh X \cdot \sinh \beta \cdot \sinh Y] \\
& + A_2 [\cosh \alpha \cdot \sinh X \cdot \sinh \beta \cdot \sinh Y] \\
& - A_3 [\sinh \alpha \cdot \cosh X \cdot \sinh \beta \cdot \sinh Y] \\
& - A_1 [\cosh \beta \cdot \cosh Y \cdot \sinh \alpha \cdot \sinh X] \\
& - A_2 [\cosh \beta \cdot \sinh Y \cdot \sinh \alpha \cdot \sinh X] \\
& + A_3 [\sinh \beta \cdot \cosh Y \cdot \sinh \alpha \cdot \sinh X] \\
& = A_1 [\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \cosh Y - \cosh \beta \cdot \sinh \alpha \cdot \cosh X \cdot \sinh Y] \\
& + A_2 [\sinh X \cdot \sinh Y \cdot (\cosh \alpha \cdot \sinh \beta - \sinh \alpha \cdot \cosh \beta)] \\
& + A_3 [\sinh \alpha \cdot \sinh \beta \cdot (\cosh Y \cdot \sinh X - \cosh X \cdot \sinh Y)] \\
& = A_1 [\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \cosh Y - \cosh \alpha \cdot \sinh \beta \cdot \sinh X \cdot \cosh Y] \\
& - A_2 \cdot \sinh X \cdot \sinh Y \cdot \sinh(\alpha - \beta) \\
& + A_3 \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& = f_1 \cdot \sinh \beta \cdot \sinh Y - f_2 \cdot \sinh \alpha \cdot \sinh X \\
& = g_2 \quad \dots (5)
\end{aligned}$$

C) Eliminate A3 from (4) and (5)

$$(4) \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) + (5) \cdot (2n+1) \cdot \sinh^2 \alpha$$

$$\begin{aligned}
& A_1 \cdot [(\sinh 2X + (n+1/2) \cdot \sinh 2\alpha) \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& + (\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y) \cdot (2n+1) \cdot \sinh^2 \alpha] \\
& + A_2 \cdot [2 \sinh^2 X \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& - \sinh X \cdot \sinh Y \cdot \sinh(\alpha - \beta) \cdot (2n+1) \sinh^2 \alpha] \\
C-1) \text{ In } A_1, \text{ the terms timed by } (2n+1) \text{ are} \\
& \sinh \alpha \cdot \cosh \alpha \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& + (\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y) \cdot \sinh^2 \alpha \\
& = \sinh^2 \alpha \cdot [\cosh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& + \cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y] \\
& = \sinh^2 \alpha \cdot [\cosh \alpha \cdot \sinh \beta \cdot \sinh X \cdot \cosh Y - \cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \\
& + \cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y] \\
& = \sinh^2 \alpha \cdot \sinh X \cdot \cosh Y (\cosh \alpha \cdot \sinh \beta - \sinh \alpha \cdot \cosh \beta) \\
& = -\sinh^2 \alpha \cdot \sinh X \cdot \cosh Y \cdot \sinh(\alpha - \beta) \\
& = \sinh \alpha \cdot \sinh X \cdot (\sinh \alpha \cdot \cosh Y \cdot \sinh(\beta - \alpha))
\end{aligned}$$

C-2) In A_1 , the terms that are not timed by $(2n+1)$

$$\begin{aligned}
& \sinh 2X \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& = \sinh X \cdot \cosh X \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\
& = \sinh \alpha \cdot \sinh X \cdot (2 \cosh X \cdot \sinh \beta \cdot \sinh(X - Y))
\end{aligned}$$

C-3) In A_2 , factor out $\sinh \alpha \cdot \sinh X$

$$\begin{aligned}
& A_2 \cdot [(\sinh \alpha \cdot \sinh X) \cdot (2 \cdot \sinh X \cdot \sinh \beta \cdot \sinh(X - Y) \\
& - \sinh Y \cdot \sinh(\alpha - \beta) \cdot (2n+1) \cdot \sinh \alpha)]
\end{aligned}$$

