

Analysis of Temporal Processing for the Poisson Inputs by More than Two Processing Neurons.

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A mathematical method has been introduced for analyzing the neural signal processing mechanisms which exists more than two. The basic concept originates in the work of Saaty (1960). We assumed that Impulses arrive by a Poisson distribution which is independently and identically distributed according to $1 - \exp(-\lambda t)$. On completing the processing, the impulse departs from the system by a sequence of independent and identically distributed random variables for $1 - \exp(-\mu t)$. The processing time courses are also identical for processing channels. For the initial condition, we set that there are i units preexisting in the system at $t=0$. The probabilistic differential-difference equations were solved by generating function technique. The present method will be available for analyzing the simultaneous neural processing system.

Neural signal, Impulse processing, Poisson series, Differential-difference equations

複数個の神経細胞に対するポアソン型入力による発射過程の過渡的経過解析

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標的神経細胞が複数個ある場合の神経インパルス処理の過渡的過程を解析する方法を Saaty (1960) の研究にもとずいて紹介した。入力としての神経インパルスはポアソン型時系列(到着確率: λ)とした。これから入力をうける受容神経細胞は2個ある場合を解析した。神経組織の任意の時刻においてそこに n 個のインパルスが処理されている(処理確率: μ) 確率 $P_n(t)$ に関する連立微分差分確率方程式を母関数展開およびラプラス積分変換を用いて解いた。本研究では簡単のため処理する神経は2個とした。 $P_0(t)$ は初期条件、すなわちすでに処理が進行しているインパルスの数によって大きく変化した。インパルス到着確率: λ が減少すると また 処理確率: μ が増加すると $P_0(t)$ は増加した。本研究方法を発展させれば多数の神経細胞が同時に神経情報を処理する場合の時間経過を分析するのに有用である。

神経細胞. インパルス処理. ポアソン型時系列. 連立微分差分確率方程式. 母関数展開

1. Introduction.

Neural impulses are processed not usually by only one neuron. Since the axons have lots of branches, the neural information will be processed simultaneously by more than one neurons. The present paper introduce temporal change in neural processing for two processors.

2. Mathematical process.

We assumed that, 1. Impulses arrive with the rate of λ , 2. and are processed with the rate of μ under the c processors, 3 the initial condition, i preexisting units at $t=0$. The forward equations are solved by using the generating function $P(z,t) = \sum P_n(t) z^n$

$$\partial P_0(t)/\partial t = -\lambda P_0(t) + \mu P_1(t) \quad (1-a)$$

$$\partial P_n(t)/\partial t = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad (1-b) \quad (1 \leq n < c)$$

$$\partial P_n(t)/\partial t = -(\lambda + c\mu) P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t) \quad (1-b) \quad (c \leq n) \quad (1-c)$$

3. Results.

For the simplest case, we analyzed two processing servers. We have separated the terms of $P_0(t)$ into the first two terms as represented $P_0(1,2)$ in the figures and the last series as $P(3)$ in the figures. Thus, the net probability is the sum of these two terms.

Fig 1 shows the temporal changes in $P_0(t)$ for $(\lambda, \mu, i) = (0.9, 0.1, i=0,1,3)$. With increase in the number of the preexisting impulses i , the probability of $P_0(t)$ is reduced.

Fig 2 shows the temporal changes in $P_0(t)$ for $(\lambda, \mu, i) = (0.5, 0.1, i=0,1,3)$. As λ decreased, $P_0(t)$ is

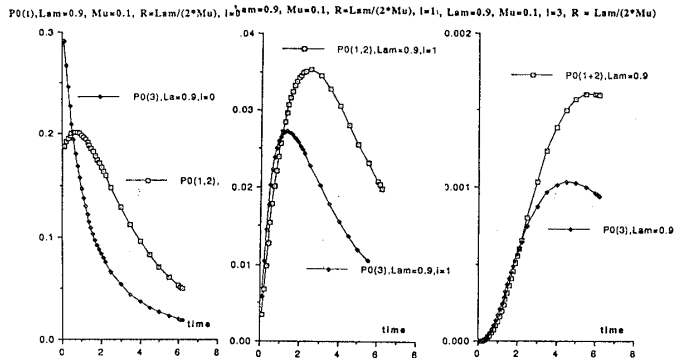


Fig 1 $(\lambda, \mu, i) = (0.9, 0.1, i=0,1,3)$.

Fig 2 $(\lambda, \mu, i) = (0.5, 0.1, i=0,1,3)$.

Fig 2

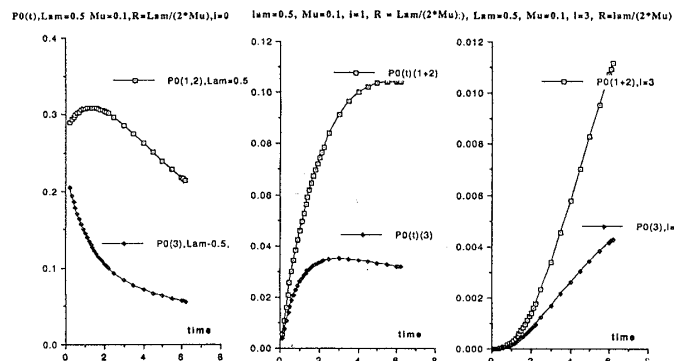
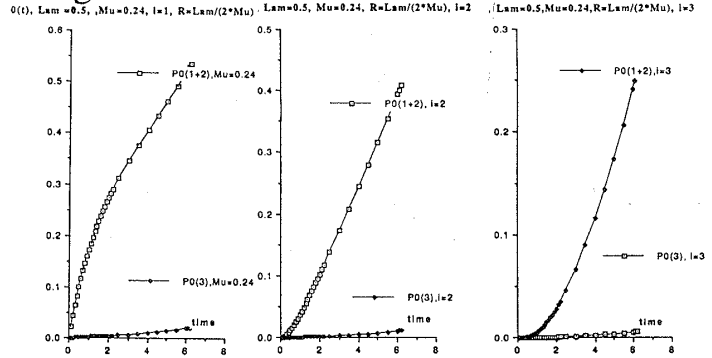


Fig 3 $(\lambda, \mu, i) = (0.5, 0.24, i=1,2,3)$.



increased.

