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Method for Predicting the Constitutional Changes around the Anastomotic Junction.

------Simulation Analysis ---

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We introduce a simulation method for evaluating the arterial wall displacement in the radial direction and the bending stress as functions of the distance from the anastomosis/ Neglecting the longitudinal vascular displacement , the equation of radial motion for an elastic tube was derived by a dynamical constitutional consideration of the cylindrical tube. Since the magnitude of wall shear stress (the bending stress) is in proportional to the third derivative of the wall displacement in the radial direction, the bending stress can be obtained in an explicit form. We analyzed the spatial distribution characters of the radial displacement of the arterial wall and the bending stress. We have discussed the availability of the present simulation method for constitutional changes around the anastomotic region of arterial system by comparing the reported biological surgical operation.

Vessel anastomosis. Constitutional change. Wall displacement in radial direction. Simulation.

血管吻合部周囲のずり応力解析法

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動脈硬化における血管吻合手術に際して吻合部周囲でどのような構造力学的変化が起こるかを予測するための理論研究をシミュレーションモデルを用いて解析した。　まず長軸方向の血管壁偏位を無視して、弾性管の半径方向の運動方程式を求めた。　つぎに　血管壁のずり応力が半径方向偏位の３次空間微分に比例することから曲げ応力をもとめる方程式を導出した。境界条件として吻合部とそこから無限遠の位置での半径方向偏位およびその一次微分を設定した。生理的な血管壁力学係数を入力して半径方向の血管壁偏位および曲げ応力が吻合部から隔たるにつれてどの様に分布するかを構造力学方程式を解析的に解くことで求めた。　動物実験で報告されている結果と対比することで　本理論的シミュレーション解析の有用性を検討した。

血管吻合手術．構造力学的変化．半径方向血管壁偏位．シミュレーション解析

1. Introduction.

Arterial graft is one of the most important medical treatment for vascular diseases such as arteriosclerosis. The important issues in surgical treatment are, 1) the impedance matching between the graft and recipient and 2) evaluation mechanical changes in the anastomotic region. For the first point, we have given optimal matching conditions for substituting the graft artery. For the second issue, we introduce a simulation technique proposed by Rodgers to predict the shear stress and radial wall displacement at the anastomotic region.

Fig 1-a shows a shell which middle surface is a cylinder as a geometric model of deformed arterial surface. The membrane force acting on the four edges must lie in tangential planes to the middle surface. They are resolved to normal and shear components as shown. The forces per unit length of section are N_x, N_ϕ and $N_{x\phi}$, $N_{\phi x}$. The loads per unit area of the shell element ha components p_x and p_ϕ in the direction of increasing x and angle. The radial component p_r is positive in out ward.

Fig 1-b is referred from Kuchar and Ostrach 1966 for the simple description of the forces on the membrane.

Fig 1-a

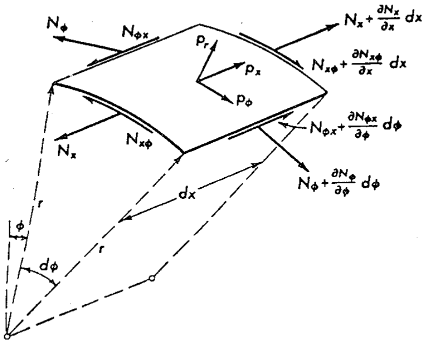


Fig 1-b

[Adapted from Kuchar and Ostrach (1966)]

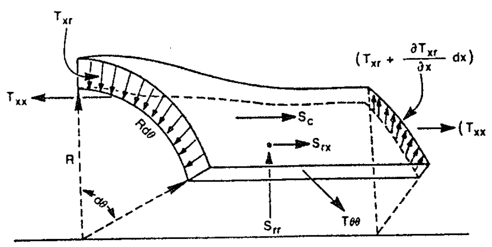
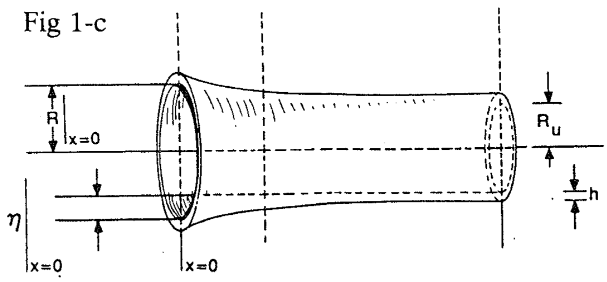


Fig 1-c



Definition of anastomotic junction and host artery. This picture shows measurable quantities of radius, wall thickness and radial wall displacement. The longitudinal position x=0 is the anastomotic junction.

2. Modeling and solution.

The model of the vessel wall proposed by Kuchar and Ostrach incorporates radial stress Srr and axial stresses Srx

$$Srr = P - 2\mu\ \partial v / \partial r \,(r = Ru) \qquad \text{------(1)*}$$

$$Srx = -\mu\ (\partial u / \partial r + \partial v / \partial x)\,(r = Ru) \qquad \text{------(2)*}$$

where
- P is transmural pressure,
- u is the longitudinal velocity of blood flow
- v is radial velocity
- μ is dynamic viscosity
- Ru is the undeformed inner radius of the vessel.