Therefore

$$\begin{aligned}
& A_1 \cdot \mu_1 \cdot [\sinh \alpha \cdot \sinh X \cdot [(2n+1) \cdot \sinh \alpha \cdot \cosh Y \cdot \sinh(\beta - \alpha) \\
& + 2 \cosh X \cdot \sinh \beta \cdot \sinh(X - Y)] \\
& + A_2 \cdot \mu_1 \cdot [\sinh \alpha \cdot \sinh X \cdot [2 \sinh X \cdot \sinh \beta \cdot \sinh(X - Y) \\
& + \sinh Y \cdot \sinh(\beta - \alpha) \cdot (2n+1) \cdot \sinh \alpha]] \\
& = \{A_1 \cdot [\mu_1 F_{11}] + A_2 \mu_1 F_{31}\} \cdot \sinh \alpha \cdot \sinh X
\end{aligned}$$

4. Computation of right side about $[\mu_1]$. Since

$$\begin{aligned}
& g_1 = f_1 \cdot [2 \cosh \alpha \cdot \sinh X + (2n+1) \cdot \cosh X \cdot \sinh \alpha] - f_3 \cdot \sinh \alpha \cdot \sinh X \\
& = \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh \alpha + 2 \cosh \alpha] \cdot [2 \cosh \alpha \cdot \sinh X + (2n+1) \cdot \cosh X \cdot \sinh \alpha] \mu_1 \\
& + \frac{U_1}{2} \cdot (2n-1)(2n+3) \mu_1 \cdot \sinh \alpha \cdot e^{-X} \cdot \sinh \alpha \cdot \sinh X \\
& = \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh 2\alpha \cdot \sinh X + 4 \cosh^2 \alpha \cdot \sinh X \\
& + (2n+1)^2 \cdot \sinh^2 \alpha \cdot \cosh X + (2n+1) \cdot \cosh X \cdot \sinh 2\alpha \\
& + (2n+1-2)(2n+1+2) \cdot \sinh^2 \alpha \cdot \sinh X] \mu_1 \\
& = \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh 2\alpha \cdot (\sinh X + \cosh X) \\
& + (2n+1)^2 \cdot \sinh^2 \alpha \cdot \cosh X + 4 \cosh^2 \alpha \cdot \sinh X \\
& + (2n+1)^2 \cdot \sinh^2 \alpha \cdot \sinh X - 4 \sinh^2 \alpha \cdot \sinh X] \mu_1 \\
& = \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh 2\alpha \cdot e^X \\
& + (2n+1)^2 \cdot \sinh^2 \alpha \cdot e^X \\
& + 4 \cdot \sinh X \cdot (\cosh^2 \alpha - \sinh^2 \alpha)] \mu_1 \\
& = \frac{U_1}{2} [(2n+1) \cdot \sinh 2\alpha + (2n+1)^2 \cdot \sinh^2 \alpha] + 2U_1 \cdot e^{-X} \cdot \sinh X \\
& = \frac{U_1}{2} \cdot \sinh \alpha \cdot (2n+1) [2 \cosh \alpha + (2n+1) \cdot \sinh \alpha] + 2U_1 \cdot e^{-X} \cdot \sinh X
\end{aligned}$$