Fig 3 shows the temporal changes in $P_0(t)$ for $(\lambda, \mu, i) = (0.5, 0.24, i=1,2,3)$. As μ increased, $P_0(t)$ increased.

4. Conclusion.

We have introduced a mathematical method for processing neural impulses under many signal processors particularly for the case of two processors.

6. Reference

1. Saaty, T.L. Operations research. 1960. pp 755-772.

7. APPENDIX.

$$P_0^* = \frac{\alpha_2^{i+1}}{\lambda + 2\mu} \left[\frac{1}{1 - \alpha_2} + \frac{\lambda \alpha_2 + \lambda - 2\mu}{4\mu - 2\mu \alpha_2 + \lambda \alpha_2^2} \right]$$

$$= \frac{1}{\lambda + 2\mu} \left[\sum_{k=0}^{\infty} \alpha_2^{k+i+1} + \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{\lambda}{2} \right)^m \left(\frac{\lambda}{2} \right)^k \alpha_2^{m+k+i+2} \right. \\ \left. + \frac{\lambda - 2\mu}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \left(\frac{\lambda}{2} \right)^m \left(\frac{\lambda}{2} \right)^k \alpha_2^{m+k+i+1} \right].$$

$$P_0(t) = \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu)t} \left[\sum_{k=0}^{\infty} (\sqrt{\rho})^{-(k+i+1)} (k+i+1) I_{k+i+1}(2\sqrt{2\lambda\mu}t) \right. \\ \left. + \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-\rho)^m \left(\frac{\lambda}{2} \right)^k (\sqrt{\rho})^{-(m+k+i+2)} (m+k+i+2) \right. \\ \left. \cdot I_{m+k+i+2}(2\sqrt{2\lambda\mu}t) + \left(\frac{\lambda}{\mu} - \frac{\lambda}{2} \right) \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m (-\rho)^m \left(\frac{\lambda}{2} \right)^k \right. \\ \left. \cdot (\sqrt{\rho})^{-(m+k+i+1)} (m+k+i+1) I_{m+k+i+1}(2\sqrt{2\lambda\mu}t) \right].$$

If we use L^{-1} to denote the inverse Laplace transform, we have

$$L^{-1}(\alpha_1^{-n}) = (2\sqrt{2\lambda\mu})^{-n} [2\sqrt{2\lambda\mu} n t^{-1} I_{n-1}(2\sqrt{2\lambda\mu}t) - n(n+1) t^{-2} I_n(2\sqrt{2\lambda\mu}t)],$$

$$L^{-1}(\alpha_1^{-n}) = (2\sqrt{2\lambda\mu})^{-n} n t^{-1} I_n(2\sqrt{2\lambda\mu}t).$$

Then:

$$P_1(t) = \frac{e^{-(\lambda+2\mu)t}}{\mu(\lambda+2\mu)} \left[\sum_{k=0}^{\infty} \rho^{-(k+i+1)} L^{-1}[(s+\lambda) \alpha_1^{-(k+i+1)}] \right. \\ \left. + \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m \left(\frac{\lambda}{2} \right)^k \rho^{-(m+k+i+2)} L^{-1}[(s+\lambda) \alpha_1^{-(m+k+i+2)}] \right. \\ \left. + \frac{\lambda - 2\mu}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \rho^m \left(\frac{\lambda}{2} \right)^k \rho^{-(m+k+i+1)} L^{-1}[(s+\lambda) \alpha_1^{-(m+k+i+1)}] \right]. \quad (38)$$

If we write the second component of the Laplace transform of $P_n(t)$ in the form

$$[(s+\lambda)/\lambda \alpha_1^{-n} - (s+\lambda)/\lambda \alpha_1^{n+1}] P_0^*(s),$$

we have:

$$P_n(t) = e^{-(\lambda+2\mu)t} \left[(\sqrt{\rho})^{n-i} I_{n-i}(2\sqrt{2\lambda\mu}t) + (\sqrt{\rho})^{n-i-1} I_{n+i+1}(2\sqrt{2\lambda\mu}t) \right. \\ \left. + (1-\rho) \rho^n \sum_{k=n+i+2}^{\infty} (\sqrt{\rho})^{-k} I_k(2\sqrt{2\lambda\mu}t) + \frac{e^{-(\lambda+2\mu)t}}{\lambda(\lambda+2\mu)} \right. \\ \left. \cdot \left[\sum_{k=0}^{\infty} \rho^{-(k+i+1)} L^{-1}[(s+\lambda)(\alpha_1^{-(n+k+i+1)} - \alpha_1^{-(n+k+i+2)})] \right. \right. \\ \left. \left. + \frac{\lambda}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m (\rho)^m \left(\frac{\lambda}{2} \right)^k \rho^{-(m+k+i+2)} \right. \right. \\ \left. \left. \cdot L^{-1}[(s+\lambda)(\alpha_1^{-(n+m+k+i+2)} - \alpha_1^{-(n+m+k+i+3)})] + \frac{\lambda - 2\mu}{\mu} \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} (-1)^m \right. \right. \\ \left. \left. \cdot \left(\frac{\lambda}{2} \right)^k \left(\frac{\lambda}{2} \right)^k \rho^{-(m+k+i+1)} L^{-1}[(s+\lambda)(\alpha_1^{-(n+m+k+i+1)} - \alpha_1^{-(n+m+k+i+2)})] \right] \right]. \quad (39)$$

