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Linear Systems Analysis of Systolic Coronary Circulation.

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Linear system analysis was applied for the coronary circulation system. The system was expressed by an equivalent electrical circuit. It is consisted of the arterial and venous sites each of which is separated to the epicardial and the end cardial regions. The mechanical properties of the vessels were expressed by resistance and capacitance which values were adopted from the reported animal experiments. The system was expressed by 4 linear differential equations. The control inputs to drive the blood during the systolic phase were set by the active elastance of cardiac muscles. At the physiological resistance and capacitance, the system was stable and controllable. Expressing the coronary arterial sclerosis by reducing the capacitance and arterial stenosis by increasing the resistance, we can evaluate the pathophysiological changes in coronary circulation.

Coronary circulation, Equivalent electrical circuit, Epicardium, Endcardiaum, Resistance, Capacitance

冠循環に対する線形システム特性

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冠循環のシステム特性を解析した。冠循環を電気的等価回路モデルで表現した。モデルは動脈側、静脈側から構成され、それぞれ心外膜近傍、心内膜近傍に分離した。血管特性は抵抗とキャパシタンスで表現した。冠血管の各部位における抵抗値とキャパシタンスの値とは動物実験で報告されている結果を採用した。系は4個の線形微分方程式で記述された。制御入力としては、収縮期心筋能動弾性を想定し、これが心外膜近傍および心内膜近傍の心筋に作用して心筋内冠血管を圧迫することで収縮期の心筋内冠血管内の逆流がおきるとした。生理的血管抵抗、キャパシタンスでは系は安定であった。また制御　であった。
本研究をより複雑にすることで　冠動脈硬化を血管キャパシタンスの低下、また冠血管狭窄を血流抵抗の上昇で表現すことで系の動特性の変化を定量的に評価することが可能である。

冠循環．電気的等価回路モデル．心外膜近傍．心内膜近傍．抵抗．キャパシタンス

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1. Introduction.

A lot of experimental studies have performed (Kajiya 1990) to characterize coronary circulation. By associating those findings and to understand intuitively, several models (Downey 1975, Spaan 1981, Krams 1989) have been proposed. In the present work, we apply linear systems analysis to evaluate an entire systemic properties of the systolic coronary circulation.

2. Mathematical method.

2-1. Construction of equivalent electrical model of coronary circulation.

Fig 1 shows the equivalent electrical circuit of the coronary circulation. Coronary vessel properties in each compartment are comprised of blood flow resistance Rn and compliance Cn. The left part describes the epicardial surface arterial system. Exogenous input for the coronary circulation system is supplied by aortic pressure Pa(t). The middle part illustrates circulation within the myocardium. It is comprised of the region near to the epicardium (upper portion) and the region near to the endocardium (lower portion). The blood flow in each region of the is controlled by the pressures produced by the active elastance of cardiac muscle (at near the epicardium : Pepi(t) and at near the endocardium : Pend(t)) and the passive pressures (P2(t), P3(t)). The left half of the middle part expresses the intramyocardial arterial system and right one the intramyocardial venous system. The right part expresses the epicardial surface venous system. Atrial pressure Pe was set to be constant. Derivation of mathematical equations are shown in Appendix1. The resistance (mmHgsec/ml) are

R1 = 1; R2 = 1; R3 = 50; R4 = 100; R5 = 25; R6 = 75; R7 = 0.5; R8 = 0.5 and

capacitance (ml/mmHg) are

C1 = 0.1; C2 = 0.05; C3 = 0.01; C4 = 0.2;

Eigen values of the system were examined by the characteristic equation of the system.

The singular values of the system $C (j \omega I - A)^{-1} B$ are shown in Fig 3. The left column is obtained by setting all the control inputs unity while the right column is obtained by setting b4, b5, b10, b11 et as exact form that were

Table1. Definition of the Electrical Circuit

Pa(t) : aortic pressure driving the coronary circulation.
P1(t) : epicardial surface arterial pressure degraded by arterial resistance R1 at the near to aorta.
F1 : coronary arterial blood flow rate from aorta.
F2 : coronary arterial flow rate at the epicardium immediately before the artery runs into myocardium.
R1 : fluid resistance for blood flow at the epicardium near to aorta,
C1 : arterial compliance at the epicardium near to aorta.
R2 : epicardial arterial resistance at the epicardium
F3 : arterial flow at near the epicardium region .
F5 : venous flow at near the epicardium region
R3 : arterial resistance for F3(t) at near the epicardium region.
R5 : venous resistance for F5(t) at near the epicardium region.
C2 : arterial compliance at near the epicardium region.
P2(t) : passive extra vascular pressure on C2(t) at the epi cardium.
Pepi(t) : active pressure generated by active contraction of cardiac muscle at near the epicardial region .
F4 : arterial flow at near the end cardium region.
F6 : venous flow at near the end cardium region.
R4 : arterial resistance for F4(t) at near the endocardium region.
R6 : venous resistance for F6(t) at near the endocardium region.
C3 : arterial compliance at near the endocardium region.
P3(t) : passive extra vascular pressure on C3(t) at the endocardium
Pend(t) : active pressure generated by active contraction of cardiac muscle fiber at near the endocardium region .
Pe : pressure at right atrium,
F8 : the extra myocardial venous flow before the venous flows
R8 : resistance for venous flow F8(t) .
C4 : compliance for the epicardial vein.
F7 : venous flow in the epicardium.
R7 : resistance for F7(t) .
P4(t) : pressure on vein at the epicardium.

Fig 2.

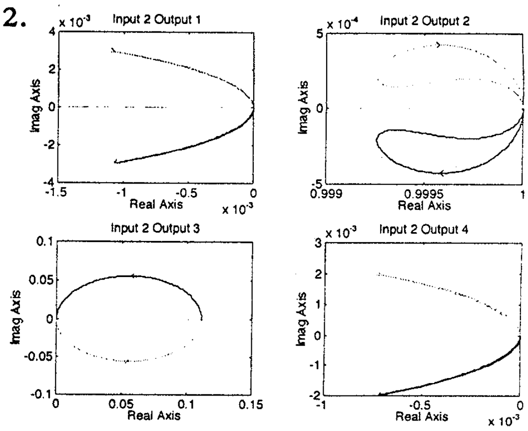

Fig 1. An equivalent electrical circuit for coronary circulation.

