

Optimal Condition for Maximizing the Blood Flow Velocity in the Elastic Arterial.

Hirayama,H., N, Kitagawa., *Y, Okita and **T, Kazui.

Department of Public Health Asahikawa Medical College
4-5 Nishi Kagura Asahikawa city 078 Japan.

E mail hirayama@asahikawa-med.ac.jp

* Department of Engineering Shizuoka University.

**The First department of Surgery Hamamatsu Medical college.

The present investigation was intended to disclose the influences of longitudinal tethering from the interstitial tissue surrounding the vessel and the dynamic elastic modulus of the wall on the blood flow velocity in circular cylindrical elastic tube with the variable combinations of radius, frequency, phases and geometric configuration within the cross section of the artery. The blood flow was described by Navier-Stokes equation and the dynamic wall motions were expressed by the constitutional equations. The tethering was expressed by numerical parameters by Womersely (1957). The strong tethering and changes in dynamic elastic properties of arterial wall produced a significant changes in pulsatile blood flow velocity distribution in the artery.

Womersely theory, Dynamic elastic arterial wall, Pulsatile blood flow, Longitudinal tethering, .

弾性型動脈における血流速度の最適化条件

平山博史、北川敬之、*沖田善光、**数井輝久

旭川医科大学公衆衛生

*静岡大学工学部

**浜松医科大学第一外科

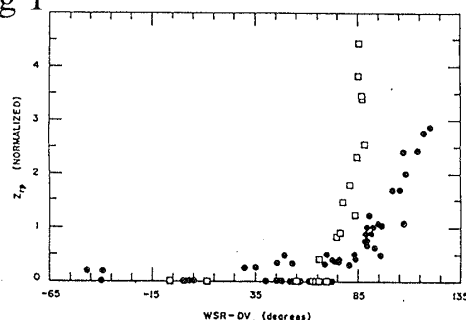
弾性円柱管内における拍動性粘性流体の速度を理論的に解析した。特に血管壁弾性および血管周囲組織からの長軸方向引っ張り力(tethering effects)を考慮した。系はナヴィエ-ストークスの連立微分方程式と血管壁の力学的運動方程式を境界条件を設定して解いた。血管壁周囲組織からの影響はWomersely(1957)の理論に基づいて数値的に設定した。血管壁弾性は生理的値の25%増倍率で表示した。血流速度はある特定の周波数で最大値に達した。血管周囲組織からの長軸方向引っ張り力(tethering effects)および血管壁弾性の変化を血流速度に顕著な影響をおよぼし、このことは血管内径、周波数、位相、血管断面内の位置のかくパラメータを変化させても観察された。動脈血流を弾性円柱管内における拍動流とみなした場合、長軸方向引っ張り力と血管壁弾性は血流速度に大きな影響をおよぼすと推定された。

弾性円柱管、拍動性粘性流、長軸方向引っ張り力、血管壁弾性 Womersely理論.

1. Introduction

The shear stress analysis was the most attractive investigation due to its close relation to arteriosclerosis. The functional relation between the elastic modulus and shear rate has been investigated by many researches. Fig 1 shows an example of normalized reversal peak wall shear rate versus the phase angle between wall shear rate and diameter variation (an index of wall elasticity) (WSR-DV) filled circles inside wall for a curved tube and squares : straight tube. (Klanchar. Cir. Res. vol 66.pp 1624. 1990). These works, however, did not involve precise description of the influences from the surrounding tissues. Present paper affords a mathematical method for analyzing the blood flow velocity in terms of tethering effects.

Fig 1



2. Mathematical method

1. Blood motion in vessel .

In actual arterial system, the pulse wave velocity C is sufficiently fast and the wave length is long enough to satisfy (a ; internal radius, ω ; angular velocity)

$$a \omega / C \ll 1$$

Above condition permits to neglect the non linear terms in the Navier-Stokes equations. Thus blood flow motions are described by linearized Navier-Stokes differential equation for radial direction

$$\frac{\partial V}{\partial t} - \frac{\partial P}{\rho \partial r} = \frac{\mu}{\rho} \left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial z^2} \right] \quad (1)$$

and that in axial direction is expressed by

$$\frac{\partial W}{\partial t} - \frac{\partial P}{\rho \partial z} = \frac{\mu}{\rho} \left[\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} - \frac{\partial^2 W}{\partial z^2} \right] \quad (2)$$

where, V : blood flow velocity in the radial direction. W : blood flow velocity in axial direction. P : input pressure, r : radial coordinate, z : axial coordinate, ν : kinematic viscosity, ρ : blood density. The law of conservation is expressed by

$$\frac{\partial (rV)}{r \partial r} = \frac{\partial W}{\partial z} \quad (3)$$

2 . The wall tension, elastic modulus and displacements relations.