The equations for the radial displacement η and longitudinal motion, γ of the vessel wall are

$$\rho_\omega h\ \partial^2 \eta / \partial t^2 = Srr\,(1 - h/Ru) - h/Ru\, T_{\theta\theta} + h\ \partial Txr / \partial x \qquad \text{------(3)*}$$

$$\rho_\omega h\ \partial^2 \gamma / \partial t^2 = Srx\,(1 + h/Ru)\, Sc + h\ \partial Txx / \partial x \qquad \text{------(4)*}$$

where
- ρ_ω is the density of the vessel wall,
- h is the wall thickness
- t is the parameter time and
- Sc is the stress component along the axial direction of the vessel wall.
- η is the difference between the inner radius at any longitudinal position R(x) and Ru.

The stresses associated with this model are

$$T_{\theta\theta} = E/(1-\sigma^2)(\varepsilon_{\theta\theta} + \sigma\,\varepsilon_{xx}) + E h^2 / [12\,Ru\,(1-\sigma^2)]]\ k\theta \qquad \text{---(5)}$$

$$Txx = E/(1-\sigma^2)(\varepsilon_{xx} + \sigma\,\varepsilon_{\theta\theta}) - E h^2 / [12\,Ru\,(1-\sigma^2)]]\ kx \qquad \text{---(6)}$$

$$Txr = -E/[12(1-\sigma^2)] * \partial[kx + \sigma k\theta - (\varepsilon_{xx} + \sigma\,\varepsilon_{\theta\theta})/Ru]/\partial x \qquad \text{--(7)}$$

where
- E is the elastic modulus of the wall and
- σ is the Poisson ratio which was assumed to 0.5

The strain and curvature changes related to wall displacement by

$$\varepsilon_{\theta\theta} = \eta / Ru \qquad \varepsilon_{xx} = \partial \gamma / \partial x \qquad \text{---(8)}$$

$$k\theta = \eta / Ru^2 \qquad kx = \partial^2 \eta / \partial x^2 \qquad \text{-----(9)}$$

Assuming that $\gamma = 0$ where the vessel is constrained from moving longitudinally and substituting the expressions for Srr, $T\theta\theta$, Txr, E and k into the equation (3), we have the wall motion equation for an elastic vessel which has been proposed by Kuchar and Ostrach ,

$$\rho_\omega h \ \partial^2 \eta / \partial t^2 = P(1 - h/Ru) - E h \ \eta / [(1-\sigma^2) Ru^2)] - E h^3 / [12 Ru (1-\sigma^2)] \ (\eta / Ru^4 + \partial^4 \eta / \partial x^4) \quad \text{----(10)}$$

when the same substitutions have been applied to the wall shear stress Txr (the equation 7),
the expression is

$$Txr = -E h^2 / [12 (1-\sigma^2)] \ \partial^3 \eta / \partial x^3 \quad \text{-----(11)}$$

The equation (10) can be simplified by performing an order-of magnitude analysis to eliminate those terms which contributes little to the radial displacement of the vessel.

The geometric coordinates (x,r), velocity variables (u,v), time (t), and radial displacement parameter (η) are converted to dimensionless form. Such dimensionless forms were substituted into the fluid continuity equation.

The boundary condition is

$$v(r = Ru) = \partial \eta / \partial t (r = Ru) \quad \text{------(12)}$$

The equation (10) can be transformed into the dimensionless form

$$\rho_\omega (1-\sigma^2) Ru^2 / (E To^2) \ \partial^2 \eta / \partial t^2 = \{ [\Delta P \ Ru \ L (1-\sigma^2) / (To \ E h \ U] P^* - 2 [\mu \ Ru (1-\sigma^2) / (E h \ To) \ \partial v / \partial r](r{=}1) \} * (1 - h/Ru) - \eta - h^2 Ru^4 / (12 Ru^2 L^4) \ \partial^4 \eta / \partial x^4 - (h^2 / Ru^2) \ \eta \quad \text{-----(13)}$$

where the * designates a dimensionless parameter and L, To, U and ΔP are constants which relate dimensional and dimensionless variable

$$x = L x^*, \quad t = To \ t^*$$
$$u = U u^* \quad \text{and} \quad P = (\Delta P) \ P^* \quad \text{------(14)}$$

The relative magnitude for each factor in the equation (13) is estimated by inputting literature values for Ru, h, E, To, ρ_ω, L, ΔP.

Neglecting the terms which contribute little to the radial displacement of the wall vessel, we have the following equation.

$$E h^3 / [12 (1-\sigma^2)] \ \partial^4 \eta / \partial x^4 + E h / (1-\sigma^2) \ \eta / Ru^3 = P(1 - h/Ru) \quad \text{---(15)}$$

The solution of the equation (15) requires that the time varying parameters η , P , h and E be prescribed.

The values for h and E may be considered constant and set equal to their mean value over the cardiac cycle. The parameter η and P are time periodic and can be expressed by a Fourier series

$$\eta = \mathrm{Re}\{ \xi \exp(i\omega t) \}$$
$$P = \mathrm{Re}\{ Pc \exp(i\omega t) \} \quad \text{---- (16)}$$

where ξ and Pc are complex constants, ω is the frequency of the pulsation and Re{} denoted the real part of the complex number. Substituting these expressions into equation (15), we have

$$E h^3 / [12 (1-\sigma^2)] \exp(i\omega t) \ \partial^4 \xi / \partial x^4 + E h / (1-\sigma^2) \exp(i\omega t) \ \xi / Ru^3 = Pc \exp(i\omega t) \ (r = Ru) \quad \text{---(17)}$$

where the factor (1- h / Ru) has been absorbed in the constant Pc. This can be further simplified into

$$\partial^4 \xi / \partial x^4 + 12 \ \xi / [h^3 Ru^2] = 12 \ Pc (1-\sigma^2) / E h^3 \quad \text{---(18)}$$