Setting

$$M_1 = (2n+1) \cdot \sinh \beta \cdot [(2n+1) \cdot \sinh \alpha + 2 \cosh \alpha] \\ = \frac{U_1}{2} \cdot \sinh \alpha \cdot \frac{M_1}{\sinh \beta} + 2U_1 \cdot e^{-X} \cdot \sinh X$$

$$g_2 = f_1 \cdot \sinh \beta \cdot \sinh Y - f_2 \cdot \sinh \alpha \cdot \sinh X \\ = \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh \alpha + 2 \cosh \alpha] \cdot \sinh \beta \cdot \sinh Y \\ + \frac{U_2}{2} e^Y \cdot [(2n+1) \cdot \sinh \beta + 2 \cosh \beta] \cdot \sinh \alpha \cdot \sinh X \\ M_2 = (2n+1) \cdot \sinh \alpha \cdot [(2n+1) \cdot \sinh \beta + 2 \cosh \beta] \\ = \frac{U_1}{2} e^{-X} \cdot \frac{M_1}{(2n+1)} \cdot \sinh Y + \frac{U_2}{2} e^Y \cdot \frac{M_2}{(2n+1)} \cdot \sinh X$$

Hence, the right side is modified by the process of eliminating A3 from (4) and (5)

$$g_1 \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X-Y) + g_2 \cdot (2n+1) \cdot \sinh^2 \alpha \\ = \left[\frac{U_1}{2} \sinh \alpha \cdot \frac{M_1}{\sinh \beta} + 2U_1 e^{-X} \cdot \sinh X \right] \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X-Y) \\ + \left[\frac{U_1}{2} e^{-X} \cdot \frac{M_1}{(2n+1)} \sinh Y + \frac{U_2}{2} e^Y \cdot \frac{M_2}{(2n+1)} \sinh X \right] \cdot (2n+1) \cdot \sinh^2 \alpha \\ = \frac{U_1}{2} \sinh^2 \alpha \cdot M_1 \cdot \sinh(X-Y) + 2U_1 e^{-X} \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh X \cdot \sinh(X-Y) \\ + \frac{U_1}{2} e^{-X} M_1 \cdot \sinh^2 \alpha \cdot \sinh Y + \frac{U_2}{2} e^Y M_2 \cdot \sinh X \cdot \sinh^2 \alpha \\ = \frac{U_1}{2} \sinh^2 \alpha \cdot M_1 \cdot (\sinh(X-Y) + e^{-X} \cdot \sinh Y) \\ + (\sinh \alpha \cdot \sinh X) \cdot (2U_1 e^{-X} \cdot \sinh \beta \cdot \sinh(X-Y)) \\ + (\sinh \alpha \cdot \sinh X) \cdot \left(\frac{U_2}{2} e^Y \cdot M_2 \cdot \sinh \alpha \right)$$

Since

$$\sinh(X-Y) + e^{-X} \cdot \sinh Y \\ = \frac{[e^{(X-Y)} - e^{-(X-Y)}]}{2} + \frac{e^{-X} \cdot (e^Y - e^{-Y})}{2} \\ = \frac{[e^{(X-Y)} - e^{-(X-Y)} + e^{-(X-Y)} - e^{-(X+Y)}]}{2} \\ = \frac{e^{-Y}}{2} \cdot (e^X - e^{-X}) = e^{-Y} \cdot \sinh X$$

The first term is

$$\frac{U_1}{2} \sinh^2 \alpha \cdot M_1 \cdot e^{-Y} \cdot \sinh X \\ = \frac{U_1}{2} (\sinh \alpha \cdot \sinh X) \cdot \sinh \alpha \cdot M_1 \cdot e^{-Y}$$

Summing them, the right side is $G_{11} \cdot 2 \sinh \alpha \cdot \sinh X \cdot \mu_1$ 3-1-B. For (2 3, 3) of terms relating to $[\mu_3]$

$$A_1 \cdot [(2n+1) \cosh X \cdot \cosh \alpha + 2 \sinh X \cdot \sinh \alpha] \mu_3 \\ + A_2 \cdot [(2n+1) \sinh X \cdot \cosh \alpha + 2 \cosh X \cdot \sinh \alpha] \mu_3 \\ - A_3 \cdot [(2n+1) \cosh X \cdot \sinh \alpha + 2 \sinh X \cdot \cosh \alpha] \mu_3 \\ - A_4 \cdot [(2n+1) \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha] \mu_3 = f_3 \quad (3)$$