The arterial wall is non linear and the " Large deformation theory " should be preferable. In actual (in vivo) arterial system, however the stress-strain-displacement relations can be regarded linear. Since in vivo state, arterial wall are dominated by potent tethering from surrounding tissue (Patel 1967) and the wall displacement due to pulsatile pressure is small enough to apply the linear deformation theory (Goldeneizer, Love). The constitutive dynamical relations among the Tension operating on the arterial wall (T_θ : wall tension in circumferential direction, T_t :

wall tension in longitudinal direction), the dynamic elastic modulus of arterial wall (E_θ : dynamic elastic modulus in circumferential direction, E_t : dynamic elastic modulus in longitudinal direction) and wall displacements (ξ : axial direction, η : radial direction) are

$$T_\theta - T_{\theta 0} = E_\theta h / (1 - \sigma_\theta \sigma_t) \eta / a + E_\theta h \sigma_t / (1 - \sigma_\theta \sigma_t) \partial \xi / \partial z \quad (4)$$

$$T_t - T_{t0} = E_t h \sigma_\theta / (1 - \sigma_\theta \sigma_t) \eta / a + E_t h / (1 - \sigma_\theta \sigma_t) \partial \xi / \partial z \quad (5)$$

Here, $T_{\theta 0}$, T_{t0} are the initial values of the wall tensions T_θ , T_t respectively. h is wall thickness, σ_t , σ_θ are the Poisson ratios in longitudinal and circumferential directions.

3. The dynamical equations for wall motion.

To describe the wall motion realistically as possible, we have incorporated the Tethering effect caused by surrounding interstitial tissue and the tensions T_θ , T_t operating on the wall. By putting M_0 : the inertia (mass effect) of surrounding tissue, C_l : the viscous coefficient, K_l the elastic coefficient of effects from surrounding tissue in longitudinal direction. The C_r and K_r are those in radial direction. Then the wall motion in longitudinal direction is described by

$$M_0 \frac{\partial^2 \xi}{\partial t^2} + C_l \frac{\partial \xi}{\partial t} + K_l \xi = -\mu \left(\frac{\partial W}{\partial r} + \frac{\partial V}{\partial z} \right)_{r=a} - \frac{\partial \eta}{\partial z} \frac{(T_{t0} - T_{\theta 0})}{a} + \frac{\partial T_t}{\partial z} \quad (6)$$

The wall motion in radial direction is expressed by

$$M_0 \frac{\partial^2 \eta}{\partial t^2} + C_r \frac{\partial \eta}{\partial t} + K_r \eta = \frac{\partial V}{\partial r(r=a)} + \frac{\eta T_{\theta 0}}{a^2} + \frac{(T_{t0} - T_{\theta 0})}{a} \frac{\partial^2 \eta}{\partial z^2} + \frac{T_{t0}}{\partial z^2} \quad (7)$$

4. The Boundary conditions.

Since there is no slippage between blood cells and the internal surface of arterial wall, the velocities of blood cells and the wall motion (the rate of change in displacement) are equivalent at the inner surface of the arterial wall. Thus

$$V(r=a) = \partial \eta / \partial t \text{ ----- B1}$$

$$W(r=a) = \partial \xi / \partial t \text{ ----- B2}$$

In the present investigation, all the systems are linearized. Thus the solutions of the equations from (1) and (7) are approximated to be linear cyclic pattern. Since the arterial pulse wave also can be expressed by linear summation of harmonic waves, they can be expressed as

$$V = V(r) e^{i\omega(t-z/C)} \text{ -----(8)}$$

$$W = W(r) e^{i\omega(t-z/C)} \text{ -----(9)}$$

$$P = P(r) e^{i\omega(t-z/C)} \text{ -----(10)}$$

Here, $\omega (= 2\pi f)$ is angular velocity, C is the unknown pulse wave velocity. α is the Womersley frequency parameters and its modified forms are

$$\alpha_0^2 = i^3 a^2 \omega / \nu = i^3 \alpha^2,$$

$$\beta_0 = i a \omega / C = i \beta$$

$$\delta_0 = (\alpha_0^2 + \beta_0^2) / 2, \quad \varepsilon = \beta_0 / a,$$

By these settings, the simultaneous equations (1), (2) and (3) are solved analytically and the solutions

$$V = [-A_1 \beta_0 a / (\mu \alpha_0^2) J_1(\varepsilon r) + A_2 \beta_0 J_1(\delta_0 r/a) / (\alpha_0 J_0(\alpha_0))] e^{i\omega(t-z/C)} \text{ -----(11*)}$$

$$W = [-A_1 \beta_0 a / (\mu \alpha_0^2) J_0(\varepsilon r) + A_2 \delta_0 J_0(\delta_0 r/a) / (\alpha_0 J_0(\alpha_0))] e^{i\omega(t-z/C)} \text{ -----(12*)}$$

$$P = A_1 J_0(\beta_0 r/a) e^{i\omega(t-z/C)} \text{ -----(13)}$$

Here A_1, A_2 are unknown coefficients to be determined. From the linear assumption, $\alpha_0^2 \gg \beta_0^2$, thus, $\alpha_0^2 + \beta_0^2 = \alpha_0^2$. Normalizing by the radius, "a" we set, $y = r/a$, $\omega = i n/a$, above expressions are simplified to

$$V = [-A_1 k a / (\mu i^3 \alpha^2) J_1(ky) + A_2 \beta_0 J_1(\alpha_0 y) / (\alpha_0 J_0(\alpha_0))] e^{i\omega(t-z/C)} \text{ -----(11)}$$

$$W = [-A_1 i \omega a / (\mu C i^3 \alpha^2) J_0(ky) + A_2 J_0(\alpha_0 y) / (\alpha_0 J_0(\alpha_0))] e^{i\omega(t-z/C)} \text{ -----(12)}$$