1. General solution in Laplace transform.

By assuming that there are l impulses preexisting in the system at $t=0$. The forward equations of the system are

$$\begin{aligned} P_0'(t) &= -(\lambda + c \cdot \mu) \cdot P_0 + c\mu P_0 + c\mu P_1(t) - (c-1)\mu \cdot P_1(t) \\ P_n'(t) &= -(\lambda + c \cdot \mu) \cdot P_n + (c-n)\mu \cdot P_n + \lambda \cdot P_{n-1} \\ &\quad + c \cdot \mu P_{n+1} - (c-n-1)\mu \cdot P_{n+1} \quad 1 \leq n \leq c \\ P_n'(t) &= -(\lambda + c\mu) \cdot P_n + \lambda \cdot P_{n-1} + c \cdot \mu \cdot P_{n+1} \quad n \geq c \end{aligned}$$

Multiplying $z^0, z^n (1 \leq n \leq c-1), z^n (n \geq c)$ on each of them,

$$\begin{aligned} z^0 \cdot P_0'(t) &= -(\lambda + c\mu) \cdot z^0 \cdot P_0 + c\mu z^0 P_0 + c\mu z^0 P_1 - (c-1)\mu \cdot z^0 P_1 \\ \sum_{n=1}^{c-1} z^n \cdot P_n'(t) &= -(\lambda + c\mu) \cdot \sum_{n=1}^{c-1} z^n \cdot P_n(t) + \mu \cdot \sum_{n=1}^{c-1} (c-n) \cdot z^n \cdot P_n(t) \\ &\quad + \lambda \cdot \sum_{n=1}^{c-1} z^n \cdot P_{n-1} + c\mu \sum_{n=1}^{c-1} z^n \cdot P_{n+1} - \mu \cdot \sum_{n=1}^{c-1} (c-n-1) \cdot P_{n+1} \cdot z^n \\ \sum_{n=c}^{\infty} z^n \cdot P_n'(t) &= -(\lambda + c\mu) \cdot \sum_{n=c}^{\infty} z^n \cdot P_n + \lambda \sum_{n=c}^{\infty} z^n \cdot P_{n-1} + c\mu \sum_{n=c}^{\infty} z^n \cdot P_{n+1} \end{aligned}$$

Adding them by side by side, we have

$$\text{The left part} = \sum_{n=0}^{\infty} z^n \cdot P_n'(t) = \partial G(t) / \partial t$$

$$\begin{aligned} \text{The first term in the right} &= -(\lambda + c\mu)(z^0 P_0 + z P_1 + \dots) \\ &= -(\lambda + c\mu) \cdot G \end{aligned}$$

$$\begin{aligned} \lambda \cdot \left[\sum_{n=1}^{c-1} z^n \cdot P_{n-1} + \sum_{n=c}^{\infty} z^n \cdot P_{n-1} \right] &= \lambda \sum_{n=1}^{\infty} z^n \cdot P_{n-1} = \lambda(z \cdot P_0 + z^2 P_1 + \dots) \\ &= \lambda z \cdot G \quad \text{and} \\ c\mu \cdot \left[z^0 \cdot P_1 + \sum_{n=1}^{c-1} z^n \cdot P_{n+1} + \sum_{n=c}^{\infty} z^n \cdot P_{n+1} \right] \\ &= c \cdot \mu \cdot z^{-1} [G - P_0] \end{aligned}$$

The residual of the right side are

$$\begin{aligned} c\mu \cdot z^0 \cdot P_0 - (c-1)\mu z^0 \cdot P_1 + \mu \sum_{n=1}^{c-1} (c-n) z^n \cdot P_n - \mu \sum_{n=1}^{c-1} (c-n-1) z^n \cdot P_{n+1} \\ = c \cdot \mu \cdot P_0 - (c-1)\mu \cdot P_1 + \mu [(c-1) z^1 \cdot P_1 + (c-2) z^2 \cdot P_2 + \dots + 1 \cdot z^{c-1} \cdot P_{c-1}] \\ - \mu [(c-2) z^1 \cdot P_2 + (c-3) z^2 \cdot P_3 + \dots + [c-c+1-1] z^{c-1} \cdot P_c] \\ = c \cdot \mu \cdot P_0 - \mu(c-1)(1-z)P_1 - \mu(c-2)z \cdot (1-z) \cdot P_2 \\ - \mu(c-3)z^2 \cdot (1-z) \cdot P_3 + \dots - \mu(c-n)z^{n-1}(1-z) \cdot P_n \\ - \mu \cdot z^{c-2} \cdot (1-z) \cdot P_{c-1} \\ = -\mu(1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n \end{aligned}$$

Hence,

$$\begin{aligned} G'(t) &= -(\lambda + c\mu)G + \lambda \cdot z \cdot G + c \cdot \mu \cdot z^{-1} (G - P_0) \\ &\quad - \mu(1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n \end{aligned}$$

For this, we have Laplace transform

$$L[G(z, t)] = G^*(s) \quad G(z, t=0) = z^i$$

$$\begin{aligned} s \cdot G^*(s) - z^i &= -(\lambda + c\mu) \cdot G^* + \lambda \cdot z \cdot G^* + c\mu(G^* - P_0^*) / z \\ &\quad - \mu(1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n^* \end{aligned}$$

Therefore,

$$\begin{aligned} G^*(s) [s + (\lambda + c\mu) - \lambda \cdot z - c \cdot \mu / z] \\ = z^i - c \cdot \mu \cdot P_0^* / z - \mu(1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^{n-1} \cdot P_n^* \end{aligned}$$

The characteristic equation of the denominator is

$$\lambda z^2 - (\lambda + c\mu + s)z + c\mu = 0$$

$$\alpha_k = \frac{\lambda + c\mu + s \pm \sqrt{(\lambda + c\mu + s)^2 - 4\lambda c\mu}}{2 \cdot \lambda}$$

$$P_0'(t) = -\lambda \cdot P_0 + \mu \cdot P_1$$

$$P_n'(t) = -(\lambda + n \cdot \mu) \cdot P_n + \lambda \cdot P_{n-1} + (n+1)\mu \cdot P_{n+1}$$