For this, similar operations are executed as μ_1 about (2 3, 1) (2 3, 2). The eliminate A_4

$$(1) \cdot [(2n+1) \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha] \mu_3 - (3) \cdot \sinh \alpha \cdot \sinh X \\ (1) : A_1 \cdot \cosh \alpha \cdot \cosh X + A_1 \cdot \cosh \alpha \cdot \sinh X - A_3 \cdot \sinh \alpha \cdot \cosh X - A_4 \cdot \sinh \alpha \cdot \sinh X = f_1$$

$$A_1 \cdot [\cosh \alpha \cdot \cosh X \cdot [(2n+1) \cdot \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha]] \mu_3 \\ + A_2 \cdot [\cosh \alpha \cdot \sinh X \cdot [(2n+1) \cdot \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha]] \mu_3 \\ - A_3 \cdot [\sinh \alpha \cdot \cosh X \cdot [(2n+1) \cdot \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha]] \mu_3$$

$$- A_1 \cdot [(2n+1) \cosh X \cdot \cosh \alpha + 2 \sinh X \cdot \sinh \alpha] \cdot \sinh \alpha \cdot \sinh X \mu_3 \\ - A_2 \cdot [(2n+1) \sinh X \cdot \cosh \alpha + 2 \cosh X \cdot \sinh \alpha] \cdot \sinh \alpha \cdot \sinh X \mu_3 \\ + A_3 \cdot [(2n+1) \cosh X \cdot \sinh \alpha + 2 \sinh X \cdot \cosh \alpha] \cdot \sinh \alpha \cdot \sinh X \mu_3$$

The coefficients of A_1 is

$$A_1 \cdot [(2n+1) \cdot (\cosh \alpha \cdot \cosh X \cdot \sinh X \cdot \sinh \alpha - \cosh X \cdot \cosh \alpha \cdot \sinh \alpha \cdot \sinh X) \\ + 2 \cdot (\cosh \alpha \cdot \cosh X \cdot \sinh X \cdot \sinh \alpha - \cosh X \cdot \cosh \alpha \cdot \sinh \alpha \cdot \sinh X)] \\ = A_1 \cdot 2 \cdot (\cosh^2 \alpha \cdot \cosh^2 X - \sinh^2 \alpha \cdot \sinh^2 X) \\ = 2 \cdot A_1 \cdot [(1 + \sinh^2 \alpha) \cdot \cosh^2 X - \sinh^2 \alpha \cdot \sinh^2 X] \\ = 2 \cdot A_1 \cdot [\cosh^2 X + \sinh^2 \alpha \cdot (\cosh^2 X - \sinh^2 X)] \\ = 2 \cdot A_1 \cdot [\cosh^2 X + \sinh^2 \alpha]$$

The coefficients of A_2 is

$$A_2 \cdot [(2n+1) \cdot (\cosh \alpha \cdot \sinh X \cdot \sinh X \cdot \sinh \alpha - \sinh X \cdot \cosh \alpha \cdot \sinh \alpha \cdot \sinh X) \\ + \cosh \alpha \cdot \sinh X \cdot 2 \cdot \cosh X \cdot \cosh \alpha - 2 \cosh X \cdot \sinh \alpha \cdot \sinh \alpha \cdot \sinh X] \\ = A_2 \cdot [2 \sinh X \cdot \cosh X \cdot (\cosh^2 \alpha - \sinh^2 \alpha)] \\ = A_2 \cdot \sinh 2X$$

The coefficients of A_3 is

$$A_3 \cdot [(2n+1) \cdot (\cosh X \cdot \sinh \alpha \cdot \sinh \alpha \cdot \sinh X - \sinh \alpha \cdot \cosh X \cdot \sinh X \cdot \sinh \alpha) \\ + 2 \cdot (\sinh X \cdot \cosh \alpha \cdot \sinh \alpha \cdot \sinh X - \sinh \alpha \cdot \cosh X \cdot \cosh X \cdot \cosh \alpha)] \\ = A_3 \cdot 2 \cdot \sinh \alpha \cdot \cosh \alpha \cdot (\sinh^2 X - \cosh^2 X) \\ = -A_3 \cdot \sinh 2\alpha$$