The wall displacements η, ξ are assumed to be synchronized with pressure and blood flow velocity

$$\eta = A_3 e^{i\omega(t-z/C)} \text{ -----(14)}$$

$$\xi = A_4 e^{i\omega(t-z/C)} \text{ -----(15)}$$

Here A_3, A_4 are unknown coefficients to be determined. Substituting equations (11), (12) and (13) into the boundary conditions B1 and B2, following algebraic equations are obtained.

$$-A_1 \beta_0 a J_1(\beta_0) / (\mu \alpha_0^2) + A_2 \beta_0 J_1(\delta_0) / (\alpha_0 J_0(\alpha_0)) - i \omega A_3 = 0 \text{ -----(16)}$$

$$-A_1 \beta_0 a J_0(\beta_0) / (\mu \alpha_0^2) + A_2 \delta_0 J_0(\delta_0) / (J_0(\alpha_0) \alpha_0) - i \omega A_4 = 0 \text{ -----(17)}$$

The wall motion equations are, from the equations (4), (5) and by putting

$$\gamma_1 = E_t / E_\theta \text{ -----(G1)}$$

$$\gamma_2 = \sigma_t / \sigma_\theta \text{ -----(G2)}$$

$T_t, T_\theta, E_t, \sigma_t$ are eliminated. As a result, the initial tension T_0 , the circumferential elastic modulus and circumferential Poisson ration are expressed in the explicit form. Thus they are

$$M_0 \frac{\partial^2 \xi}{\partial t^2} + C L \frac{\partial \xi}{\partial t} + K I \xi = -\mu \left[\frac{\partial W}{\partial r} + \frac{\partial V}{\partial z(r=a)} \right]$$

$$+ \frac{\partial \eta}{\partial z} \frac{(T_{t0} - T_{\theta 0})}{a} + \frac{\gamma_1 E_\theta h}{1 - \gamma_2 \sigma_\theta^2}$$

$$+ \frac{\partial^2 \xi}{\partial z^2} \frac{\sigma_\theta}{a} + \frac{\partial \eta}{\partial z} \text{ -----(18)}$$

$$M_0 \frac{\partial^2 \eta}{\partial t^2} + C r \frac{\partial \eta}{\partial t} + K r \eta = [P - 2\mu \frac{\partial V}{\partial r(r=a)}] + \frac{\eta T_{\theta 0}}{a^2}$$

$$+ T_{t0} \frac{\partial^2 \eta}{\partial z^2} + \frac{E_\theta h}{1 - \gamma_2 \sigma_\theta^2} \frac{\gamma_2 \sigma_\theta}{a} \frac{\partial \xi}{\partial z} + \frac{\eta}{a^2} \text{ -----(19)}$$

Substituting equations (11)-(15) into equations (18)(19) results in the algebraic equations for A_1 to A_4

$$\begin{aligned} & A_1 [J_0(\beta_0) + \beta_0^2 / (\alpha \alpha_0^2) ((J_0(\beta_0) - J_2(\beta_0)) \\ & - A_2 \mu \beta_0 / (\alpha \alpha_0 J_0(\alpha_0)) \\ & [J_0(\delta_0) - J_2(\delta_0)] \\ & + A_3 [(T_{t0} \beta_0^2 + T_{\theta 0} - E \theta h / (1 - \gamma_2 \sigma \theta^2)) / a^2 \\ & + \omega^2 (M_0 - i C r / \omega - K r / \omega^2)] \\ & + A_4 E \theta h / (1 - \gamma_2 \sigma \theta^2) \gamma_2 \sigma \theta \beta_0 / a^2 = 0 \quad \text{-----(20)} \end{aligned}$$

$$\begin{aligned} & - A_1 2 \beta_0^2 / (\alpha \alpha_0^2 J_1(\beta_0)) \\ & + A_2 \mu (\alpha \alpha_0^2 + 2 \beta_0^2) J_1(\delta_0) / (\alpha \alpha_0 a J_0(\alpha_0)) \\ & - A_3 \beta_0 / a^2 (\gamma_1 E \theta h \sigma \theta / (1 - \gamma_2 \sigma \theta^2) \\ & + T_{t0} - T_{\theta 0}) \\ & + A_4 [\gamma_1 E \theta h \beta_0^2 / ((1 - \gamma_2 \sigma \theta^2) (a^2)) \\ & + \omega^2 (M_0 - i C l / \omega - K l / \omega^2)] = 0 \quad \text{-----(21)} \end{aligned}$$

Because in actual biological system, pulse wave velocity is sufficiently larger than blood flow velocity $a \omega / C < 1$, above Bessel functions are approximated as following

$$\begin{aligned} J_0(\beta_0) &= 1, J_1(\beta_0) = \beta_0/2 \\ J_2(\beta_0) &= \beta_0^2/8, \end{aligned}$$