By multiplying z^0 and z^n on both sides of these

equations, we have the relations for $\sum_{n=0}^{c-2} P_n(t) \cdot z^n$

$$\begin{aligned} z^0 \cdot P_0'(t) &= -\lambda \cdot z^0 \cdot P_0 + \mu \cdot z^0 \cdot P_1 \\ \sum_{n=1}^{c-2} z^n \cdot P_n'(t) &= -\sum_{n=1}^{c-2} (\lambda + n\mu) z^n \cdot P_n + \lambda \sum_{n=1}^{c-2} z^n \cdot P_{n-1} \\ &\quad + \mu \sum_{n=1}^{c-2} (n+1) \cdot P_{n+1} \cdot z^n \end{aligned}$$

Adding side by side

$$\begin{aligned} \sum_{n=0}^{c-2} z^n \cdot P_n'(t) &= -\lambda \cdot \left(z^0 \cdot P_0 + \sum_{n=1}^{c-2} z^n \cdot P_n \right) \\ &\quad + \lambda \cdot \left(z^1 \cdot P_0 + z^2 \cdot P_1 + \dots + z^{c-2} \cdot P_{c-3} \right) \\ &\quad - \mu \cdot [1 \cdot z^1 \cdot P_1 + 2 \cdot z^2 \cdot P_2 + 3 \cdot z^3 \cdot P_3 + (c-2) \cdot z^{c-2} \cdot P_{c-2}] \\ &\quad + \mu \cdot [1 \cdot P_1 \cdot z^0 + 2 \cdot z \cdot P_2 + 3 \cdot z^2 \cdot P_3 + \dots + (c-1) \cdot z^{c-2} \cdot P_{c-1}] \end{aligned}$$

Here setting

$$Q(z, t) = \sum_{n=0}^{c-2} z^n \cdot P_n(t)$$

Then, we have

$$\frac{\partial Q}{\partial z} = P_1 + 2zP_2 + 3z^2P_3 + \dots + (c-2)z^{c-3}P_{c-2}$$

Therefore

$$\begin{aligned} Q' &= -\lambda \cdot Q + \lambda \cdot z \cdot Q - \mu z \cdot \partial Q / \partial z + \mu \cdot \partial Q / \partial z \\ &\quad - \lambda \cdot z^{c-1} P_{c-2} + \mu(c-1)z^{c-2} P_{c-1} \end{aligned}$$

$$\lambda(z \cdot P_0 + z^2 \cdot P_1 + \dots + z^{c-2} \cdot P_{c-3})$$

$$= \lambda z \cdot Q - \lambda z^{c-1} \cdot P_{c-2}$$

$$\begin{aligned} \mu[1 \cdot z^0 \cdot P_1 + 2 \cdot z \cdot P_2 + 3 \cdot z^2 \cdot P_3 + \dots + (c-2) \cdot z^{c-3} \cdot P_{c-2} + (c-1)z^{c-2} \cdot P_{c-1}] \\ = \mu \cdot \partial Q / \partial z + \mu(c-1)z^{c-2} \cdot P_{c-1} \end{aligned}$$

As a result, we have

$$-\frac{\partial Q}{\partial t} + \mu \cdot \frac{\partial Q}{\partial z} \cdot (1-z) = \lambda \cdot Q \cdot (1-z) + \lambda z^{c-1} \cdot P_{c-2} - \mu \cdot (c-1)z^{c-2} \cdot P_{c-1}$$

The characteristic equation is

$$\frac{dt}{(-1)} = \frac{dz}{\mu \cdot (1-z)} = \frac{dQ}{\lambda(1-z) \cdot Q + \lambda \cdot z^{c-1} \cdot P_{c-2} - \mu(c-1)z^{c-2} \cdot P_{c-1}}$$

1. For the first two equations, we have

$$-\mu \cdot dt = (1-z)^{-1} \cdot dz$$

$$z = 1 - c_1 e^{\mu t}$$

2. Substitute this to the third equation.

$$\frac{dt}{(-1)} = \frac{dQ}{\lambda \cdot c_1 e^{\mu t} \cdot Q + \lambda(1 - c_1 e^{\mu t})^{-1} \cdot P_{c-2} - \mu(c-1)(1 - c_1 e^{\mu t})^{c-2} \cdot P_{c-1}}$$

For this, putting $f(x) = \lambda c_1 e^{\mu x}$ then,

$$\int_0^t f(x) dx = \frac{\lambda c_1}{\mu} e^{\mu t}$$

Putting

$$g(x) = \mu(c-1)(1 - c_1 e^{\mu x})^{c-2} \cdot P_{c-1} - \lambda(1 - c_1 e^{\mu x})^{c-1} \cdot P_{c-2}$$

we have

$$\begin{aligned} Q(t) &= e^{-\lambda c_1 / \mu e^{\mu t}} \left[\int_0^t [g(t) e^{\lambda c_1 / \mu e^{\mu t}}] dt + c \right] \\ &= e^{-\lambda c_1 / \mu e^{\mu t}} \cdot \int_0^t \left[\mu(c-1)(1 - c_1 e^{\mu x})^{c-2} \cdot P_{c-1} - \lambda(1 - c_1 e^{\mu x})^{c-1} \cdot P_{c-2} \right] e^{\lambda c_1 / \mu e^{\mu x}} dx \\ &\quad + c_2 \left(c \cdot e^{-\lambda c_1 / \mu e^{\mu t}} \right) \quad \text{----- (14)} \end{aligned}$$