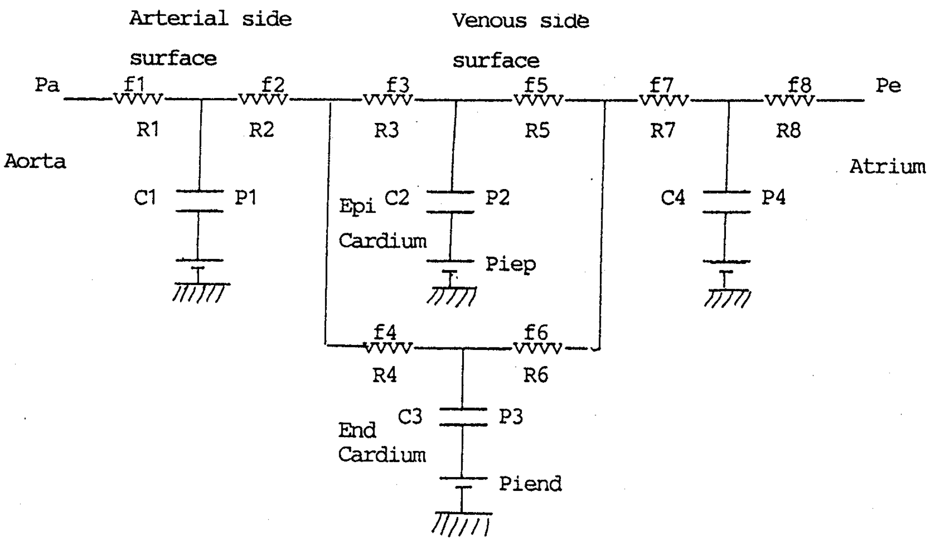

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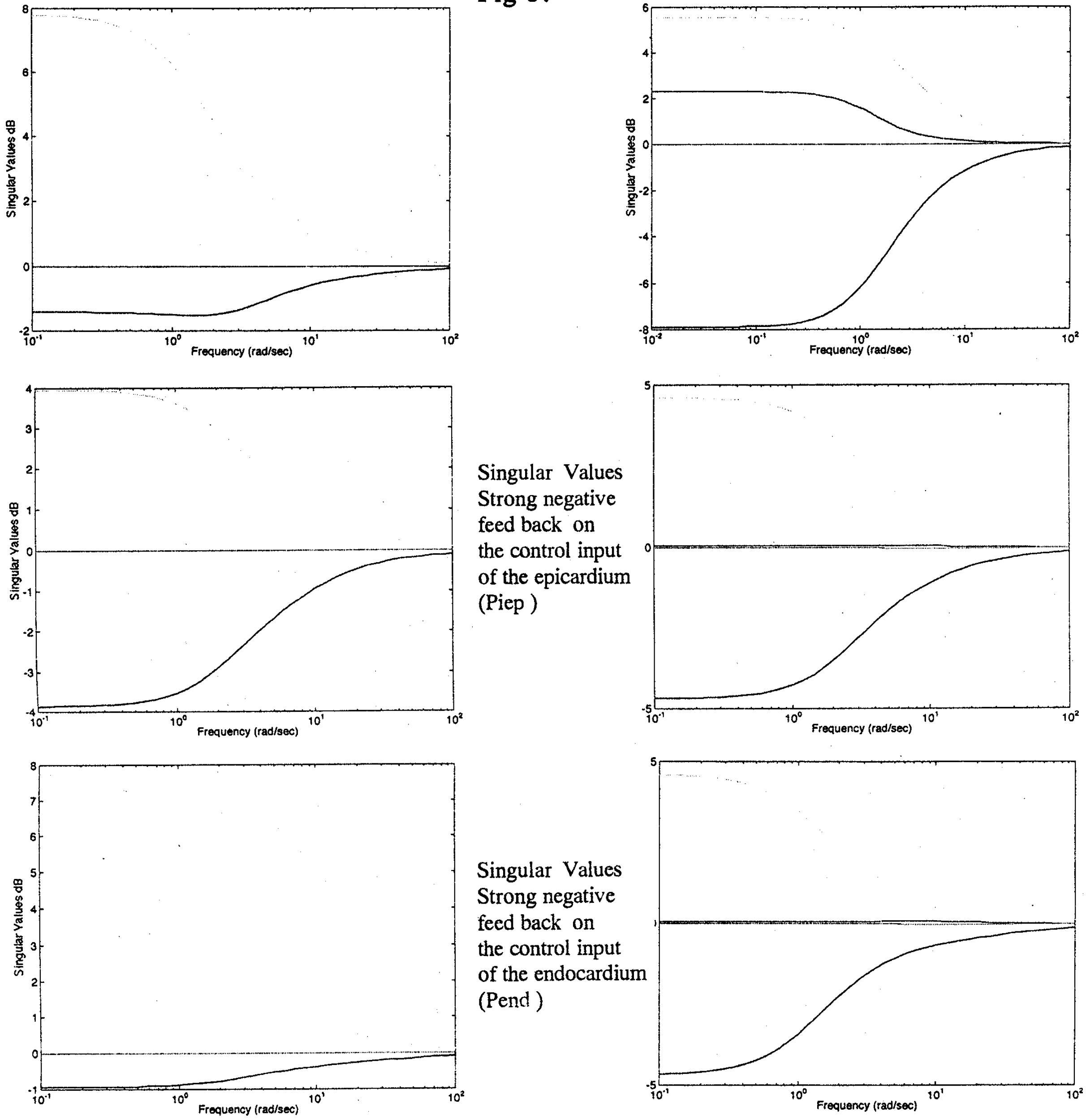
Fig 3.



Singular Values Strong negative feed back on the control input of the epicardium (Piep)

Singular Values Strong negative feed back on the control input of the endocardium (Pend)

given in the APPENDIX 1 for the coefficients of the control inputs Pei (epicardial active pressure) and Pend (endocardial one).

3. Results and Discussion.

The system was judged to be stable and controllable. All the real parts of the eigen values were negative. The nyquist loops escaped the critical point -1 + 0j. The singular values converged to 0 as the frequency (rad/sec) increased. As the control inputs have realized to be physiological ones (right column), all the singular values decreased. The magnitude of the feed back on the control inputs influenced strongly on the singular values.

A mathematical method for computing the optimized present system characterized by minimizing the performance function

$$L = \int_{0}^{te} \left[\sum_{n=1}^{4} \alpha n \, Pn'(t)^2 + \beta 1 \, U0(t)^2 + \beta 2 \; Ue(t)^2 + \beta 3 \, Ui(t)^2 + \beta 4 \, Un(t)^2 \right] dt$$

is shown in **APPENDIX 2.**

4. Conclusion.

The coronary circulation may be stable and controllable but an intense comparison to experiments is necessary.

6.References.

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[APPENDIX 1].