Then equations (16)(17)(20)(21) are simplified to following forms

$$-\beta_0^2 a A_1 / (2 \mu \alpha_0^2) + \beta_0 F_{10} A_2 / 2 - i \omega A_3 = 0 \quad \text{-----(22)}$$

$$-\beta_0 a A_1 / (2 \mu \alpha_0^2) + A_2 - i \omega A_4 = 0 \quad \text{-----(23)}$$

$$\begin{aligned} & A_1 - \mu \beta_0 (2 - F_{10}) A_2 / a \\ & + A_3 [(T_{t0} \beta_0^2 / a^2 + T_{\theta 0} / a^2 - E \theta h / ((1 \\ & - \gamma_2 \sigma \theta^2) a^2) + \omega^2 (M_0 - i C r / \omega - K r / \omega^2)] \\ & + A_4 E \theta h / (1 - \gamma_2 \sigma \theta^2) (\gamma_2 \sigma \theta \beta_0) / a^2 = 0 \quad \text{-----(24)} \end{aligned}$$

$$\begin{aligned} & -\beta_0^3 A_1 / \alpha_0^2 + \mu \alpha_0^2 F_{10} A_2 / (2 a) \\ & - A_3 \beta_0 [(\gamma_1 E \theta h \sigma \theta / (1 - \gamma_2 \sigma \theta^2) \\ & + T_{t0} - T_{\theta 0})] / a^2 \\ & + A_4 [\gamma_1 E \theta h / (1 - \gamma_2 \sigma \theta^2) \beta_0^2 / (a^2) \\ & + \omega^2 (M_0 - i C l / \omega - K l / \omega^2)] = 0 \quad \text{-----(25)} \end{aligned}$$

where $F_{10} = 2 J_1(\alpha_0) / (\alpha_0 J_0(\alpha_0))$

The equations (22) to (24) are simultaneous algebraic equations but all the right sides are 0. Thus only three unknown are expressed by a given A_n . The only relation we should seek to obtain blood flow velocity is the A_1 and A_2 . Here for the simplicity, we assume that the tethering in axial direction is considerably stronger than

radial one. The visco-elastic modulus of arterial wall were set to be isotropic. Thus we set $Edt = Ed\theta = E$ and $dt = d\theta$. Here we put

$$\lambda = h/R H/h (1 - KL/\omega^2),$$

$$\nu = \lambda E / (1 - \sigma^2) 1 / (\rho C^2)$$

Solving above simultaneous equation for A_1 and A_2

$$\begin{aligned} A_2/A_1 &= [2 / (\nu (F_{10}(\alpha) - 2 \sigma)) \\ & - (1 - 2 \sigma) / (F_{10}(\alpha) - 2 \sigma)] / (\rho_0 C) \end{aligned}$$

Now putting

$$\chi = \rho_0 C A_2/A_1$$

Then, the axial flow velocity can be obtained

$$W = A_1 / (\rho_0 C) [1 + \chi J_0(\alpha^{3/2} y) / J_0(\alpha^{3/2})] e^{i w(t-z/C)}$$

On the other hand, the input pressure gradient can be expanded by Fourier expansion at any arbitrary points on the axial coordinate. Thus it can be expanded at $z=0$ by

$$-\partial p / \partial z = A e^{i w t}$$

On the other hand, the input pressure can be obtained from linearized Navier-Stokes equation

$$P = A_1 J_0(i w r / C) e^{i w(t-z/C)}$$

Thus

$$-\partial (A_1 J_0(i w r / C) e^{i w(t-z/C)}) / \partial z = A e^{i w t}$$

From this relation, we get $A_1 i \omega / C = A$. Utilizing the definition of the frequency parameter

$$a^2 \omega / \nu = \alpha^2, \text{ and the relation}$$

$$A_1 / \rho_0 C = A R^2 / (i \mu \alpha^2)$$

We have the axial flow velocity

$$W = A R^2 / (i \mu \alpha) [1 + \chi J_0(\alpha^{3/2} y) / J_0(\alpha^{3/2})] e^{i w(t-z/C)}$$

Where the τ_θ , τ_t are non dimensional parameter defined as

$$\tau_\theta = T_{\theta 0} / [E \theta h / (1 - \gamma_2 \sigma \theta^2)] \quad \text{-----(28)}$$

$$\tau_t = T_{t0} / [E \theta h / (1 - \gamma_2 \sigma \theta^2)] \quad \text{-----(29)}$$

$$\text{Further} \quad k = M - i C / \alpha^2 + K / \alpha^4$$

in which the effects each components from the surrounding tissue are normalized as

$$\begin{aligned} M &= M_0 / (a \rho), C = C L a / \mu, \\ K &= K L a^3 \rho / \mu^2 \end{aligned}$$