The appropriate boundary conditions are

η (x ---> ∞) is bounded
$\partial \eta / \partial x$ (x -- > oo) is bounded
η (x ---> 0) is measured value and
$\partial \eta / \partial x$ (x -- > 0) is approximated numerically.--(19)

The solution of (18) under the present form of the boundary conditions are

$$\eta(x) = (\eta o - \eta oo) \exp(-C x) \{ \cos(Cx) + \sin(Cx) \} + (D / C) \sin(C x) \exp(-C x) + \eta oo \quad \text{-----(20)}$$

where
η o is the radial displacement functions evaluated at x =0, the anastomotic junction. η oo is the radial displacement sufficiently proximal or distal to the anastomosis.
Substituting the equation (20) into Txr results

$$Txr(x) = -E h^2 / [12 (1-\sigma^2)] \ [4 C^3 (\eta o - \eta oo) \exp(-C x) \cos(Cx) + 2 C^2 D \exp(-C x) \{ \cos(Cx) + \sin(Cx) \} \quad \text{---- (21)}$$

Equations (20) and (21) provide the theoretical framework for analyzing the experimental data. In the investigation reported by Rodgers et in 1987, they set either the theoretical elastic tube model prediction for C and D,

$$C = 3^{1/4} (h \ Ru)^{(-1/2)} \text{ and}$$

$$D = \partial \eta / \partial x (x \to 0)$$

to calculate η (x) and Txr(x). Another method to determine C and D are regression analysis in which C and D are estimated by numerically regressing the experimental contours against equation (20). The resultant values for C and D are input into the equation (21) for calculating Txr(x).

The apparatus proposed by Rodgers (1986) is mainly mechano hemo dynamical one. They are consisted of the natural hemo dynamic environment of tethered host artery-graft combination are simulated in vitro. They measured outer radius of a pulsatiling vessel at multiple longitudinal pulsating inside radius R(x) and wall thickness h. The inside radius and wall thickness of a relaxed non perfused vessel segment are measured following the completion of each perfusion experiment.

The data of resting geometry related to the measurement of the outer radius of the tethered vessel by applying the conservation law and constant density property of the wall.

To evaluate the elastic property of the wall, the dynamic incremental elastic modulus E as Bergel (1961) has introduced. E was computed by measuring the maximum and minimum transmural pressure and wall distension difference. Moreover, E was independent from the longitudinal position.

Fig 2-a

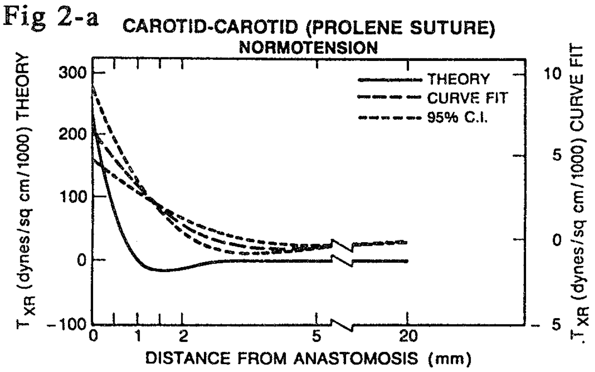
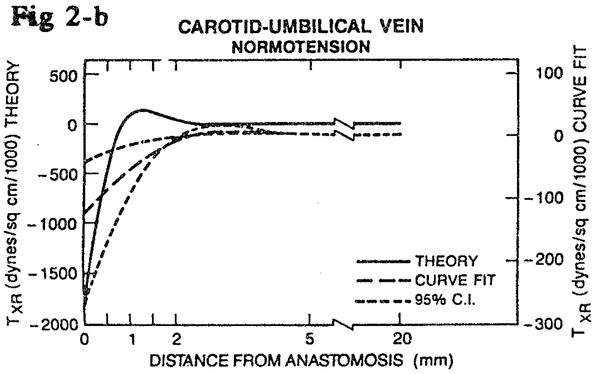


Fig 2-b



3.Results.

Fig 2 compares the elastic model prediction for Txr (solid curves) versus the values derived from the numerical regression (dashed curves). The95% confidence intervals of the curve-fit predictions for Txr are included (short dashed curves). Shear stress by the theoretical elastic model have over estimated than the numerical curve fit to the experimental data.

DISCUSSION

In Rodgers paper, as an alternative approach, a curve fitting or regression model has been used. It is advantageous to use the elastic model and regress on the parameters C and D in equation (20). The parameters C and D have to be adjusted to fit the experimental data. The regression technique affords effective comparison between the measured and regressed parameters C and D.

In the Rodgers boundary conditions, η(x=0) was set equal to the measured radius immediately adjacent to the anastomosis. Ro minus Ru. As such the true boundary value is reflected in the displacement function. Moreover Rodgers boundary condition approximates the first derivative using a two point divided difference between η (x=0) and immediately adjacent experimental longitudinal data point. These two rigorous boundary condition afford realistic application of the elastic tube model to the anastomotic junction.

Both the end points of the experimental displacement curve η(x=0) and η(x=oo) are selected based on experimental criterion. η(x=0) must be identically correspond to the longitudinal position most immediately adjacent to the vessel surface interface.

We have introduced Rodgers simulation method to predict mechanical changes around the anastomotic region. Their method, however still lack rigorous mechanical consideration such as tethering effects from the surrounding tissue, initial wall tension, viscous retardation force from the blood flow. Incorporating such properties will improve the prediction capability of their method. In the APPENDIX, we give a rigorous mathematical treatment for analyzing the wall displacement, shear stress and moments on the arterial wall.