The differential equations for the coronary blood flow are

$$f1(t) - f2(t) = C1 \ dP1(t) / dt \quad \text{-------(1)}$$
$$f2(t) = f3(t) + f4(t) \quad \text{-------(2)}$$
$$f3(t) - f5(t) = C2 \ dP2(t) / dt \quad \text{-------(3)}$$
$$f4(t) - f6(t) = C3 \ dP3(t) / dt \quad \text{-------(4)}$$
$$f7(t) = f5(t) + f6(t) \quad \text{-------(5)}$$
$$f7(t) - f8(t) = C4 \ dP4(t) / dt \quad \text{-------(6)}$$

The equations for the coronary vessel pressures are

$$P0(t) - P1(t) = R1 \ f1(t) \quad \text{-------(7)}$$
$$P1(t) - (P2(t) + Piep(t)) = R2 \ f2(t) + R3 \ f3(t) \quad \text{---(8)}$$
$$P1(t) - (P3(t) + Pied(t)) = R2 \ f2(t) + R4 \ f4(t) \quad \text{---(9)}$$
$$(P2(t) + Piep(t)) - P4(t) = R5 \ f5(t) + R7 \ f7(t) \quad \text{---(10)}$$
$$(P3(t) + Pied(t)) - P4(t) = R6 \ f6(t) + R7 \ f7(t) \quad \text{----(11)}$$
$$P4(t) - Pe(t) = R8 \ f8(t) \quad \text{-----(12)}$$

To obtain the differential equations for the pressure, we express fn(t) by pn(t). From equation (2),(8) and (9), we have equations containing only f3(t) and f4(t). From (8) and (9),

$$R2 \ f2 + R3 \ f3 = P1 - P2 - Piepi$$
$$R2 \ f2 + R4 \ f4 = P1 - P3 - Pied$$

Substitute

$$f2 = f3+f4$$

and rearrange

$$(R2 + R3) \ f3 + R2 \ f4 = P1 - P2 - Piepi$$
$$R2 \ f3 + (R2 + R4) \ f4 = P1 - P3 - Pied$$

Here, we set

$a1 = R2 + R3$, $a2=R2$

$a3=R2$, $a4=R2+R4$

and

$$g1(t) = P1(t) - P2(t) - Piep(t)$$
$$g2(t) = P1(t) - P3(t) - Pied(t)$$

Then, we have

$$a1 \ f3 + a2 \ f4 = g1$$
$$a3 \ f3 + a4 \ f4 = g2$$

Here we set

$a5 = a1 \ a4 - a3 \ a2$, $a6=a4 / a5$, $a7= - a2 / a5$

We eliminate f4(t). Then we have

$$f3(t) = a6 \ g1(t) + a7 \ g2(t)$$
$$= (a6+a7) \ P1 - a6 \ P2 - a7 \ P3 - a6 \ Piepi - a7 \ Pied \quad \text{--(13)}$$

$$f4(t) = a9 \ g1(t) + a10 \ g2(t)$$
$$= (a9 + a10) \ P1 - a9 \ P2 - a10 \ P3 - a9 \ Piepi - a10 \ Pied \quad \text{---(14)}$$

from equations (5), (10) and (11), we have

$$(R5 + R7) \ f5 + R7 \ f6 = P2 + Piepi - P4$$
$$R7 \ f5 + (R6 + R7) \ f6 = P3 + Pied - P4$$

Setting

$a11= R5 + R7$, $a12 = R7$

$a13 = R7$, $a14 = R6 + R7$

and

$$g3(t) = P2(t) + Piepi(t) - P4(t)$$
$$g4(t) = P3(t) + Pied(t) - P4(t)$$

We have two simultaneous equations

$$a11 \ f5(t) + a12 \ f6(t) = g3(t)$$
$$a13 \ f5(t) + a14 \ f6(t) = g4(t)$$

Putting

$a15 = a14 \ a11 - a12 \ a13$, $a16= a14 / a15$

$a17= - a12 / a15$

We obtain

$$f5(t) = a16 \ g3(t) + a17 \ g4(t)$$
$$= a16 \ P2 + a17 \ P3 + a16 \ Piep + a17 \ Pied + P4 \ (-a16 - a17) \quad \text{----------(15)}$$
$$f6(t) = a19 \ g3(t) + a20 \ g4(t)$$
$$= a19 \ P2 + a20 \ P3 + a19 \ Piep + a20 \ Pied + P4 \ (-a19 - a20) \quad \text{----------(16)}$$

Thus, the differential equations for P1(t) is from the equations (1) and (2).

$$C1 \ P'1(t)= f1 - f3 - f4$$

Substitute this by the equations (7), (13) and (14)

$$C1 \ P'1 = (P0-P1) / R1$$
$$-(a6+a7) \ P1+a6 \ P2+a7 \ P3+a6 \ Piep+a7 \ Pied$$
$$-(a9+a10) \ P1+a9 \ P2+a10 \ P3+a9 \ Piep+a10 \ Pied$$

Putting

$b0= 1 / (R1 \ C1)$

$b1 = [- 1/R1 -(a6+a7)-(a9+a10)]/C1$

$b2=(a6+a9)/C1$, $b3=(a7+a10)/ C1$

$b4=(a6+a9)/C1$, $b5=b3$

Then we have

$$P'1(t)=b0 \ P0(t)+b1 \ P1(t) + b2 \ P2(t) + b3 \ P3(t) +b4 \ Piep(t)+b5 \ Pied(t) \quad \text{--------(17)}$$

The differential equation for P2(t) is

$$C2 \ P'2(t)=f3(t)-f5(t)$$
$$=(a6 + a7) \ P1 - a6 \ P2 - a7 \ P3 - a6 \ Piep - a7 \ Pied$$
$$- a16 \ P2 - a17 \ P3 + (a16 + a17) \ P4$$
$$- a16 \ Piep - a17 \ Pied$$

Putting

$b6=(a6+a7)/C2$, $b7= - (a6+a16) / C2$

$b8= -(a7+a17)/C2$, $b9= -(a16+a17)/C2$

$b10=b7$, $b11=b8$

We have

$$P'2(t) = b6 \ P1(t) + b7 \ P2(t) + b8 \ P3(t) + b4 \ P4(t) + b10 \ Piep(t) + b11 \ Pied(t) \quad \text{--------(18)}$$

The differential equation for P3(t) is

$$C3 \ P'3=f4(t) - f6(t)$$

$$P'3(t)=b12 \ P1(t) + b13 \ P2(t) + b14 \ P3(t) + b15 \ P4(t) + b16 \ Piep(t) + b17 \ Pied(t) \quad \text{---------(19)}$$

The differential equation for P4(t) is

$$C4 \ P'4=f5(t) + f6(t) - f8(t)$$

$$P'4(t) =b18 \ P2(t) + b19 \ P3(t) + b20 \ P4(t) + b21 \ Piep(t) + b22 \ Pied(t) + b23 \ Pe(t) \quad \text{--------(20)}$$

[APPENDIX 2].

The terms involved in the performance function are expanded as

$$\alpha 1 \ (P'1)^2$$
$$= \alpha 1 \ (b0 \ U0 + b1 \ P1 + b2 \ P2 + b3 \ P3 + b4 \ Ui + b5 \ Un)^2$$

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$= \alpha 1\, [b0^2\, U0^2 + b1^2\, P1^2 + b2^2\, P2^2 + b3^2\, P3^2 + b4^2\, Ui^2 + b5^2\, Un^2$
$+2\, (b0\, b1\, U0\, P1 + b0\, b2\, U0\, P2 + b0\, b3\, U0\, P3 + b0\, b4\, U0\, Ui + b0\, b5\, U0\, Un$
$+ b1\, b2\, P1\, P2 + b1\, b3\, P1\, P3 + b1\, b4\, P1\, Ui + b1\, b5\, P1\, Un$
$+ b2\, b3\, P2\, P3 + b2\, b4\, P2\, Ui + b2\, b5\, P2\, Un$
$+ b3\, b4\, P3\, Ui + b3\, b5\, P3\, Un$
$+ b4\, b5\, Ui\, Un\,]$