According to Ataback (1968), the magnitude of longitudinal tethering was set to be

$$M=0.5, CL=10 \text{ and } KL=100$$

Fig 2 Effect of frequency and longitudinal tethering

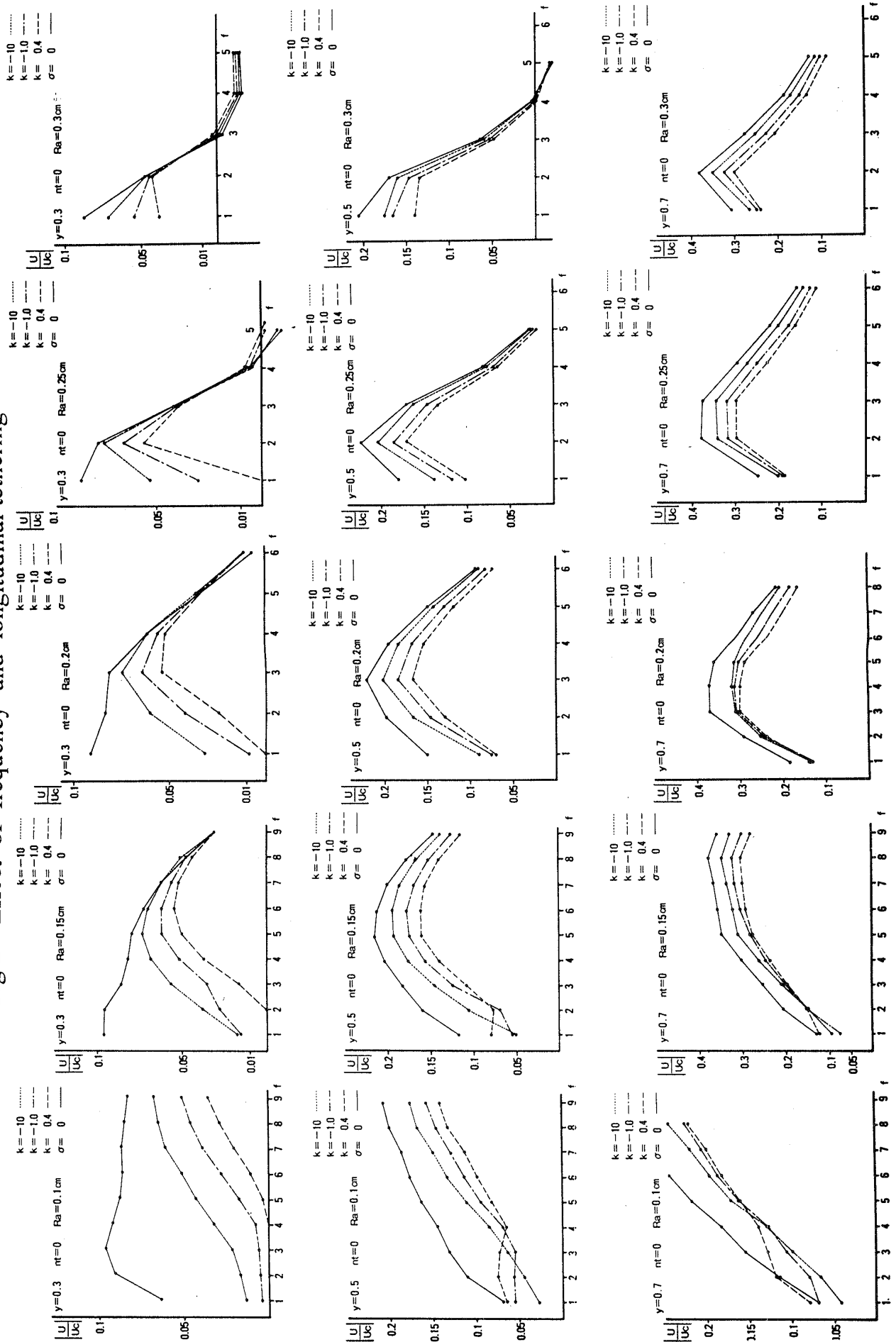
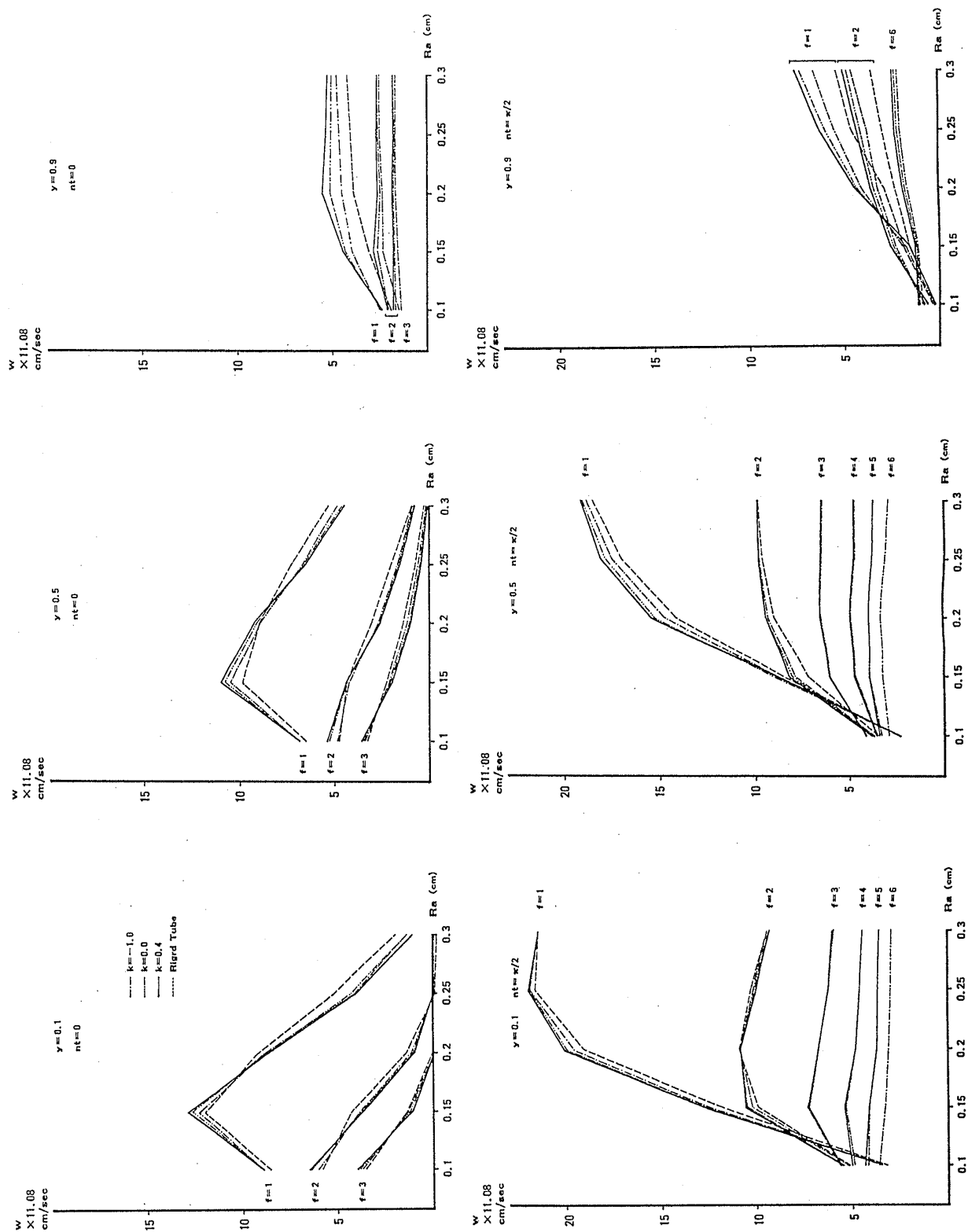