5. Conclusion.

We introduced a simulation method proposed by Rodgers (1987) to predict radial wall displacement and shear stress around the anastomotic junctional region. By incorporating more rigorous biomechanical properties of arterial wall, their method will available for evaluating the deformation of arterial wall by graft operation.

6. References.

1. Dodgers V.G., Teodoti MF and Boorovetz. H S. Experimental determination of mechanical shear stress about an anastomotic junction. J. Biomechanics. vol 20. pp 795-803. 1987.
2. Kuchar N R and Ostrach. S. Flow in the entrance region of circular elastic tube. Biomedical fluid mechanics Symposium. N.Y. pp 45-69. 1966.
3. Bergel D H. J. Physiology. vol 156. pp 445-457. 1961.

APPENDIX

1] Axial displacement Ua and U.

Fig 1 shows the axial displacement selectively. Fig 2 is the enlarged picture of Fig 1 from holizontal view.
∠AAoD=∠A'Ao'B'=90',and assumption 2
∠AoDB= ∠C'Ao'C=180'. So ∠Ao'A'C' = ∠B'Ao'C.

The Ao'B' changes dw/dx per unit length of the x axis against Ao'C. Then
Ua = U -C'A'o =U-A'Ao' sin (∠Ao'A'C')
= U-z sin (∠B'Ao'C) =U-z sin(dw/dx).
Since dw/dx is small, sin(dw/dx)=dw/dx. So
Ua =U-z dw/dx. (1)

2] Circumferential displacement Va, V.

Fig 3 illustrate the circumferential displacement. Fig 4 is the magnified picture of Fig 3 from holizontal view.. On the circle of radius R, $2\pi R : V = 2\pi : \phi$. Then ϕ = v/R.
As Va=AA"-NA".
Firstly we calculate AA" then NA"= NP + PA"
Since point A is on the circle whose radius is R + z, then AA" = (R+z) ϕ = (R+z) v/R. (2)- a

Now we seek NA". According to the assumption 1, ∠AAoE=∠A'Ao'E'=90'. Since
R+z >> Wa and 1 >> v/R= ϕ,

one can regard JA"// A'N.
Then

$$90' = \angle A'JA'' = \angle JA''N = \angle PNA' = \angle NA'J. \quad (2)\text{-b}$$

$dw/d\phi$ is the ratio of change in W around the central angle ϕ. Then

$$\angle E'Ao'F = 1/R\ dw/d\phi \qquad (2)\text{-c}$$

W can be regarded constant between F'F and GH. So F'Ao'F//GEH. Then

$$\angle JAo'F = \angle Ao'EH = 90'. \text{ So}$$

$$\angle PA'N = \angle A'Ao'A'' = \angle E'Ao'F = 1/R\ dw/d\phi.$$

As PA'N is a right angled triangle and NA' = Wa, then $\tan(\angle PA'N) = NP/NA'$. So

$$NP = Wa \tan(\angle PA'N) = Wa\,(1/R\ dw/d\phi). \quad (2)\text{-d}$$

Since A'Ao' = z, and assume z does not change around circumference,

$$PA'' = PAo' \sin(\angle PAo'A'')$$
$$= (A'Ao' - PA') \sin(\angle A'Ao'A'')$$
$$= (z - PA') \sin(1/R\ dw/d\phi).$$

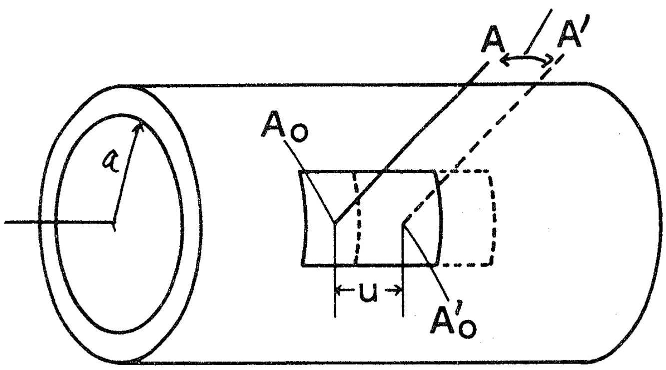
As $\cos(\angle PA'N) = NA'/PA'$. Then

$$PA' = Wa/\cos(1/R\ dw/d\phi). \qquad (2)\text{-e}$$

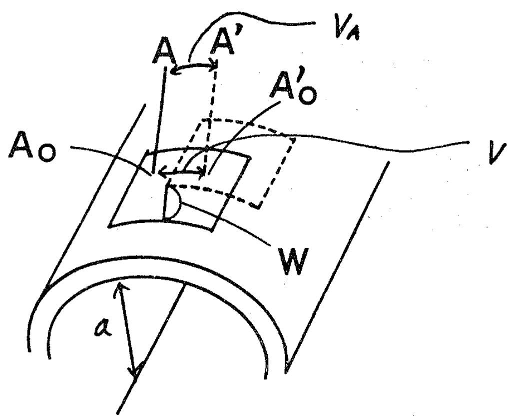
So

$$PA'' = (z - Wa/\cos(1/R\ dw/d\phi)) \sin(1/R\ dw/d\phi).$$

Axial displacement



Circumferential displacement



Therefore

$$NA'' = NP + PA''$$
$$= Wa \tan(1/R\ dw/d\phi) + z \sin(1/R\ dw/d\phi)$$
$$- Wa \tan(1/R\ dw/d\phi)$$
$$= z \sin(1/R\ dw/d\phi).$$