$\alpha 2\, (P'2)^2$
$= \alpha 2\, (b9\, P4 + b6\, P1 + b7\, P2 + b8\, P3 + b10\, Ui + b11\, Un\,)^2$
$= \alpha 2\, [b9^2\, P4^2 + b6^2\, P1^2 + b7^2\, P2^2 + b8^2\, P3^2 + b10^2\, Ui^2 + b11^2\, Un^2$
$+2\, (b9\, b6\, P4\, P1 + b9\, b7\, P4\, P2 + b9\, b8\, P4\, P3 + b9\, b10\, P4\, Ui + b9\, b11\, P4\, Un$
$+ b6\, b7\, P1\, P2 + b6\, b8\, P1\, P3 + b6\, b10\, P1\, Ui + b6\, b11\, P1\, Un$
$+ b7\, b8\, P2\, P3 + b7\, b10\, P2\, Ui + b7\, b11\, P2\, Un$
$+ b8\, b10\, P3\, Ui + b8\, b11\, P3\, Un$
$+ b10\, b11\, Ui\, Un\,]$

$\alpha 3\, (P'3)^2$
$= \alpha 3\, (b15\, P4 + b12\, P1 + b13\, P2 + b14\, P3 + b16\, Ui + b17\, Un\,)^2$
$= \alpha 3\, [b15^2\, P4^2 + b12^2\, P1^2 + b13^2\, P2^2 + b14^2\, P3^2 + b16^2\, Ui^2 + b17^2\, Un^2$
$+2\, (b15\, b12\, P4\, P1 + b15\, b13\, P4\, P2 + b15\, b14\, P4\, P3 + b15\, b16\, P4\, Ui + b15\, b17\, P4\, Un$
$+ b12\, b13\, P1\, P2 + b12\, b14\, P1\, P3 + b12\, b16\, P1\, Ui + b12\, b17\, P1\, Un$
$+ b13\, b14\, P2\, P3 + b13\, b16\, P2\, Ui + b13\, b17\, P2\, Un$
$+ b14\, b16\, P3\, Ui + b14\, b17\, P3\, Un$
$+ b16\, b17\, Ui\, Un\,]$

$\alpha 4\, (P'4)^2$
$= \alpha 4\, (b20\, P4 + b23\, Ue + b18\, P2 + b19\, P3 + b21 Ui + b22\, Un\,)^2$
$= \alpha 4\, [b20^2\, P4^2 + b23^2\, Ue^2 + b18^2\, P2^2 + b19^2\, P3^2 + b21^2\, Ui^2 + b22^2\, Un^2$
$+2\, (b20\, b23\, P4\, Ue + b20\, b18\, P4\, P2 + b20\, b19\, P4\, P3 + b20\, b21\, P4\, Ui + b20\, b22\, P4\, Un$
$+ b23\, b18\, Ue\, P2 + b23\, b19\, Ue\, P3 + b23\, b21\, Ue\, Ui + b23\, b22\, Ue\, Un$
$+ b18\, b19\, P2\, P3 + b18\, b21\, P2\, Ui + b18\, b22\, P2\, Un$
$+ b19\, b21\, P3\, Ui + b19\, b22\, P3\, Un$
$+ b21\, b22\, Ui\, Un\,]$

By gathering the similar terms such that

$U0^2\, (\alpha 1\, b0^2 + \beta 1)$
$Ui^2\, (\alpha 1\, b4^2 + \alpha 2\, b10^2 + \alpha 3\, b16^2 + \alpha 4\, b21^2 + \beta 3)$
$Un^2\, (\alpha 1\, b5^2 + \alpha 2\, b11^2 + \alpha 3\, b17^2 + \alpha 4\, b22^2 + \beta 4)$
$Ue^2\, (\alpha 4\, b23^2 + \beta 2)$

$P1^2\, (\alpha 1\, b1^2 + \alpha 2\, b6^2 + \alpha 3\, b12^2)$
$P2^2\, (\alpha 1\, b2^2 + \alpha 2\, b7^2 + \alpha 3\, b13^2 + \alpha 4\, b18^2)$
$P3^2\, (\alpha 1\, b3^2 + \alpha 2\, b8^2 + \alpha 3\, b14^2 + \alpha 4\, b19^2)$
$P4^2\, (\qquad \alpha 2\, b9^2 + \alpha 3\, b15^2 + \alpha 4\, b20^2)$

$2\, U0\, P1\, (\alpha 1\, b0\, b1\,)$
$2\, U0\, P2\, (\alpha 1\, b0\, b2\,) + 2\, U0\, P3\, (\alpha 1\, b0\, b3\,)$
$2\, U0\, Ui\, (\alpha 1\, b0\, b4\,) \;+2\, U0\, Un\, (\alpha 1\, b0\, b51\,)$

$2\, P1\, P2\, (\alpha 1\, b1\, b2 + \alpha 2\, b6\, b7 + \alpha 3\, b12\, b13\,)$
$2\, P1\, P3\, (\alpha 1\, b1\, b3 + \alpha 2\, b6\, b8 + \alpha 3\, b12\, b14\,)$
$2\, P1\, P4\, (\qquad \alpha 2\, b6\, b9 + \alpha 3\, b12\, b15\,)$

$2\, P2\, P3\, (\alpha 1\, b2\, b3 + \alpha 2\, b7\, b8 + \alpha 3\, b13\, b14 + \alpha 4\, b18\, b19)$
$2\, P2\, P4\, (\alpha 2\, b7\, b9 + \alpha 3\, b13\, b15 + \alpha 4\, b18\, b20\,)$
$2\, P3\, P4\, (\alpha 2\, b8\, b9 + \alpha 3\, b14\, b15 + \alpha 4\, b19\, b20\,)$

$2\, P1\, Ui\, (\alpha 1\, b1\, b4 + \alpha 2\, b6\, b10 + \alpha 3\, b12\, b16\,)$
$2\, P2\, Ui\, (\alpha 1\, b2\, b4 + \alpha 2\, b7\, b10 + \alpha 3\, b13\, b16 + \alpha 4\, b18\, b21\,)$
$2\, P3\, Ui\, (\alpha 1\, b3\, b4 + \alpha 2\, b8\, b10 + \alpha 3\, b14\, b16 + \alpha 4\, b19\, b21\,)$
$2\, P4\, Ui\, (\alpha 2\, b9\, b10 + \alpha 3\, b15\, b16 + \alpha 4\, b20\, b21\,)$

$2\, P1\, Un\, (\alpha 1\, b1\, b5 + \alpha 2\, b6\, b11 + \alpha 3\, b12\, b17\,)$
$2\, P2\, Un\, (\alpha 1\, b2\, b5 + \alpha 2\, b7\, b11 + \alpha 3\, b13\, b17 + \alpha 4\, b18\, b22\,)$
$2\, P3\, Un\, (\alpha 1\, b3\, b5 + \alpha 2\, b8\, b11 + \alpha 3\, b14\, b17 + \alpha 4\, b19\, b22\,)$
$2\, P4\, Un\, (\alpha 2\, b9\, b11 + \alpha 3\, b15\, b17 + \alpha 4\, b20\, b22\,)$