Therefore (3) is

$$A_1 \cdot 2 \cdot (\cosh^2 X + \sinh^2 \alpha) + A_2 \cdot \sinh 2X - A_3 \cdot \sinh 2\alpha = (4)$$

On the other hand, by

$$(1) \sinh \beta \cdot \sinh Y - (2) \cdot \sinh \alpha \cdot \sinh X$$

$$A_1 \cdot [\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y] \\ - A_2 \cdot \sinh X \cdot \sinh Y \cdot \sinh(\alpha - \beta) \\ + A_3 \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X-Y) = f_1 \cdot \sinh \beta \cdot \sinh Y - f_2 \cdot \sinh \alpha \cdot \sinh X = g_2 \quad (5)$$

To eliminate A3 from (4), (5)

$$(4) \sinh \alpha \cdot \sinh \beta \cdot \sinh(X-Y) + (5) \cdot \sinh 2\alpha$$

The coefficients of A_1

$$2 \cdot (\cosh^2 X - \sinh^2 \alpha) \cdot [\sinh \alpha \cdot \sinh \beta \cdot \sinh(X-Y)] \\ + [\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y] \cdot \sinh 2\alpha \\ = 2 \cdot \sinh \alpha \cdot [(\cosh^2 X + \sinh^2 \alpha) \cdot \sinh \beta \cdot \sinh(X-Y) \\ + (\cosh \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh \alpha \cdot \cosh \beta \cdot \sinh X \cdot \cosh Y) \cdot \cosh \alpha] \\ = 2 \cdot \sinh \alpha \cdot [\\ \cosh^2 X \cdot \sinh \beta \cdot \sinh X \cdot \cosh Y \quad (1) \\ + \sinh^2 \alpha \cdot \sinh \beta \cdot \sinh X \cdot \cosh Y \quad (2) \\ - \cosh^2 X \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \quad (3) \\ - \sinh^2 \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \quad (4) \\ + \cosh^2 \alpha \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \quad (5) \\ - \sinh \alpha \cdot \cosh \beta \cdot \cosh \alpha \cdot \sinh X \cdot \cosh Y] \quad (6)$$

Summing 4th and 5th terms

$$: \sinh \beta \cdot \cosh X \cdot \sinh Y \cdot (\cosh^2 \alpha - \sinh^2 \alpha) \\ = \sinh \beta \cdot \cosh X \cdot \sinh Y$$

Associating the 1, 2 and 6 terms

$$\begin{aligned} & \sinh X \cdot \cosh Y \cdot (\cosh^2 X \cdot \sinh \beta + \sinh^2 \alpha \cdot \sinh \beta - \sinh \alpha \cdot \cosh \beta \cdot \cosh \alpha) \\ &= \sinh X \cdot \cosh Y \cdot [(1 + \sinh^2 X) \sinh \beta + \sinh^2 \alpha \cdot \sinh \beta - \sinh \alpha \cdot \cosh \beta \cdot \cosh \alpha] \\ &= \sinh X \cdot \cosh Y \cdot \sinh \beta + \sinh X \cdot \cosh Y \cdot \sinh^2 X \cdot \sinh \beta \\ &+ \sinh X \cdot \cosh Y \cdot \sinh^2 \alpha \cdot \sinh \beta - \sinh X \cdot \cosh Y \cdot \sinh \alpha \cdot \cosh \beta \cdot \cosh \alpha \end{aligned}$$