Because $dw/d\phi$ is small, $\sin(1/R\ dw/d\phi) = 1/R\ dw/d\phi$.
Then $Va = (R+z)\,v/R - z/R\ dw/d\phi$. (2)-f

II] The stress- strain-displacement relations

The axial εx, circumferential $\varepsilon \phi$, and shearing $\varepsilon x \phi$, strains are expressed by the displacements (Ua,Va,Wa) of the point Ao on normal.

$$\varepsilon x = \frac{\partial Ua}{\partial x} \quad (3) \qquad \varepsilon \phi = \frac{1}{(R+z)}\left(\frac{\partial Va}{\partial \phi} + Wa\right) \quad (4)$$

$$\gamma x\phi = \frac{1}{(R+Z)}\frac{\partial Ua}{\partial \phi} + \frac{\partial Va}{\partial x} \quad (5)$$

Inputting the equ(1),(2) into the equ (3) (4) (5) and putting Wa=W,the strains were expanded by the displacements of the point Ao on the inner surface of the arterial wall.

$$\varepsilon x = \frac{U'}{R} - Z\frac{W''}{R^2} \quad (6) \qquad \varepsilon \phi = \frac{V^{\circ}}{R} - Z\frac{\ddot{W}}{R(R+Z)} + \frac{W}{(R+Z)} \quad (7)$$

$$\gamma x\phi = \frac{U^{\circ}}{R+z} + \frac{(R+Z)V'}{R^2} - \frac{W'^{\circ}}{R}\left(\frac{Z}{R} + \frac{Z}{(R+Z)}\right) \quad (8)$$

(Here we use the differential operaters as followings.

$$R\frac{\partial f}{\partial x} = f', \qquad \frac{\partial g}{\partial \phi} = g^{\circ} \qquad \frac{\partial^2 g}{\partial \phi^2} = \ddot{g}$$

On the other hand, the stress-strain relations in a given compartment of the arterial segments were expressed as

$$\sigma x = E\ (\varepsilon x + \varepsilon \phi)/(1-\sigma^2) \quad \text{axial stress} \quad (9)$$
$$\sigma \phi = E\ (\varepsilon \phi + \varepsilon x)/(1-\sigma^2) \quad \text{circumferential stress} \quad (10)$$
$$\tau x\phi = E\ \gamma x\phi / 2(1+\sigma) \quad \text{shearing stress} \quad (11)$$

Where E is the young' elastic modulus, σ is the Poisson ratio.

III] The expression of the forces and moments by using the displacement (U,V,W) of the point on the inner surface of the arterial wall.

1] Nx (the axial resultant force)

we set the standard point of the integration at point A on the normal.Setting the thickness of the arterial wall h, then $-h/2 < z < h/2$. As a result, the stress resultant is revealed by following integration

$$\int_{-h/2}^{h/2} \sigma x\,(R+Z)\,dz\,d\phi$$

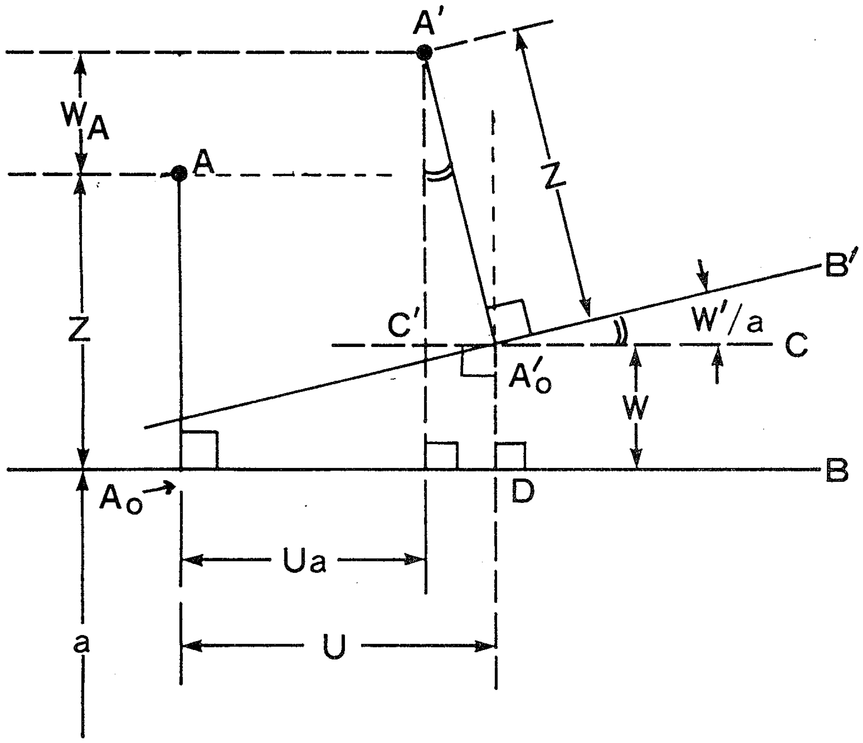
This is what the force operating longitudinally along the x axis,

$$Nx\,R\,d\phi, \text{ Therefore } \int Nx^*(R\,d\phi) = \int^{h/2} \sigma x\,(R+Z)\,dz\,d\phi,$$