$2\, Ue\, P2\, (\alpha 4\, b18\, b23\,) \;+2\, Ue\, P3\, (\alpha 4\, b19\, b23\,)$
$2\, Ue\, P4\, (\alpha 4\, b20\, b23\,)$
$2\, Un\, Ui\, (\alpha 1\, b5\, b4 + \alpha 2\, b10\, b11 + \alpha 3\, b16\, b17 + \alpha 4\, b21\, b22)$
$2\, Ui\, Ue\, (\alpha 4\, b21\, b23\,) \;+2\, Un\, Ue\, (\alpha 4\, b22\, b23\,)$

Then, the Hamiltonian function is obtained as following by setting co-state variables as qn (n=1,2,3,4)

$H = d1\, U0^2 + d2\, Ui^2 + d3\, Un^2 + d4\, Ue^2$
$+ d5\, P1^2 + d6\, P2^2 + d7\, P3^2 + d8\, P4^2$
$+ d9\, U0\, P1 + d10\, U0\, P2 + d11\, U0\, P3 + d12\, U0\, Ui$
$+ d13\, U0\, Un$
$+ d14\, P1\, P2 + d15\, P1\, P3 + d16\, P1\, P4 + d17\, P2\, P3$
$+ d18\, P2\, P4 + d19\, P3\, P4$
$+ d20\, P1\, Ui + d21\, P2\, Ui + d22\, P3\, Ui + d23\, P4\, Ui$
$+ d24\, P1\, Un + d25\, P2\, Un + d26\, P3\, Un + d27\, P4\, Un$
$+ d28\, P2\, Ue + d29\, P3\, Ue + d30\, P4\, Ue$
$+ d31\, Ui\, Un + d32\, Ui\, Ue + d33\, Un\, Ue$
$+ q1\, (b0\, U0 + b1\, P1 + b2\, P2 + b3\, P3 + b4\, Ui + b5\, Un)$
$+ q2\, (b6\, P1 + b7\, P2 + b8\, P3 + b9\, P4 + b10\, Ui + b11\, Un)$
$+ q3\, (b12\, P1 + b13\, P2 + b14\, P3 + b15\, P4 + b16\, Ui + b17\, Un)$
$+ q4\, (b18\, P2 + b19\, P3 + b20\, P4 + b21\, Ui + b22\, Un + b23\, Ue)$

[APPENDIX 3].

The optimal control Ui, Un, Ue and U0 can be obtained by differentiating the Hamiltonian with respect to them.

$$\frac{\partial H}{\partial U0} = 2\, d1\, U0 + d9\, P1 + d10\, P2 + d11\, P3 + d12\, Ui + d13\, Un + q1\, b0 = 0$$

Then, we have

$$2\, d1\, U0 + d12\, Ui + d13\, Un = G1$$

where

$$G1 = -\, d9\, P1 - d10\, P2 - d11\, P3 - b0\, q1$$

$$\frac{\partial H}{\partial Ui} = 2\, d2\, Ui + d12\, U0 + d20\, P1 + d21\, P2 + d22\, P3 + d23\, P4 + d31\, Un + d32\, Ue + b4\, q1 + b10\, q2 + b16\, q3 + b21\, q4 = 0$$

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Then, we have

$$d12\,U0 + 2\,d2\,Ui + d31\,Un + d32\,Ue = G2$$

where

$$G2 = -(d20\,P1 + d21\,P2 + d22\,P3 + d23\,P4 + b4\,q1 + b10\,q2 + b16\,q3 + b21\,q4)$$

$$\frac{\partial H}{\partial Un} = 2\,d3\,Un + d13\,U0 + d24\,P1 + d25\,P2 + d26\,P3 + d27\,P4 + d31\,Ui + d33\,Ue + b5\,q1 + b11\,q2 + b17\,q3 + b22\,q4 = 0$$

Then, we have

$$d13\,U0 + 2\,d3\,Un + d31\,Ui + d33\,Ue = G3$$

where

$$G3 = -(2\,d3\,Un + d24\,P1 + d25\,P2 + d26\,P3 + d27\,P4 + b5\,q1 + b11\,q2 + b17\,q3 + b22\,q4$$

$$\frac{\partial H}{\partial Ue} = 2\,d4\,Ue + d28\,P2 + d29\,P3 + d30\,P4 + d32\,Ui + d33\,Un + b23\,q4 = 0$$

Then, we have

$$d\,32\,Ui + d33\,Un + 2\,d4\,Ue = G4$$

where

$$G4 = -(d28\,P2 + d29\,P3 + d30\,P4 + b23\,q4)$$

Therefore we have four linear simultaneous differential equations for Un, Ui, U0 and Ue

$$\begin{aligned} &2\,d1\,U0 + d12\,Ui + d13\,Un = G1 \\ &d12\,U0 + 2\,d2\,Ui + d31\,Un + d32\,Ue = G2 \\ &d13\,U0 + 2\,d3\,Un + d31\,Ui + d33\,Ue = G3 \\ &d\,32\,Ui + d33\,Un + 2\,d4\,Ue = G4 \end{aligned}$$

By solving these linear algebraic equations, we have

$$\begin{aligned} &Un = d55\,G1 + d56\,G2 + d57\,G3 + d58\,G4 \\ &Ui = d46\,G1 + d47\,G2 + d48\,G3 - d45\,Un \\ &U0 = (G1 - d12\,Ui - d13\,Un)\,/\,(2\,d1) \\ &Ue = (G2 - d12\,U0 - 2\,d2\,Ui - d31\,Un)\,/\,d32 \end{aligned}$$

$$\begin{aligned} G2 &= -(d20\,P1 + d21\,P2 + d22\,P3 + d23\,P4 + b4\,q1 + b10\,q2 + b16\,q3 + b21\,q4) \\ G3 &= -(d24\,P1 + d25\,P2 + d26\,P3 + d27\,P4 + b5\,q1 + b11\,q2 + b17\,q3 + b22\,q4) \\ G4 &= -(d28\,P2 + d29\,P3 + d30\,P4 + b23\,q4) \end{aligned}$$

Un is thus, obtained by substituting Gn(n=1,2,3,4)

$$\begin{aligned} Un = &-d55\,(d9\,P1 + d10\,P2 + d11\,P3 + b0\,q1) \\ &-d56\,(d20\,P1 + d21\,P2 + d22\,P3 + d24\,P4 + b4\,q1 + b10\,q2 + b16\,q3 + b21\,q4) \\ &-d57\,(\ldots + b22\,q4) - d58\,(d28\,P2 + \ldots + b23\,q4) \end{aligned}$$

Rearrange

$$\begin{aligned} Un = &P1(-d55\,d9 - d56\,d20 - d57\,d24) \\ &+ P2(-d55\,d10 - d56\,d21 - d57\,d25 - d58\,d28) \\ &+ P3(-d55\,d11 - d56\,d22 - d57\,d26 - d58\,d29) \\ &+ P4(-d56\,d23 - d57\,d27 - d58\,d30) \\ &+ q1(-d55\,b0 - d56\,b4 - d57\,b5) \\ &+ q2(-d56\,b10 - d57\,b11) \\ &+ q3(-d56\,b16 - d57\,b17) \\ &+ q4(-d56\,b21 - d57\,b22 - d58\,b23) \end{aligned}$$