The 3rd term can be

$$\begin{aligned} & -\cosh^2 X \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \\ &= -(1 + \sinh^2 X) \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \\ &= -\sinh \beta \cdot \cosh X \cdot \sinh Y - \sinh^2 X \cdot \sinh \beta \cdot \cosh X \cdot \sinh Y \end{aligned}$$

Sum with this and (1) + (2) + (3)

$$\begin{aligned} &= \sinh \beta \cdot (\sinh X \cdot \cosh Y - \cosh X \cdot \sinh Y) \\ &+ \sinh^2 X \cdot \sinh \beta \cdot (\sinh X \cdot \cosh Y - \cosh X \cdot \sinh Y) \\ &+ \sinh X \cdot \cosh Y \cdot \sinh \alpha \cdot (\sinh \alpha \cdot \sinh \beta - \cosh \beta \cdot \cosh \alpha) \\ &= \sinh \beta \cdot \sinh(X - Y) + \sinh^2 X \cdot \sinh \beta \cdot \sinh(X - Y) \\ &- \sinh X \cdot \cosh Y \cdot \sinh \alpha \cdot \cosh(\alpha - \beta) \end{aligned}$$

Add (4) and (5) to above

$$\begin{aligned} & \sinh \beta \cdot \cosh X \cdot \sinh Y + \sinh \beta \cdot [\sinh X \cdot \cosh Y - \cosh X \cdot \sinh Y] \\ &+ \sinh^2 X \cdot \sinh \beta \cdot \sinh(X - Y) - \sinh X \cdot \cosh Y \cdot \sinh \alpha \cdot \cosh(\alpha - \beta) \\ &= \sinh X \cdot [\sinh \beta \cdot \cosh Y + \sinh X \cdot \sinh \beta \cdot \sinh(X - Y) \\ &\quad - \cosh Y \cdot \sinh \alpha \cdot \cosh(\alpha - \beta)] \\ &= \sinh X \cdot [\cosh Y \cdot (\sinh \beta - \sinh \alpha \cdot \cosh(\alpha - \beta)) \\ &\quad - \sinh X \cdot \sinh \beta \cdot \sinh(X - Y)] \end{aligned}$$

Since

$$\begin{aligned} & \sinh \beta - \sinh \alpha \cdot \cosh(\alpha - \beta) \\ &= \sinh \beta - \sinh \alpha \cdot [\cosh \alpha \cdot \cosh \beta - \sinh \alpha \cdot \sinh \beta] \\ &= \sinh \beta - \sinh \alpha \cdot \cosh \alpha \cdot \cosh \beta + \sinh^2 \alpha \cdot \sinh \beta \\ &= \sinh \beta \cdot (1 + \sinh^2 \alpha) - \sinh \alpha \cdot \cosh \alpha \cdot \cosh \beta \\ &= \sinh \beta \cdot \cosh^2 \alpha - \sinh \alpha \cdot \cosh \alpha \cdot \cosh \beta \\ &= \cosh \alpha \cdot (\cosh \alpha \cdot \sinh \beta - \sinh \alpha \cdot \cosh \beta) \\ &= \cosh \alpha \cdot (-1) \cdot \sinh(\alpha - \beta) \end{aligned}$$

Hence

$$\begin{aligned} &= \sinh X \cdot [-\cosh Y \cdot \cosh \alpha \cdot \sinh(\alpha - \beta) \\ &\quad + \sinh X \cdot \sinh \beta \cdot \sinh(X - Y)] \end{aligned}$$

Thus, the coefficient of A_1 is

$$= F_{21} \cdot \sinh X \cdot \mu_3$$

The coefficients of A_2

$$\begin{aligned} & \sinh 2X \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\ &- \sinh X \cdot \sinh Y \cdot \sinh(\alpha - \beta) \cdot \sinh 2\alpha \\ &= 2 \cdot \sinh X \cdot \sinh \alpha \cdot [\cosh X \cdot \sinh \beta \cdot \sinh(X - Y) - \sinh Y \cdot \sinh(\alpha - \beta) \cdot \cosh \alpha] \\ &= 2 \cdot \sinh X \cdot \sinh \alpha \cdot F_{41} \cdot \mu_3 \end{aligned}$$