3. Since $c_2 = f(c_1) = f[(1-z)e^{-\mu t}]$ is unknown function

For $i > c-2$: $Q(z, 0) = 0$ and

For $i \leq c-2$: $Q(z, 0) = z^i$

$$f(1-z) = z^i e^{\lambda \mu(1-z)}$$

Then, setting $y = 1-z$, we have

$$f(y) = (1-y)^i e^{\lambda \mu y} \quad \text{and}$$

$$f[(1-z)e^{-\mu t}] = [1 - (1-z)e^{-\mu t}] e^{\lambda \mu(1-z)e^{-\mu t}}$$

Therefore, in equation (14), setting $c_1 = (1-z)e^{-\mu t}$

$$\text{and, we get } c_2 = f(c_1) = [1 - (1-z)e^{-\mu t}] e^{\lambda \mu(1-z)e^{-\mu t}}$$

Substitute this to equation (14)

$$\begin{aligned} Q &= e^{-\lambda \mu(1-z)} \cdot \int_0^t \left[\mu(c-1) \{1 - (1-z)e^{-\mu(t-x)}\}^{c-2} P_{c-1}(x) \right. \\ &\quad \left. - \lambda \{1 - (1-z)e^{-\mu(t-x)}\}^{c-1} \cdot P_{c-2}(x) \right] e^{\lambda \mu(1-z)} e^{-\mu(t-x)} dt \\ &\quad + e^{-\lambda \mu(1-z)} \cdot [1 - (1-z)e^{-\mu t}] \quad \text{----- (15)} \end{aligned}$$

For the first term in the integration of (15)

$$L \left[\left(1 - e^{-t}\right)^{v-1} \cdot \left(1 - \lambda e^{-t}\right)^{-\mu} e^{ae^{-t}} \right] = \frac{\Gamma(v) \cdot \Gamma(s)}{\Gamma(v+s)} \Phi_1[s, \mu, v, \lambda, a]$$

Setting following in the above

$$v = 1, \quad \lambda = 1-z, \quad \mu = 2-c, \quad a = \lambda/\mu(1-z)$$

Then,

$$L \left[[1 - (1-z)e^{-t}]^{-2} e^{\lambda \mu(1-z)e^{-t}} \right] = \frac{\Gamma(1) \cdot \Gamma(s)}{\Gamma(s+1)} \Phi_1[s, 2-c, 1, 1-z, \lambda(1-z)/\mu]$$

Hence, putting t to μt , then $s \rightarrow s/\mu$

$$\begin{aligned} L[F_{c-2}(t-x)] \\ = L \left[[1 - (1-z)e^{-\mu t}]^{-2} e^{\lambda \mu(1-z)e^{-\mu t}} \right] = \frac{1}{\mu} \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 2-c, 1, 1-z, \lambda/\mu(1-z)] \end{aligned}$$

Similarly for the second term in (15) $P_{c-2}(x)$

$$\begin{aligned} L[F_{c-1}(t-x)] \\ = L \left[[1 - (1-z)e^{-\mu t}]^{-1} e^{\lambda \mu(1-z)e^{-\mu t}} \right] = \frac{1}{\mu} \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1[s/\mu, 1-c, 1, 1-z, \lambda/\mu(1-z)] \end{aligned}$$

Hence, about

$$L[F_{c-2}(t-x)] \quad \& \quad L[P_{c-1}(x)]$$

$$L[F_{c-1}(t-x)] \quad \& \quad L[P_{c-2}(x)]$$

integration with respect to dx

$$L[Q(z, t)] = e^{-\lambda \mu(1-z)} \cdot [\mu(c-1) \cdot L[P_{c-1}(t)] \cdot \Phi_1[s/\mu, 2-c, 1, 1-z, \lambda(1-z)/\mu]]$$

$$\begin{aligned} = e^{-\lambda \mu(1-z)} \cdot \left[\frac{\mu(c-1)}{\mu} \cdot L[P_{c-1}(t)] \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_1 \left[s/\mu, 2-c, 1, 1-z, \frac{\lambda}{\mu}(1-z) \right] \right. \\ \left. - \frac{\lambda}{\mu} \cdot L[P_{c-2}(t)] \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \Phi_2 \left[s/\mu, 1-c, 1, 1-z, \frac{\lambda}{\mu}(1-z) \right] \right] \end{aligned} \quad (17)$$

In the following, we proceed by the steps of

□ Integration of $L[P_{c-2}(t)]$

□ Express this by $P_{c-1}(s)$

□ (Substitute them into the equation (17))

1 Laplace transform of $L[P_{c-2}(t)]$

$$\text{From equation (9)} \quad Q(z, t) = \sum_{n=0}^{c-2} P_n(t) \cdot z^n$$

$$\frac{\partial^{c-2} Q}{\partial z^{c-2}} = (c-2)! P_{c-2}(s)$$

Hence,

$$L[P_{c-2}(t)] = \frac{1}{(c-2)!} \frac{\partial^{c-2} L[Q(z, t)]}{\partial z^{c-2}} \Big|_{(z=0)}$$

In the equation (17), there are products of

$$g(z) = e^{-\lambda/\mu(1-z)}$$

$$h(z) = \Phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)]$$

$$\frac{\partial^{(n)} [f \cdot g]}{\partial x^n} = \sum_{r=0}^n C_r \cdot \frac{\partial^{(n-r)} f(x)}{\partial x^{(n-r)}} \cdot \frac{\partial^{(r)} g(x)}{\partial x^r}$$

$$\frac{\partial^{c-2}}{\partial z^{c-2}} \left[e^{\lambda/\mu(1-z)} \cdot \Phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] \right]$$

$$= \sum_{k=0}^{c-2} C_k \cdot \frac{\partial^{(k)}}{\partial z^{(k)}} \cdot \Phi_1 \cdot \frac{\partial^{(c-2-k)}}{\partial z^{(c-2-k)}} e^{\lambda/\mu(1-z)} \Big|_{(z=0)}$$

$$D^{(c-k-2)} e^{\lambda/\mu(1-z)} = (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu}$$