Fig 3 Flow velocity vs Ra effects of tethering



3. Results.

Fig 2 show the normalized axial blood flow velocity U/U_c where U_c is the center-line velocity. From the left column to the right column, the internal radius R_a was set to 0.1 cm, 0.15 cm, 0.2 cm, 0.25 cm and 0.3 cm. From the top row to the bottom row, the radial position y on the cross section of arterial plane was $y=0.3$ (near the central position), $y=0.5$ (at the middle position) and $y=0.7$ (near the wall). The phase was uniformly set at $nt=0.0$. In each figure, we have examined the effects of changes in tethering effects expressed by $k=-10$ (strong tethering), $k=-1.0$ and $k=0.4$ (the weak tethering near to free ending). Another change was imposed on the Poisson ratio to set 0.0001.

Fig 3 shows the absolute values (cm/sec) of blood flow velocities as a function of radius. The top row shows those when the phase is $nt=0$ and bottom $nt=\pi/2$. Data denoted by --- are those for rigid tube.

Fig 4 shows the normalized axial blood flow velocity when the dynamic elastic modulus has been increased by 25 %.

4. Discussion.

The present investigation reported the influence of longitudinal tethering from the interstitial tissue surrounding the vessel and dynamic elastic modulus of the wall under the variable combinations of radius, frequency, phases and geometric configuration within the cross section of the artery. We discuss several assumptions required for solving the equations.

1. Linearization

Arteries in biological system are essentially non linear. Therefore to express the blood flow in solvable form, the linearization was unavoidable. The non linear terms in Navier- Stokes equations can be neglected when the pulse wave velocity C is sufficiently large so as to satisfy $\omega \ll C$. Indeed, the pulse wave velocity is 4 to 6 m/sec in aorta and 6-12 m/sec in femoral artery (Li 1980) while the radius of aorta is 0.45 cm and of femoral artery is 0.2 cm to 0.27 cm (Hirayama 1988). This condition is applicable to arteries smaller than lower abdominal aorta such as lower part of iliac artery, femoral artery. From the theoretical analysis of pulse wave velocity LI (1981), linearized model can approximate. The tapering of vessel can be neglected (Li, 1981, Hirayama 1988).