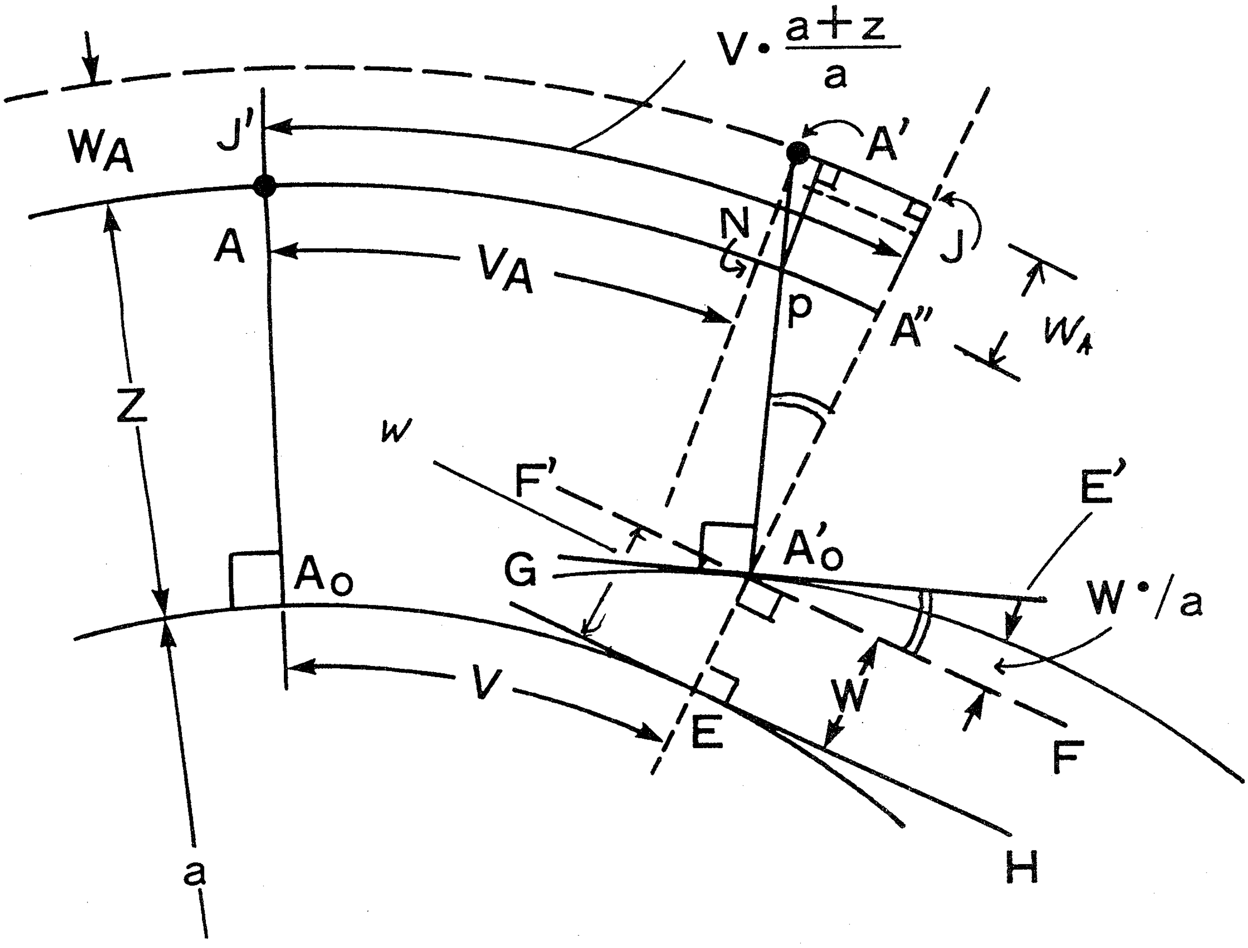
Then

$$Nx = \int_{-h/2}^{h/2} \sigma x\,(R+Z)/R\,dz\,d\phi \qquad (12)$$

Relation between Ua and U



Relation between Va and V



Putting the equation 6,7 into equ9

$$N_x=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{u'}{R}-z\frac{w''}{R^2}+\frac{\sigma v^\circ}{R}-\frac{\sigma z}{R^2}\frac{\ddot{w}}{(1+z/R)}$$

$$+\frac{\sigma w}{R\,(1+z/R)}+\frac{u'}{R}\frac{z}{R}-\frac{z w''}{R^2}\frac{z}{R}+\frac{\sigma v^\circ}{R}\frac{z}{R}$$

$$-\frac{\sigma z}{R^2}\frac{\ddot{w}}{(1+z/R)\,R}\frac{z}{R}+\frac{\sigma w}{(1+z/R)}\frac{z}{R}\Big)\,dz$$

Since $R>>z$, we expand it by power series of z/R as

$$\sum_{n=0}^{\infty}(-z/R)^n=1-\frac{z}{R}+\frac{z^2}{R^2}\qquad(13)$$

then

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{u'}{R}+\frac{\sigma v^\circ}{R}-\frac{\sigma z\ddot{w}}{R^2}\sum_{n=0}^{\infty}(-z/R)^n$$

$$+\frac{\sigma w}{R}\sum_{n=0}^{\infty}(-z/R)^n-\frac{z^2 w''}{R^3}-\frac{\sigma z^2\ddot{w}}{R^3}\sum_{n=0}^{\infty}(-z/R)^n$$

$$+\frac{\sigma w z}{R^2}\sum_{n=0}^{\infty}(-z/R)\Big)\,dz$$

the 3,6,9,10th terms, the higher order terms of z^n $(n>2)$ are neglected.