Therefore, z differentiation in the first term of $Q(z, t)$

$$\frac{\partial^{c-2} L[Q(z, t)]}{\partial z^{c-2}} \Big|_{(z=0)} = \sum_{k=0}^{c-2} C_k \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(2-c) \cdot P_{c-1}^*(s) \cdot (c-1)}{\partial z^{(k)}} \Big|_{(z=0)}$$

In the second term

$$\frac{\partial^{c-2} L[Q(z, t)]}{\partial z^{c-2}} \Big|_{(z=0)} = - \sum_{k=0}^{c-2} C_k \left(\frac{\lambda}{\mu} \right)^{c-k-2} e^{-\lambda/\mu} \cdot \frac{\partial^{(k)} \Phi_1(1-c) \cdot P_{c-2}^*(s) \cdot (\lambda/\mu)}{\partial z^{(k)}} \Big|_{(z=0)}$$

Hence,

$$P_{c-2}^*(s) = \frac{1}{(c-2)!} \frac{\partial^{c-2} Q^*(z, s)}{\partial z^{c-2}} \Big|_{z=0}$$

$$= \frac{1}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu}\right)^{c-k-2} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right.$$

$$\left. - \left(\frac{\lambda}{\mu}\right) \cdot P_{c-2}^*(s) \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu}\right)^{c-k-2} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \Big|_{z=0} \right\}$$

Solving this with respect to $P_{c-2}^*(s)$

$$P_{c-2}^*(s) \left[1 + \left(\frac{\lambda}{\mu}\right) \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu}\right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \cdot e^{-\lambda/\mu} \frac{\Gamma(s/\mu)}{(c-2)! \Gamma(s/\mu+1)} \right]$$

$$= \frac{1}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} e^{-\lambda/\mu} \cdot (c-1) \cdot P_{c-1}^*(s) \sum_{k=0}^{c-2} C_k \left(\frac{\lambda}{\mu}\right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \Big|_{z=0}$$

Hence,

$$P_{c-2}^*(s) = \frac{(c-1)}{(c-2)!} \cdot \frac{\Gamma(s/\mu)}{\Gamma(s/\mu+1)} P_{c-1}^*(s) \cdot e^{-\lambda/\mu} \cdot \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu}\right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \Big|_{z=0}$$

$$\cdot \left[1 + \frac{\Gamma(s/\mu)}{(c-2)!} \frac{(\lambda/\mu) e^{-\lambda/\mu}}{\Gamma(s/\mu+1)} \sum_{k=0}^{c-2} C_k \cdot \left(\frac{\lambda}{\mu}\right)^{c-k-2} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \Big|_{z=0} \right]^{-1}$$

2. Next we seek $P_n^*(s)$, since

$$\frac{\partial^{(n)} Q^*(z, s)}{\partial z^{(n)}} = n! P_n(s) \Big|_{z=0}$$

Then, the differentiation for $Q(z, s)$ is turned from $c-2$ to n . Similarly for the terms of $n \leq c-2$,

$$P_n^*(s) = \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right.$$

$$\left. - \left(\frac{\lambda}{\mu}\right) \cdot P_{c-2}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \Big|_{z=0} \right\} \quad (21)$$

Here utilizing the equation (7)

$$\sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot P_n^* = \alpha_2^{i+1} \mu (1-\alpha_2)$$

Multiplying on the both sides of (21) on $(c-n) \cdot \alpha_2^n$ and

summing $\sum_{n=0}^{c-1}$

$$\sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot P_n^*(s)$$

$$= \sum_{n=0}^{c-1} (c-n) \cdot \alpha_2^n \cdot \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \left\{ (c-1) \cdot P_{c-1}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right.$$

$$\left. - \frac{\lambda}{\mu} \cdot P_{c-2}^*(s) \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \Big|_{z=0} \right\}$$

Since

$$\Phi_1(2-c) = f_1[\Gamma(m+2-c)/\Gamma(2-c)]$$

Then, for the terms of $n=c-2$, $c-1$ are dropped and the series until $c-3$ is

$$\frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} = \sum_{n=0}^{c-3} (c-n) \cdot \alpha_2^n \cdot \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \cdot P_{c-1}^*(s)$$

$$\cdot \left\{ (c-1) \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(2-c)}{\partial z^{(k)}} \right.$$

$$\left. - \frac{\lambda}{\mu} \cdot \frac{P_{c-2}^*(s)}{P_{c-1}^*(s)} \cdot \sum_{k=0}^n C_k \cdot \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{\partial^{(k)} \Phi_1(1-c)}{\partial z^{(k)}} \right\}$$

$$+ f(n=c-1, c-2)$$

$$f(n=c-1, c-2) = P_{c-1}^* \left[\alpha_2^{k-1} + 2\alpha_2^{c-2} \frac{P_{c-2}^*}{P_{c-1}^*} \right]$$

$$P_{c-2}^*(s) = \frac{\Gamma(s/\mu)}{(c-2)! \Gamma(s/\mu+1)} \left\{ (c-1) P_{c-1}^*(s) \right.$$

$$\cdot \sum_{k=0}^{c-2} \binom{c-2}{k} \left(\frac{\lambda}{\mu}\right)^{c-k-2} e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0}$$

$$- \frac{\lambda}{\mu} P_{c-2}^*(s) \sum_{k=0}^{c-2} \binom{c-2}{k} \left(\frac{\lambda}{\mu}\right)^{c-k-2} e^{-\lambda/\mu}$$

$$\cdot \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \Big\}. \quad (19)$$