5. Calculation of the right side in terms of (23.3) $[\mu_3]$

$$\begin{aligned} g_1^* &= f_1 \cdot [(2n+1) \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha] \mu_3 - f_1 \sinh \alpha \cdot \sinh X \\ &= \frac{U_1}{2} e^{-X} \cdot [(2n+1) \sinh \alpha + 2 \cosh \alpha] [(2n+1) \sinh X \cdot \sinh \alpha + 2 \cosh X \cdot \cosh \alpha] \\ &- \frac{U_1}{2} \cdot e^{-X} (2n+1) \cdot (2n+3) \mu_3 \cdot \sinh \alpha \cdot e^{-X} \cdot \sinh \alpha \cdot \sinh X \\ &= \frac{U_1}{2} e^{-X} \cdot [(2n+1)^2 \sinh \alpha \cdot \sinh X \cdot \sinh \alpha + (2n+1) \sinh X \cdot \sinh 2\alpha \\ &\quad + 4 \cosh X \cdot \cosh^2 \alpha + (2n+1) \cdot \cosh X \cdot \sinh 2\alpha \\ &\quad - (2n+1-2)(2n+1+2) \sinh^2 \alpha \cdot \sinh X] \mu_3 \\ &= \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh 2 \cdot (\cosh X + \sinh X) \\ &\quad + 4 \cosh X \cdot \cosh^2 \alpha + 4 \sinh^2 \alpha \cdot \sinh X] \mu_3 \end{aligned}$$

$$\begin{aligned} &= \frac{U_1}{2} e^{-X} \cdot [(2n+1) \cdot \sinh 2\alpha \cdot e^X \\ &\quad + 4 \cosh X \cdot \cosh^2 \alpha + 4 \cdot (\cosh^2 \alpha - 1) \cdot \sinh X] \mu_3 \\ &= \frac{U_1}{2} \cdot [(2n+1) \cdot \sinh 2\alpha \\ &\quad + e^{-X} [4 \cosh^2 \alpha \cdot (\cosh X + \sinh X) - 4 \cdot \sinh X]] \mu_3 \\ &= \frac{U_1}{2} \cdot [(2n+1) \cdot \sinh 2\alpha + 4 \cdot \cosh^2 \alpha - 4 \cdot e^{-X} \cdot \sinh X] \mu_3 \\ &= [(2n+1) \cdot \sinh \alpha + 2 \cosh \alpha] \cdot 2 \cosh \alpha - 4 e^{-X} \cdot \sinh X \mu_3 \end{aligned}$$

$$g_1^* = \left\{ \frac{M_1}{(2n+1)} \cdot \frac{2 \cdot \cosh \alpha}{\sinh \beta} - 4 \cdot e^{-X} \cdot \sinh X \right\} \mu_3 \cdot \frac{U_1}{2}$$

About g_2^*

$$\begin{aligned} g_2^* &= f_1 \cdot \sinh \beta \cdot \sinh Y - f_2 \cdot \sinh \alpha \cdot \sinh X \\ &= \frac{U_1}{2} e^{-X} \cdot \frac{M_1}{(2n+1)} \sinh Y + \frac{U_2}{2} e^Y \cdot \frac{M_2}{(2n+1)} \sinh X \end{aligned}$$