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{u'}{R}+\frac{\sigma V^\circ}{R}+\frac{\sigma z^2\ddot{w}}{R^3}+\frac{\sigma w}{R^3}$$

$$-\frac{z^2 w''}{R^3}-\frac{\sigma z^2 w}{R^3}-\frac{\sigma w\,z^2}{R^3}+\frac{\sigma v^\circ}{R}\Big)\,dz$$

Then we get the final form as

$$=\frac{E}{(1-\sigma^2)}\Big(\frac{u'}{R}+\frac{\sigma v^\circ}{R}+\frac{\sigma w}{R}\Big)h-\frac{w''}{R^3}\frac{h^3}{12}\qquad(14)$$

2] $N\phi$ (the circumferential resultant)

The $N\phi$ is expressed by the integration of $\sigma\phi$ as

$$N\phi=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}(\varepsilon\phi+\sigma\varepsilon x)\Big(1+\frac{z}{R\phi}\Big)dz$$

In the cylindrical shell, the $R\phi$ should be regarded infinite . Then the resultant forces $N\phi$ is simply expressed as $N\phi=\int\sigma\phi\,dz$. So

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{v^\circ}{R}-\frac{z\ddot{w}}{R^2(R+z)}+\frac{w}{(R+z)}+\frac{\sigma u'}{R}\Big)dz$$

Neglecting the 5th term,and utilyzing equ(13)

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{v^\circ}{R}+\frac{\sigma u'}{R}-\frac{z\ddot{w}}{R^2}\sum_{n=0}^{\infty}(-z/R)^n$$

$$+\frac{w}{R}\sum_{n=0}^{\infty}(-z/R)^n\Big)dz$$

neglecting the higher order terms over z^3

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{v^\circ}{R}+\frac{\sigma u'}{R}+\frac{w}{R}+(\ddot{w}+w)\frac{z^2}{R^3}\Big)dz$$

$$=\frac{E}{(1-\sigma^2)}\Big(\Big(\frac{v^\circ}{R}+\frac{\sigma u'}{R}+\frac{w}{R}\Big)h+(\ddot{w}+w)\frac{h^3}{12\,R^3}\Big)\qquad(15)$$

3] $Nx\phi$ (the shearing force)

The $Nx\phi$ is calculated by $\tau x\phi$ as

$$=\int_{-h/2}^{h/2}\frac{E}{2\,(1+\sigma)}\gamma x\phi\Big(1+\frac{z}{R}\Big)dz$$

$$=\frac{E}{2\,(1+\sigma)}\int_{-h/2}^{h/2}\Big(\frac{u^\circ}{(R+z)}+\frac{(R+z)}{R^2}V'$$

$$-\frac{w^{\circ\prime}}{R}\Big(\frac{z}{R}+\frac{z}{(R+z)}\Big)\Big)(1+z/R)\,dz$$

the 3,4,7th terms are calculated to 0. Utilyzing equ(11)

$$=\frac{E}{2(1+\sigma)}\int_{-h/2}^{h/2}\Big(\frac{u^\circ}{R}(1-z/R)+\frac{v'}{R}-\frac{z w'^\circ}{R^2}(1-z/R+z^2/R^2)$$

$$+\frac{z u^\circ}{R^2}(1-z/R)+(v'-w'^\circ)\frac{z^2}{R^3}-\frac{z^2 w^{\circ\prime}}{R^3}(1-z/R)\Big)dz$$

the higher order terms over z^2 are neglected. Then

$$=\frac{E}{2\,(1+\sigma)}\int_{-h/2}^{h/2}\Big(\frac{u^\circ}{R}+\frac{v'}{R}+(v'-w'^\circ)\frac{z^2}{R^3}\Big)dz$$

$$=\frac{E}{2\,(1+\sigma)}\Big(\Big(\frac{u^\circ}{R}+\frac{v'}{R}\Big)h+\frac{h^3}{12R^3}(v'-w'^\circ)\Big)\qquad(16)$$

4] Mx (the bending moment for the axial direction)

The moment Mx is given by the integration of the stress σx as

$$Mx=-\int_{-h/2}^{h/2}\sigma x\,(1+z/R)\,z\,dz$$

$$=\frac{E}{(1-\sigma^2)}\int_{-h/2}^{h/2}\Big(\frac{z^2 w''}{R^2}+\frac{\sigma\ddot{w}}{R^2}\frac{z^2}{(1+z/R)}-\frac{\sigma w}{R}\frac{z}{(1+z/R)}$$

$$-\frac{z^2 u'}{R^2}-\frac{\sigma z^2 v^\circ}{R^2}+\frac{\sigma z^3\ddot{w}}{R^3(1+z/R)}-\frac{\sigma w}{R^2}\frac{z^2}{(1+z/R)}\Big)dz$$

Utilyzing equ(13)

$$= \frac{E}{(1-\sigma^2)} \int_{-h/2}^{h/2} \Big(\frac{z^2 w''}{R^2} + \frac{\sigma z^2 \ddot{w}}{R^2}(1-z/R) - \frac{\sigma w z}{R}(1-z/R)$$

$$-(u' + \sigma v^\circ)\frac{z^2}{R^2} + \frac{\sigma z^3 \ddot{w}}{R^3}(1-z/R) - \frac{\sigma w z^2}{R^3}(1-z/R)\, dz$$

$$- \frac{\sigma w z^2}{R^3}(1-z/R)\, dz$$

the 3,8 th terms are canceled each other .

$$= \frac{E}{(1-\sigma^2)} \int_{-h/2}^{h/2} \Big(\frac{z^2 w''}{R^2} + \frac{\sigma z^2 \ddot{w}}{R^2} - (u' + \sigma v^\circ)(z^2/R^2) \Big) dz$$

$$= \frac{E}{(1-\sigma^2)} \Big(\frac{h^3}{12R^2} \big(w'' + \sigma \ddot{w} - (u' + \sigma v^\circ) \big) \Big) \quad (17)$$

5] Mφ (the bending moment for the circumferential direction)

By integrating the stress $\sigma\phi$ as $M\phi = -\int_{-h/2}^{h/2} \sigma\phi(1+z/R\phi)\, z\, dz$.