Solving for $P_{c-2}^*(s)$ yields

$$P_{c-2}^*(s) = \frac{\Gamma(s/\mu)(c-1)}{(c-2)! \Gamma(s/\mu+1)} P_{c-1}^*(s) \sum_{k=0}^{c-2} \binom{c-2}{k} \left(\frac{\lambda}{\mu}\right)^{c-k-2}$$

$$\cdot e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} /$$

$$\left\{ 1 + \frac{\lambda}{\mu} \frac{\Gamma(s/\mu)}{(c-2)! \Gamma(s/\mu+1)} \sum_{k=0}^{c-2} \binom{c-2}{k} \left(\frac{\lambda}{\mu}\right)^{c-k-2} \right.$$

$$\cdot e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \Big\}. \quad (20)$$

We also have, for $n \leq c-2$,

$$P_n^*(s) = \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \left\{ (c-1) P_{c-1}^*(s) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu}$$

$$\cdot \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} - \frac{\lambda}{\mu} P_{c-2}^*(s) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^{n-k}$$

$$\cdot e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \Big\}. \quad (21)$$

Thus,

$$P_{c-1}^*(s) = \frac{\alpha_2^{i+1}}{1-\alpha_2} / \left(\alpha_2^{c-1} + \sum_{n=0}^{c-3} (c-n) \alpha_2^n \frac{\Gamma(s/\mu)}{n! \Gamma(s/\mu+1)} \right)$$

$$\cdot \left\{ (c-1) \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \right.$$

$$\left. - \frac{\lambda}{\mu} \frac{P_{c-2}^*(s)}{P_{c-1}^*(s)} \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\lambda/\mu} \frac{d^k}{dz^k} \Phi_1 \left[\frac{s}{\mu}, -(c-1), 1, (1-z), \frac{\lambda}{\mu} (1-z) \right] \Big|_{z=0} \right\}$$

$$+ 2\alpha_2^{c-2} \frac{P_{c-2}^*(s)}{P_{c-1}^*(s)}.$$

This expression must now be substituted in (20) and (21) to obtain $P_n^*(s)$ for $0 \leq n \leq c-2$ explicitly.

$$P_n^*(s) = \frac{\mu}{\lambda} \left[\sum_{j=0}^{c-1} (c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j-2}} \frac{1-(\alpha_2/\alpha_1)^{n-j+1}}{1-\alpha_2/\alpha_1} \right.$$

$$\left. - (c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j-3}} \frac{1-(\alpha_2/\alpha_1)^{n-j}}{1-\alpha_2/\alpha_1} \right] - \frac{1}{c\mu\alpha_2^{n-3}} \frac{1-(\alpha_2/\alpha_1)^{n-i}}{1-\alpha_2/\alpha_1} \quad (n \leq c) \quad (24)$$

We replace $P_j^*(s)$, for $0 \leq j \leq c-1$ by the expression that we determined previously.

** $P_n(s) * (24)$

Returning to the equation (5), we modify $P_n^*(s)$

$$P^*(z, s) = \frac{[z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n) z^n \cdot P_n^*(s)]}{-\lambda(z - \alpha_1)(z - \alpha_2)}$$

From the Leibnitz formula

$$\begin{aligned}
 & \frac{1}{m!} \frac{d^m}{dz^m} \left[(z - \alpha_1)^{-1} (z - \alpha_2)^{-1} \right]_{z=0} \\
 &= \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} \frac{d^k (z - \alpha_1)^{-1}}{dz^k} \cdot \frac{\partial^{(m-k)} (z - \alpha_2)^{-1}}{\partial z^{(m-k)}} \quad (z=0) \\
 &= \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} \frac{k!}{(\alpha_1)^{k+1}} \cdot \frac{(m-k)!}{(\alpha_2)^{m-k+1}} \\
 &= \frac{1}{m!} \sum_{k=0}^m \frac{m!}{(m-k)! k!} \cdot \frac{k!}{(\alpha_1)^{k+1}} \cdot \frac{(m-k)!}{(\alpha_2)^{m-k+1}} \\
 &= \sum_{k=0}^m (\alpha_2/\alpha_1)^{k+1} \cdot (\alpha_2)^{-(m+2)} \\
 &= 1/(\alpha_2)^{m+2} \cdot (\alpha_2/\alpha_1) \sum_{k=0}^m (\alpha_2/\alpha_1)^k \\
 &= \frac{1}{\alpha_2^{m+1}} \cdot \frac{1}{\alpha_1} \left[\frac{1 - (\alpha_2/\alpha_1)^{m+1}}{1 - \alpha_2/\alpha_1} \right] \\
 & \quad \text{where } \alpha_1 \alpha_2 = c \cdot \mu \quad \therefore 1/\alpha_1 = \alpha_2/(c\mu) \\
 &= \frac{\lambda}{c \cdot \mu \cdot \alpha_2^{m-2}} \left[\frac{1 - (\alpha_2/\alpha_1)^{m+1}}{1 - \alpha_2/\alpha_1} \right]
 \end{aligned}$$

As a result, we obtain the equation (24)

The case $c=2$ *****

For $c=2$, the Laplace transform of the generating function (5) is given by

$$\begin{aligned}
 P^*(z, s) &= \frac{z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n) \cdot z^n \cdot P_n^*(s)}{sz - (1-z)(c\mu - \lambda z)} \\
 &= \frac{z^{i+1} - \mu(1-z)(2P_0^* + zP_1^*)}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)} \quad (25)
 \end{aligned}$$