Putting

$$\begin{aligned} h1 &= -d55\,d9 - d56\,d20 - d57\,d24 \\ h2 &= -d55\,d10 - d56\,d21 - d57\,d25 - d58\,d28 \\ h3 &= -d55\,d11 - d56\,d22 - d57\,d26 - d58\,d29 \\ h4 &= -d56\,d23 - d57\,d27 - d58\,d30 \\ h5 &= -d55\,b0 - d56\,b4 - d57\,b5 \\ h6 &= -d56\,b10 - d57\,b11 \\ h7 &= -d56\,b16 - d57\,b17 \\ h8 &= -d56\,b21 - d57\,b22 - d58\,b23 \end{aligned}$$

Then, we have

$$Un = h1\,P1 + h2\,P2 + h3\,P3 + h4\,P4 + h5\,q1 + h6\,q2 + h7\,q3 + h8\,q4$$

$$\begin{aligned} Ui = &-d46\,(d9\,P1 + d10\,P2 + d11\,P3 + b0\,q1) \\ &-d47\,(d20\,P1 + d21\,P2 + d22\,P3 + d23\,P4 + b4\,q1 + b10\,q2 + b16\,q3 + b21\,q4) \\ &-d48\,(d24\,P1 + d25\,P2 + d26\,P3 + 27\,P4 + b5\,q1 + b11\,q2 + b17\,q3 + b22\,q4) \\ &-d45\,(h1\,P1 + h2\,P2 + h3\,P3 + h4\,P4 + h5\,q1 + h6\,q2 + h7\,q3 + h8\,q4) \end{aligned}$$

$$\begin{aligned} Ui = &P1(-d46\,d9 - d47\,d20 - d48\,d24 - d45\,h1) \\ &+ P2(-d46\,d10 - d47\,d21 - d48\,d25 - d45\,h2) \\ &+ P3(-d46\,d11 - d47\,d22 - d48\,d26 - d45\,h3) \\ &+ P4(-d47\,d23 - d48\,d27 - d45\,h4) \\ &+ q(-d46\,b0 - d47\,b4 - d48\,b5 - d45\,h5) \\ &+ q2(-d47\,b10 - d48\,b11 - d45\,h6) \\ &+ q3(-d47\,b16 - d48\,b17 - d45\,h7) \\ &+ q4(-d47\,b21 - d48\,b22 - d45\,h8) \end{aligned}$$

Putting

$$\begin{aligned} h9 &= -d46\,d9 - d47\,d20 - d48\,d24 - d45\,h1 \\ h10 &= -d46\,d10 - d47\,d21 - d48\,d25 - d45\,h2 \\ h11 &= -d46\,d11 - d47\,d22 - d48\,d26 - d45\,h3 \\ h12 &= -d47\,d23 - d48\,d27 - d45\,h4 \\ h13 &= -d46\,b0 - d47\,b4 - d48\,b5 - d45\,h5 \\ h14 &= -d47\,b10 - d48\,b11 - d45\,h6 \\ h15 &= -d47\,b16 - d48\,b17 - d45\,h7 \\ h16 &= -d47\,b21 - d48\,b22 - d45\,h8 \end{aligned}$$

Thus, we have

$$Ui = h9\,P1 + h10\,P2 + h11\,P3 + h12\,P4 + h13\,q1 + h14\,q2 + h15\,q3 + h16\,q4$$

$$\begin{aligned} U0 &= (G1 - d12\,Ui - d13\,Un)/(2\,d1) \\ &= [-d9\,P1 - d10\,P2 - d11\,P3 - b0\,q1 \\ &\quad - d12\,(h9\,P1 + h10\,P2 + h11\,P3 + h12\,P4 + h13\,q1 + h14\,q2 + h15\,q3 + h16\,q4) \\ &\quad - d13\,(h1\,P1 + h2\,P2 + h3\,P3 + h4\,P4 + h5\,q1 + h6\,q2 + h7\,q3 + h8\,q4)]/(2\,d1) \\ &= P1(-d9 - d12\,h9 - d13\,h1)/(2d1) \\ &+ P2(-d10 - d12\,h10 - d13\,h2)/(2d1) \\ &+ P3(-d11 - d12\,h11 - d13\,h3)/(2d1) \\ &+ P4(-d12\,h12 - d13\,h4)/(2d1) \\ &+ q1(b0 - d12\,h13 - d13\,h5)/(2d1) \\ &+ q2(-d12\,h14 - d13\,h6)/(2d1) \\ &+ q3(-d12\,h15 - d13\,h7)/(2d1) \\ &+ q4(-d12\,h16 - d13\,h8)/(2d1) \\ &= h17\,P1 + h18\,P2 + h19\,P3 + h20\,P4 + h21\,q1 + h22\,q2 + h23\,q3 + h24\,q4 \end{aligned}$$

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Ue = (G2 - d12 U0 - 2 d2 Ui - d31 Un)/d32
= [- (d20 P1 + d21 P2 + d22 P3 + d23 P4
+ b4 q1+ b10 q2 + b16 q3 + b21 q4)
- d12 (h17 P1+ h18 P2 + h19 P3 + h20 P4 + h21 q1
+ h22 q2 + h23 q3 + h24 q4)
- 2 d2 (h9 P1+ h10 P2 + h11 P3 + h12 P4 + h13 q1
+ h14 q2 + h15 q3 + h16 q4)
- d31 (h1 P1+h2 P2+h3 P3+h4 P4+h5 q1+h6 q2
+h7 q3+h8 q4)]/d32
= P1 (- d20 - d12 h17 -2 d2 h9 - d31 h1) /(d32)
+ P2 (- d21 - d12 h18 -2 d2 h10 - d31 h2) /(d32)
+ P3 (- d22 - d12 h19 -2 d2 h11 - d31 h3) /(d32)
+ P4 (- d23 - d12 h20 -2 d2 h12 - d31 h4) /(d32)
+ q1 (- b4 - d12 h21 -2 d2 h13 - d31 h5) /(d32)
+ q2 (- b10 - d12 h22 -2 d2 h14 - d31 h6) /(d32)
+ q3 (- b16 - d12 h23 -2 d2 h15 - d31 h7) /(d32)
+ q4 (- b21 - d12 h24 -2 d2 h16 - d31 h8) /(d32)
=h25 P1+ h26 P2 + h27 P3 + h28 P4 + h29 q1
+ h30 q2 + h31 q3 + h32 q4

[APPENDIX 4].