However in the cylindrical coordinates, $R\phi = \infty$. Then

$$M\phi = -\int_{-h/2}^{h/2} \sigma x\, z\, dz$$

$$= \frac{-E}{(1-\sigma^2)} \int_{-h/2}^{h/2} \Big(\frac{z v^\circ}{R} - \frac{z^2 \ddot{w}}{R(R+z)} + \frac{w z}{(R+z)} + \frac{\sigma z u'}{R} - \frac{\sigma z^2 w''}{R^2} \Big) dz$$

$$= \frac{E}{(1-\sigma^2)} \int_{-h/2}^{h/2} \Big(\frac{z^2 \ddot{w}}{R^2} \big(1 - \frac{z}{R} + \frac{z^2}{R^2}\big) - \frac{w z}{R}\big(1 - \frac{z}{R}\big) + \frac{\sigma z^2 w''}{R^2} \Big) dz$$

$$= \frac{E}{(1-\sigma^2)} \int_{-h/2}^{h/2} \Big(\frac{z^2 \ddot{w}}{R^2} + \frac{w z^2}{R^2} + \frac{\sigma z^2 w''}{R^2} \Big) dz$$

$$= \frac{E}{(1-\sigma^2)} \frac{h^3}{12 R^2} (w + \ddot{w} + \sigma w'') \quad (18)$$

6] Mxφ (the twisting moment)

By utilyzing the shearing stress $R\phi = oo$, the shearing moment $Mx\phi$ is

$$Mx\phi = -\int_{-h/2}^{h/2} \tau_{x\phi}(1 + z/R)\, z\, dz$$

$$= \frac{-E}{2(1+\sigma)} \int_{-h/2}^{h/2} \Big(\frac{u^\circ}{(R+z)} + \frac{(R+z) v'}{R^2}$$

$$- \frac{w'^\circ}{R} \Big(\frac{z}{R} + \frac{z}{(R+z)} \Big) \Big(z + \frac{z^2}{R} \Big) \Big) dz$$

$$= \frac{-E}{2(1+\sigma)} \int_{-h/2}^{h/2} \Big(\frac{z u^\circ}{R} \sum_{n=0}^{\infty} (-z/R)^n + \frac{z^2 v'}{R^2} - \frac{z^2 w'^\circ}{R^2}$$

$$- \frac{z^2 w'^\circ}{R^2} \sum_{n=0}^{\infty} (-z/R)^n + \frac{z^2 u^\circ}{R^2} \sum_{n=0}^{\infty} (-z/R)^n + \frac{v' z^2}{R^2}$$

$$- \frac{z^3 w'^\circ}{R^3} \sum_{n=0}^{\infty} \Big(-\frac{z}{R} \Big)^n \Big) dz$$

the 1,3,7,9,11th terms and the higher order term of z are neglected

$$= \frac{E}{2(1+\sigma)} \int_{-h/2}^{h/2} 2 (w^{\circ\prime} - v') \frac{z^2}{R^2} dz$$

$$= \frac{E}{(1+\sigma)} \frac{h^3}{12} \frac{1}{R^2} (w^{\circ\prime} - v') \quad (19)$$

Similar calculation brings us to

$$M\phi x = -\int_{-h/2}^{h/2} \tau\phi x\, z\, dz =$$

$$= \frac{E}{12(1+\sigma)} \frac{h^3}{R^2} \Big(w'^\circ + \frac{u^\circ}{2} - \frac{v'}{2} \Big) (20)$$

At last we have established the mathematical expression of the forces (equation 14,15,16) and moments (equation 17,18,19,20) utilyzing the displacements of the arbitrary point just on the inner surface of the arterial wall.

IV] Biological consideration

Now we simplify these equations according to the biological data of the human femoral artery in vivo by Learoyd & Taylor(1966). the range of the ratio of the wall thickness h and radius R was $0.0561 < h/2R < 0.12$. Therefore $0.0048 > h2/12*R2 > h3/12*R2$. Consequently the terms that are mutiplied by $h3/12*R2$ are neglected. So the $Nx, N\phi, Nx\phi, N\phi x$ are all simplified and the moments are all neglected. Then the following equations are establishied.

$$Nx = \frac{Eh}{1-\sigma^2} \Big(\frac{\partial u}{\partial x} + \frac{\sigma w}{R} + \frac{\sigma}{R} \frac{\partial v}{\partial \phi} \Big)$$

$$N\phi = \frac{Eh}{1-\sigma^2} \Big(\frac{\partial v}{R \partial \phi} + \frac{w}{R} + \frac{\sigma \partial u}{\partial x} \Big)$$

$$Nx\phi = \frac{Eh}{2(1+\sigma)} \Big(\frac{u^\circ}{R} + \frac{v'}{R} + \frac{h^2 (v' - w'^\circ)}{12 R^3} \Big)$$

$$N\phi x = \frac{Eh}{2(1+\sigma)} \Big(\frac{u^\circ}{R} + \frac{v'}{R} + \frac{h^2 (u^\circ + w'')}{12 R^3} \Big)$$

As mentioned above $h^2/12*R^2 \to 0$, then $Nx\phi = N\phi x$.