2. Setting of tethering effects.

In the investigations for artificial vessel or theoretical analysis of blood flow in arteries, the effects of the tethering by the surrounding interstitial tissues were not taken into consideration. Several theoretical studies (Atabek 1968, Jones 1971) revealed the effects of tethering on the pulse wave velocities. By the simulation study (Patel 1967), the effects of the tethering from the surrounding interstitial tissue can be decomposed to inertial, viscous and elastic effects. Certainly, more complex models were examined, the simplest three model can express at least 6 Hz which is sufficient for physiological blood flow. This modeling, however is only for longitudinal component and radial one has yet been measured because of its difficulty. Theoretical

investigation disclosed that the effects of radial tethering on the pulse wave velocity were serious (Jones 1972). Thus for the simplicity, we set the radial component of tethering effect equivalent to longitudinal one.

3. Effects of frequency dependency of visco elastic properties of arterial wall.

The dynamic visco-elastic modulus of arterial wall heavily depends on the frequency (Learoyd 1967). For the human arterial system below 35 years old, the dynamic elastic modulus E_d of artery increases rapidly from $F=1$ Hz to 2 Hz. For example, the E_d of thoracic aorta increases 40 %, of abdominal aorta 10 %, of iliac artery 20 % and of femoral artery 50 %. For the range of $F>3$ Hz, the E_d increases very gradually. Simultaneously, the product of wall viscosity and angular velocity $\eta \omega$ increases 20 % during $1 < F < 2$ (Hz) (Learoyd 1967). To incorporate these frequency dependencies of arterial wall properties to the present analysis, however, makes the understanding of the effects of results complex. Thus we have set them independent.

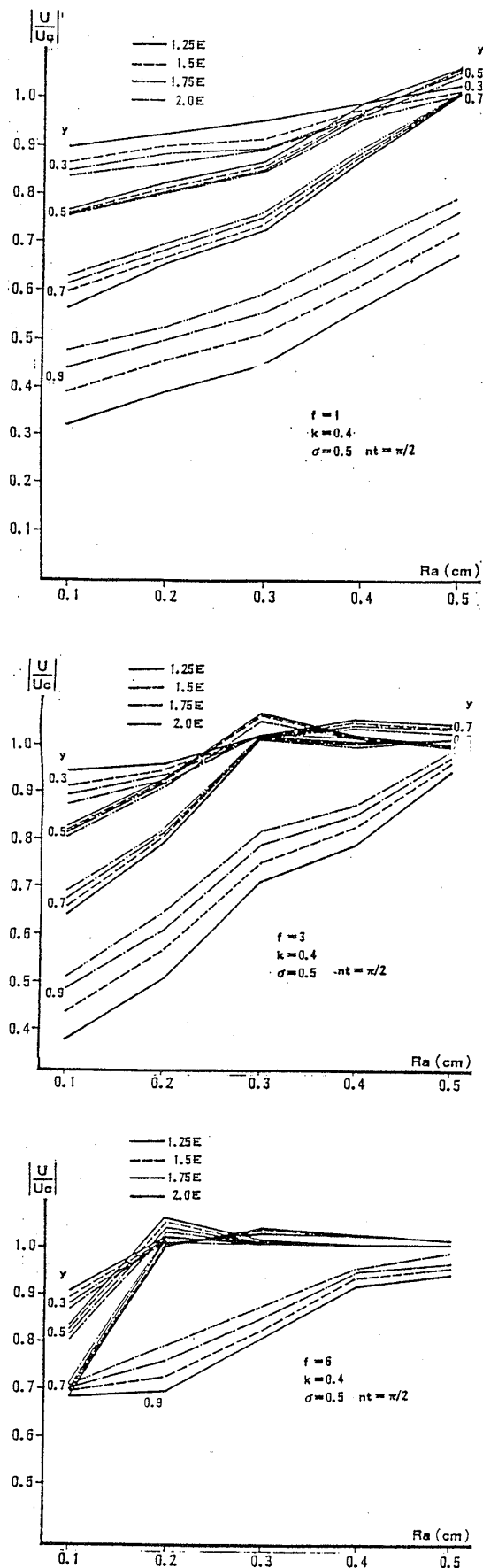
5. Conclusion

1. The frequency that produced the peak blood flow velocity at small radius ranged in high frequency while that at large radius ranged in low frequency. The arterial system may be organized effectively to maximize the velocity under the physiological conditions.
2. Dynamic elastic modulus must play a significant role in physiological blood flow velocity.

6. References

1. Li, J.K., Melbin, R., Riffle, A and Noordergraaf, A. : Pulse wave propagation. *Cir. Res.* vol 49. pp 441-452. 1981.
2. Bergel D.H. : The dynamic elastic properties of the arterial wall. *J. Physiol.* vol 156. pp 465-472. 1961.
3. Learoyd, B.M and Taylor, M.G. : Alteration with age in the visco-elastic properties of human arterial system. *Cir. Res.* vol 28. pp 278-292. 1966.
4. Atabek H.B. : Wave propagation through a viscous fluid contained in a tethered, initially stressed orthotropic elastic tube. *Biophysical Journal.* vol 8. pp 626-649. 1968.
5. Jones E, Anliker M and Chang I. : Effects of viscosity and constraints on the dispersion and dissipation of waves in large blood vessels. *Biophysical Journal* vol 11. pp 1085-1120. 1971.
6. Attinger F M L. : Two dimensional in vitro studies of femoral arterial walls of the dog. *Cir. Res.* vol 22. pp 829-840. 1968.
7. Dobrin P B and Doyle J M. : Vascular smooth muscle and the anisotropy of dog carotid artery. *Cir. Res.* vol 27. pp 105-119. 1970.
8. Cox. R.H. : Pressure dependence of the mechanical properties of arteries in vivo. *Am J. Physiol.* vol 229. pp 1371-1375. 1975.
9. Patel D J., Janicki J S., Vaishnav R N and Young J T. : Dynamic anisotropic visco-elastic properties of the aorta in living dogs. *Cir. Res.* vol 32. pp 93-107. 1973.
10. Womersley J.R. An elastic tube theory of pulse transmission and oscillatory flow in arteries. WADC. Technical Report. 56. 614. 1957.