The optimized differetial equations for the pressures are

P1'= b0 U0 + b1 P1 + b2 P2 + b3 P3 + b4 Ui + b5 Un
P'1=b0 (h17 P1+h18 P2+...+h24 q4)+b1 P1+b2 P2+b3 P3
+b4 (h9 P1+h10 P2+...+h16 q4)+b5 (h1 P1+...+h8q4)

P'1=P1(b0 h17 + b4 h9 + b5 h1 + b1)
+ P2(b0 h18 + b4 h10 + b5 h2 + b2)
+ P3 (b0 h19 + b4 h11 + b5 h3 + b3)
+ P4 (b0 h20 + b4 h12 + b5 h4)
+ q1 (b0 h21 + b4 h13 + b5 h5)
+ q2 (b0 h22 + b4 h14 + b5 h6)
+ q3 (b0 h23 + b4 h15 + b5 h7)
+ q4 (b0 h24 + b4 h16 + b5 h8)
= J1 P1 + J2 P2 + J3 P3 + J4 P4 + J5 q1 + J6 q2
+ J7 q3 + J8 q4

P'2(t) = b6 P1(t) + b7 P2(t) + b8 P3(t) + b4 P4(t)
+ b10 Ui + b11 Un
= P1 (b10 h9 + b11 h1 + b6)
+ P2 (b10 h10 + b11 2 + b7)
+ P3 (b10 h11 + b11 h3 + b8)
+ P4 (b10 h12 + b11 h4 + b96)
+ q1 (b10 h13 + b11 h5)
+ q2 (b10 h14 + b11 h6)
+ q3 (b10 h15 + b11 h7)
+ q4 (b10 h16 + b11 h8)
= J9 P1 + J10 P2 + J11 P3 + J12 P4 + J13 q1 + J14 q2
+ J15 q3 + J16 q4

P'3(t)=b12 P1(t) + b13 P2(t) + b14 P3(t) + b15 P4(t)
+ b16 Ui + b17 Un
= P1 (b16 h9 + b17 h1 + b12)
+ P2 (b16 h10 + b17 h2 + b13)
+ P3 (b16 h11 + b17 h3 + b14)
+ P4 (b16 h12 + b17 h4 + b15)
+ q1 (b16 h13 + b17 h5)
+ q2 (b16 h14 + b17 h6)
+ q3 (b16 h15 + b17 h7)
+ q4 (b16 h16 + b17 h8)
= J17 P1 + J18 P2 + J19 P3 + J20 P4 + J21 q1
+ J22 q2 + J23 q3 + J24 q4

P'4(t) =b18 P2(t) + b19 P3(t) + b20 P4(t)
+ b21 Ui + b22 Un + b23 Ue
= P1 (b21 h9 + b22 h1 + b23 h25)
+ P2 (b21 h10 + b22 h2 + b23 h26 + b18)
+ P3 (b21 h11 + b22 h3 + b23 h27 + b19)
+ P4 (b21 h12 + b22 h4 + b23 h28 + b20)
+ q1 (b21 h13 + b22 h5 + b23 h29)
+ q2 (b21 h14 + b22 h6 + b23 h30)
+ q3 (b21 h15 + b22 h7 + b23 h31)
+ q4 (b21 h16 + b22 h8 + b23 h32)
= J25 P1 + J26 P2 + J27 P3 + J28 P4 + J29 q1
+ J30 q2 + J31 q3 + J32 q4

[APPENDIX 5].

The optimized differential equations for the co-state variables.

$$q'1 = -\frac{\partial H}{\partial P1} = -(2\,d5\ P1 + d9\ U0 + d14\ P2 + d15\ P3 + d16\ P4 + d20\ Ui + d24\ Un + b1\ q1 + b6\ q2 + b12\ q3)$$

Substituting the optimized control inputs

Un=h1 P1+h2 P2+h3 P3+h4 P4+h5 q1+h6 q2
+h7 q3+h8 q4
Ui=h9 P1+ h10 P2 + h11 P3 + h12 P4 + h13 q1
+ h14 q2 + h15 q3 + h16 q4
U0 =h17 P1+ h18 P2 + h19 P3 + h20 P4 + h21 q1
+ h22 q2 + h23 q3 + h24 q4
Ue =h25 P1+ h26 P2 + h27 P3 + h28 P4 + h29 q1
+ h30 q2 + h31 q3 + h32 q4

We have
= - d9 (h17 P1+ h18 P2 + h19 P3 + h20 P4 + h21 q1
+ h22 q2 + h23 q3 + h24 q4)
-d20 (h9 P1+ h10 P2 + h11 P3 + h12 P4 + h13 q1
+ h14 q2 + h15 q3 + h16 q4)
-d24 (h1 P1+h2 P2+h3 P3+h4 P4+h5 q1+h6 q2
+h7 q3+h8 q4)
-2 d5 P1-d14 P2 - d15 P3-d16 P4
-b1 q1 - b6 q2 -b12 q3

Rearrange
q'1=P1(-d9 h17 - d20 h9 - d24 h1 - 2 d5)
+ P2(-d9 h18 - d20 h10 - d24 h2 - d14)
+ P3(-d9 h19 - d20 h11 - d24 h3 - d15)
+ P4(-d9 h20 - d20 h12 - d24 h4 - d16)
+ q1(-d9 h21 - d20 h13 - d24 h5 - b1)
+ q2(-d9 h22 - d20 h14 - d24 h6 - b6)
+ q3(-d9 h23 - d20 h15 - d24 h7 - b12)
+ q4(-d9 h24 - d20 h16 - d24 h8)

q1' = J33 p1 + J34 P2 + J35 P3 + J36 P4
+ J37 q1 + J38 q2 + J39 q3 + J40 q4

q2' = d H / dP2
=- (2 d6 P2 + d10 U0 + d14 P1 + d17 p3 + d18 P4
+ d21 Ui + d25 Un + d28 Ue + b2 q1 + b7 q2
+ b13 q3 + b18 q4)

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Substitute the optimized control inputs

$$= - d10 (h17\ P1 + h18\ P2 + h19\ P3 + h20\ P4 + h21\ q1 + h22\ q2 + h23\ q3 + h24\ q4)$$
$$- d21 (h9\ P1 + h10\ P2 + h11\ P3 + h12\ P4 + h13\ q1 + h14\ q2 + h15\ q3 + h16\ q4)$$
$$- d25 (h1\ P1 + h2\ P2 + h3\ P3 + h4\ P4 + h5\ q1 + h6\ q2 + h7\ q3 + h8\ q4)$$
$$- d28 (h25\ P1 + h26\ P2 + h27\ P3 + h28\ P4 + h29\ q1 + h30\ q2 + h31\ q3 + h32\ q4)$$
$$- (2\ d6\ P2 + d14\ P1 + d17\ p3 + d18\ P4 + b2\ q1 + b7\ q2 + b13\ q3 + b18\ q4)$$