To eliminate A_3 by computing g_1^* and g_2^*

$$\begin{aligned} g_1^* \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) + g_2^* \cdot \sinh 2\alpha \\ &= \left[\frac{M_1 \cdot \cosh \alpha}{(2n+1) \cdot \sinh \beta} - 2e^{-X} \cdot \sinh X \right] U_1 \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) \\ &+ \left[\frac{U_1}{2} e^{-X} \cdot \frac{M_1}{(2n+1)} \sinh Y + \frac{U_2}{2} e^Y \cdot \frac{M_2}{(2n+1)} \sinh X \right] \cdot \sinh 2\alpha \\ &= U_1 \cdot \left[\frac{M_1 \cdot \cosh \alpha}{(2n+1)} - 2e^{-X} \cdot \sinh X \cdot \sinh \beta \right] \sinh \alpha \cdot \sinh(X - Y) \\ &+ U_1 \cdot e^{-X} \cdot \frac{M_1}{(2n+1)} \cdot \sinh Y \cdot \cosh \alpha \cdot \sinh \alpha \\ &+ U_2 \cdot e^Y \cdot \frac{M_2}{(2n+1)} \cdot \sinh X \cdot \sinh \alpha \cdot \cosh \alpha \end{aligned}$$

Rearrange with respect to U_1

$$\begin{aligned} & \frac{M_1 \cdot \cosh \alpha}{(2n+1)} \cdot \sinh \alpha \cdot \sinh(X - Y) - 2e^{-X} \sinh X \cdot \sinh \beta \cdot \sinh \alpha \cdot \sinh(X - Y) \\ &+ e^{-X} \cdot \frac{M_1}{(2n+1)} \cdot \sinh Y \cdot \cosh \alpha \cdot \sinh \alpha \\ &= \frac{M_1 \cdot \cosh \alpha \cdot \sinh \alpha}{(2n+1)} \cdot \left[\frac{e^{(X-Y)} - e^{-(X-Y)}}{2} + e^{-X} \cdot \sinh Y \right] \\ &- 2e^{-X} \cdot \sinh \beta \cdot \sinh(X - Y) \cdot [\sinh X \cdot \sinh \alpha] \\ &= \frac{M_1 \cdot \cosh \alpha \cdot \sinh \alpha}{(2n+1)} \cdot \left[\frac{e^{(X-Y)} - e^{-(X-Y)}}{2} + \frac{e^{-X} \cdot (e^Y - e^{-Y})}{2} \right] \\ &- 2e^{-X} \cdot \sinh \beta \cdot \sinh(X - Y) \cdot [\sinh X \cdot \sinh \alpha] \\ &= \frac{M_1 \cdot \cosh \alpha \cdot \sinh \alpha}{(2n+1)} \cdot \frac{1}{2} [e^{X-Y} - e^{-X-Y}] \\ &- 2e^{-X} \cdot \sinh \beta \cdot \sinh(X - Y) \cdot [\sinh X \cdot \sinh \alpha] \end{aligned}$$

$$\begin{aligned} &= \frac{M_1 \cdot \cosh \alpha \cdot \sinh \alpha}{(2n+1)} \cdot e^{-Y} \cdot \sinh X - 2e^{-X} \cdot \sinh \beta \cdot \sinh(X - Y) \cdot [\sinh X \cdot \sinh \alpha] \\ &= \sinh \alpha \cdot \sinh X \cdot \left[\frac{M_1 \cdot \cosh \alpha \cdot e^{-Y}}{2n+1} - 2e^{-X} \cdot \sinh \beta \cdot \sinh(X - Y) \right] \end{aligned}$$

Hence

$$\begin{aligned} g_1^* \cdot \sinh \alpha \cdot \sinh \beta \cdot \sinh(X - Y) + g_2^* \cdot \sinh 2\alpha \\ &= \sinh \alpha \cdot \sinh X \cdot \left[U_2 \cdot \frac{e^Y \cdot M_2 \cdot \cosh \alpha}{(2n+1)} \right. \\ &\quad \left. + U_1 \cdot \left\{ \frac{M_1 \cdot \cosh \alpha \cdot e^{-Y}}{2n+1} - 2e^{-X} \sinh \beta \cdot \sinh(X - Y) \right\} \right] \mu_3 \\ &= \sinh X \cdot \sinh \alpha \cdot G_{12} \cdot \mu_3 \end{aligned}$$