Fig 4 Effect of dynamic elastic modulus



APPENDIX.

The $F_{10}(\alpha)$ is given in the Table calculated by Womersely (1958) and can be put as

$$1 - F_{10}(\alpha) = fR + i fJ$$

where $i^2 = -1$ Therefore

$$\begin{aligned} F_{10} [(\gamma_1 + \gamma_2)(\sigma d \theta) + \tau t - \tau \theta/2 - 1/2] - 2 \gamma_1 \\ = (1 - fR - i fJ) [(Re \gamma_1 + Re \gamma_2 + i (Im \gamma_1 + Im \gamma_2)) \\ (\sigma d \theta + i \sigma j \theta) + (Re \tau t) + (Im \tau t) - (Re \tau \theta)/2 \\ - 1/2 - i (Im \tau \theta)/2] - 2 (Re \gamma_1 + i Im \gamma_1) \\ = [(Re \gamma_1 + Re \gamma_2)(\sigma d \theta) - (Im \gamma_1 + Im \gamma_2) \\ (\sigma j \theta) + i \{ (Re \gamma_1 + Re \gamma_2)(\sigma j \theta) + (Im \gamma_1 \\ + Im \gamma_2)(\sigma d \theta) \} + (Re \tau t) - (Re \tau \theta)/2 - 1/2 \\ + i (Im \tau t) - (Im \tau \theta)/2] (1 - fR - i fJ) \\ - 2 (Re \gamma_1 + i Im \gamma_1) \end{aligned}$$

Putting

$$\begin{aligned} ReC_{10} &= (Re \gamma_1 + Re \gamma_2)(\sigma d \theta) \\ &\quad - (Im \gamma_1 + Im \gamma_2)(\sigma j \theta) \\ ImC_{10} &= (Re \gamma_1 + Re \gamma_2)(\sigma j \theta) \\ &\quad + (Im \gamma_1 + Im \gamma_2)(\sigma d \theta) \\ ReC_{11} &= (Re \tau t) - (Re \tau \theta)/2 - 1/2 \\ ImC_{11} &= (Im \tau t) - (Im \tau \theta)/2 \end{aligned}$$

Then

$$= (ReC_{10} + ReC_{11} + i (ImC_{10} + ImC_{11})) (1 - fR - i fJ) - 2 (Re \gamma_1 + i Im \gamma_1)$$

Putting

$$\begin{aligned} ReC_{12} &= ReC_{10} + ReC_{11} \\ ImC_{12} &= ImC_{10} + ImC_{11} \\ &= ReC_{12} (1 - fR) + ImC_{12} fJ - 2 Re \gamma_1 \\ &\quad + i [ImC_{12} (1 - fR) - ReC_{12} fJ] + i (-2 Im \gamma_1). \end{aligned}$$

Putting

$$\begin{aligned} ReC_{13} &= ReC_{12} (1 - fR) + ImC_{12} fJ - 2 Re \gamma_1 \\ ImC_{13} &= ImC_{12} (1 - fR) - ReC_{12} fJ - 2 Im \gamma_1 \end{aligned}$$

Therefore

$$= 2 [ReC_1 - i ImC_1] [ReC_9 + ReC_{13} + i (ImC_9 + ImC_{13})] / (ReC_1^2 + ImC_1^2)$$

Putting

$$\begin{aligned} ReC_{14} &= ReC_9 + ReC_{13} \\ ImC_{14} &= ImC_9 + ImC_{13} \\ &= 2 [ReC_1 - i ImC_1] [ReC_{14} + i ImC_{14}] \\ &\quad / (ReC_1^2 + ImC_1^2) \end{aligned}$$

Putting

$$\begin{aligned} ReB_2 &= (ReC_1 ReC_{14} + ImC_1 ImC_{14}) / (ReC_1^2 + ImC_1^2) \\ ImB_2 &= (ReC_1 ImC_{14} - ImC_1 ReC_{14}) / (ReC_1^2 + ImC_1^2) \end{aligned}$$

About $F_{10} + 2k$

$$= ReB_3 + i ImB_3 \quad \text{and} \quad 1 - F_{10} = fR + i fJ$$

Then

$$\begin{aligned} F_{10} + 2k &= 1 - fR - i fJ + 2k + i 2k \\ &= 1 - fR + 2 ReB_3 + i (-fJ + 2 ImB_3) \\ &= ReB_3 + i ImB_3 \end{aligned}$$