Rearrange

$$= P1 (- d10\ h17 - d21\ h9 - d25\ h1 - d28\ h25 - d14)$$
$$+ P2 (- d10\ h18 - d21\ h10 - d25\ h2 - d28\ h26 - 2\ d6)$$
$$+ P3 (- d10\ h19 - d21\ h11 - d25\ h3 - d28\ h27 - d17)$$
$$+ P4 (- d10\ h20 - d21\ h12 - d25\ h4 - d28\ h28 - d18)$$
$$+ q1 (- d10\ h21 - d21\ h13 - d25\ h5 - d28\ h29 - b2)$$
$$+ q2 (- d10\ h22 - d21\ h14 - d25\ h6 - d28\ h30 - b7)$$
$$+ q3 (- d10\ h23 - d21\ h15 - d25\ h7 - d28\ h31 - b13)$$
$$+ q4 (- d10\ h24 - d21\ h16 - d25\ h8 - d28\ h32 - b18)$$

$$q2' = J41\ P1 + J42\ P2 + J43\ P3 + J44\ P4 + J45\ q1 + J46\ q2 + J47\ q3 + J48\ q4$$

$$q3' = d H / d P3$$
$$= -(2\ d7\ P3 + d11\ U0 + d15\ P1 + d17\ P2 + d19\ P4 + d22\ Ui + d26\ Un + d29\ Ue + b3\ q1 + b8\ q2 + b14\ q3 + b19\ q4)$$

Substitute the optimized control inputs

$$= - d11 (h17\ P1 + h18\ P2 + h19\ P3 + h20\ P4 + h21\ q1 + h22\ q2 + h23\ q3 + h24\ q4)$$
$$- d22 (h9\ P1 + h10\ P2 + h11\ P3 + h12\ P4 + h13\ q1 + h14\ q2 + h15\ q3 + h16\ q4)$$
$$- d26 (h1\ P1 + h2\ P2 + h3\ P3 + h4\ P4 + h5\ q1 + h6\ q2 + h7\ q3 + h8\ q4)$$
$$- d29 (h25\ P1 + h26\ P2 + h27\ P3 + h28\ P4 + h29\ q1 + h30\ q2 + h31\ q3 + h32\ q4)$$
$$- (2\ d7\ P3 + d15\ P1 + d17\ P2 + d19\ P4 + b3\ q1 + b8\ q2 + b14\ q3 + b19\ q4)$$

Rearrange

$$= P1 (- d11\ h17 - d22\ h9 - d26\ h1 - d29\ h25 - d15)$$
$$+ P2 (- d11\ h18 - d22\ h10 - d26\ h2 - d29\ h26 - d17)$$
$$+ P3 (- d11\ h19 - d22\ h11 - d26\ h3 - d29\ h27 - 2\ d7)$$
$$+ P4 (- d11\ h20 - d22\ h12 - d26\ h4 - d29\ h28 - d19)$$
$$+ q1 (- d11\ h21 - d22\ h13 - d26\ h5 - d29\ h29 - b3)$$
$$+ q2 (- d11\ h22 - d22\ h14 - d26\ h6 - d29\ h30 - b8)$$
$$+ q3 (- d11\ h23 - d22\ h15 - d26\ h7 - d29\ h31 - b14)$$
$$+ q4 (- d11\ h24 - d22\ h16 - d26\ h8 - d29\ h32 - b19)$$

$$q3' = J49\ P1 + J50\ P2 + J51\ P3 + J52\ P4 + J53\ q1 + J54\ q2 + J55\ q3 + J56\ q4$$

$$q4' = - d H / d P4$$
$$= - (2\ d8\ P4 + d16\ P1 + d18\ P2 + d19\ P3 + d23\ Ui + d27\ Un + d30\ Ue + b9\ q2 + b15\ q3 + b20\ q4)$$

Substitute the optimized control inputs

$$= - d23 (h9\ P1 + h10\ P2 + h11\ P3 + h12\ P4 + h13\ q1 + h14\ q2 + h15\ q3 + h16\ q4)$$
$$- d27 (h1\ P1 + h2\ P2 + h3\ P3 + h4\ P4 + h5\ q1 + h6\ q2 + h7\ q3 + h8\ q4)$$
$$- d30 (h25\ P1 + h26\ P2 + h27\ P3 + h28\ P4 + h29\ q1 + h30\ q2 + h31\ q3 + h32\ q4)$$
$$- (2\ d8\ P4 + d16\ P1 + d18\ P2 + d19\ P3 + b9\ q2 + b15\ q3 + b20\ q4)$$

Rearrange

$$= P1 (- d23\ h9 - d27\ h1 - d30\ h25 - d16)$$
$$+ P2 (- d23\ h10 - d27\ h2 - d30\ h26 - d18)$$
$$+ P3 (- d23\ h11 - d27\ h3 - d30\ h27 - d19)$$
$$+ P4 (- d23\ h12 - d27\ h4 - d30\ h28 - 2\ d8)$$
$$+ q1 (- d23\ h13 - d27\ h5 - d30\ h29 \quad)$$
$$+ q2 (- d23\ h14 - d27\ h6 - d30\ h30 - b9)$$
$$+ q3 (- d23\ h15 - d27\ h7 - d30\ h31 - b15)$$
$$+ q4 (- d23\ h16 - d27\ h8 - d30\ h32 - b20)$$

$$q4' = J57\ P1 + J58\ P2 + J59\ P3 + J60\ P4 + J61\ q1 + J62\ q2 + J63\ q3 + J64\ q4$$

Associatingly, we have eight linear simultaneous otimized differential equations

$$P1' = J1\ P1 + J2\ P2 + J3\ P3 + J4\ P4 + J5\ q1 + J6\ q2 + J7\ q3 + J8\ q4$$

$$P2' = J9\ P1 + J10\ P2 + J11\ P3 + J12\ P4 + J13\ q1 + J14\ q2 + J15\ q3 + J16\ q4$$

$$P3' = J17\ P1 + J18\ P2 + J19\ P3 + J20\ P4 + J21\ q1 + J22\ q2 + J23\ q3 + J24\ q4$$

$$P4' = J25\ P1 + J26\ P2 + J27\ P3 + J28\ P4 + J29\ q1 + J30\ q2 + J31\ q3 + J32\ q4$$

$$q1' = J33\ p1 + J34\ P2 + J35\ P3 + J36\ P4 + J37\ q1 + J38\ q2 + J39\ q3 + J40\ q4$$

$$q2' = J41\ P1 + J42\ P2 + J43\ P3 + J44\ P4 + J45\ q1 + J46\ q2 + J47\ q3 + J48\ q4$$

$$q3' = J49\ P1 + J50\ P2 + J51\ P3 + J52\ P4 + J53\ q1 + J54\ q2 + J55\ q3 + J56\ q4$$

$$q4' = J57\ P1 + J58\ P2 + J59\ P3 + J60\ P4 + J61\ q1 + J62\ q2 + J63\ q3 + J64\ q4$$

and four control inputs.

$$Un = h1\ P1 + h2\ P2 + h3\ P3 + h4\ P4 + h5\ q1 + h6\ q2 + h7\ q3 + h8\ q4$$

$$Ui = h9\ P1 + h10\ P2 + h11\ P3 + h12\ P4 + h13\ q1 + h14\ q2 + h15\ q3 + h16\ q4$$

$$U0 = h17\ P1 + h18\ P2 + h19\ P3 + h20\ P4 + h21\ q1 + h22\ q2 + h23\ q3 + h24\ q4$$

$$Ue = h25\ P1 + h26\ P2 + h27\ P3 + h28\ P4 + h29\ q1 + h30\ q2 + h31\ q3 + h32\ q4$$

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