The condition on which the numerator becomes zero

$z = \alpha_2$ is

$$\alpha_2^{i+1} = \mu(1 - \alpha_2)(2P_0^* + \alpha_2 P_1^*)$$

The Laplace transform of equation (1) - a is

$$s \cdot P_0^* = -\lambda \cdot P_0^* + \mu \cdot P_1^*$$

$$(s + \lambda) \cdot P_0^* = \mu \cdot P_1^*$$

hence P_1^*, P_0^* are

$$\begin{aligned}
 2P_0^* + \alpha_2 \cdot P_1^* &= \alpha_2^{i+1} / [\mu(1 - \alpha_2)] \\
 &= 2 \cdot P_0^* + \alpha_2 \cdot (s + \lambda) \cdot P_0^* / \mu
 \end{aligned}$$

$$\therefore P_0^* = \frac{\alpha_2^{i+1}}{(1 - \alpha_2)[2\mu + \alpha_2(s + \lambda)]} \quad (26)$$

$$P_1^* = \frac{(s + \lambda)}{\mu} \frac{\alpha_2^{i+1}}{(1 - \alpha_2)[2\mu + \alpha_2(s + \lambda)]} \quad (27)$$

Substitute (26) (27) to (25)

$$P^*(z, s) = \left\{ z^{i+1} - \frac{(1-z)\alpha_2^{i+1} \cdot (2\mu + z(s + \lambda))}{(1 - \alpha_2)(2\mu + (s + \lambda) \cdot \alpha_2)} \right\} \frac{1}{[-\lambda(z - \alpha_1)(z - \alpha_2)]}$$

to modify (32)

$$2\mu + (s + \lambda) \cdot z = 2\mu + (s + \lambda) \cdot \alpha_2 + (s + \lambda)(z - \alpha_2)$$

$$P^*(z, s) = \frac{z^{i+1}}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)}$$

$$\begin{aligned}
 & - \frac{(1-z) \cdot \alpha_2^{i+1} [2\mu + (s + \lambda) \cdot \alpha_2 + (s + \lambda)(z - \alpha_2)]}{(1 - \alpha_2)(2\mu + (s + \lambda) \cdot \alpha_2) [-\lambda(z - \alpha_1)(z - \alpha_2)]} \\
 &= \frac{z^{i+1}}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)} - \frac{(1-z) \cdot \alpha_2^{i+1} [2\mu + (s + \lambda) \cdot \alpha_2]}{(1 - \alpha_2)(2\mu + (s + \lambda) \cdot \alpha_2) [-\lambda \cdot (z - \alpha_1)(z - \alpha_2)]} \\
 & \quad - \frac{(1-z) \cdot \alpha_2^{i+1} [(s + \lambda)(z - \alpha_2)]}{(1 - \alpha_2)(2\mu + (s + \lambda) \cdot \alpha_2) [-\lambda(z - \alpha_1)(z - \alpha_2)]} \\
 &= \frac{z^{i+1}}{-\lambda \cdot (z - \alpha_1)(z - \alpha_2)} + \frac{(1-z) \cdot \alpha_2^{i+1}}{(1 - \alpha_2) \cdot \lambda \cdot (z - \alpha_1)(z - \alpha_2)} \\
 & \quad + \frac{(1-z) \cdot \alpha_2^{i+1} (s + \lambda)}{(1 - \alpha_2)(2\mu + (s + \lambda) \cdot \alpha_2) \cdot \lambda \cdot (z - \alpha_1)} \\
 &= \frac{z^{i+1}(1 - \alpha_2) - (1-z) \cdot \alpha_2^{i+1}}{-\lambda(z - \alpha_1)(z - \alpha_2)(1 - \alpha_2)} - \frac{(1-z) \cdot \alpha_2^{i+1} (s + \lambda)}{\lambda \cdot \alpha_1(1 - z/\alpha_1)[2\mu + (s + \lambda) \cdot \alpha_2][1 - \alpha_2]}
 \end{aligned}$$

The numerator of the first term in (33) - (33)

$$z^{i+1} \cdot (1 - \alpha_2) - (1 - z) \cdot \alpha_2^{i+1}$$

$$= z^{i+1} \cdot [1 - (\alpha_2/z)^{i+1}] - z^{i+1} \cdot \alpha_2 [1 - (\alpha_2/z)^i]$$

since

$$\sum_{r=0}^i x^r = \frac{1 - x^{i+1}}{1 - x} \quad \text{putting } x = \alpha_2/z$$

$$(1 - (\alpha_2/z)^{i+1}) = (1 - \alpha_2/z) \cdot \sum_{r=0}^i (\alpha_2/z)^r$$

$$\begin{aligned}
 &= z^{i+1} \cdot (1 - \alpha_2/z) \cdot \sum_{r=0}^i (\alpha_2/z)^r - z^{i+1} \cdot \alpha_2 \cdot (1 - \alpha_2/z) \cdot \sum_{r=0}^{i-1} (\alpha_2/z)^r \\
 &= z^i (z - \alpha_2) (1 + \alpha_2/z + \alpha_2^2/z^2 + \alpha_2^3/z^3 + \dots + \alpha_2^i/z^i) \\
 & \quad - z^i \cdot \alpha_2 (z - \alpha_2) (1 + \alpha_2/z + \alpha_2^2/z^2 + \dots + \alpha_2^{i-1}/z^{i-1}) \\
 &= (z - \alpha_2) (z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \alpha_2^3 \cdot z^{i-3} + \dots + \alpha_2^i) \\
 & \quad - z \cdot \alpha_2 (z - \alpha_2) (z^{i-1} + \alpha_2 \cdot z^{i-2} + \alpha_2^2 \cdot z^{i-3} + \dots + \alpha_2^{i-1})
 \end{aligned}$$

Therefore the first term is

$$\begin{aligned}
 &= \frac{(z - \alpha_2)(z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \dots + \alpha_2^i)}{\lambda \alpha_1 (1 - z/\alpha_1)(z - \alpha_2)(1 - \alpha_2)} \\
 & \quad - \frac{z \cdot \alpha_2 \cdot (z - \alpha_2)(z^{i-1} + \alpha_2 \cdot z^{i-2} + \dots + \alpha_2^{i-1})}{\lambda \alpha_1 (1 - z/\alpha_1)(z - \alpha_2)(1 - \alpha_2)} \\
 &= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \alpha_2^2 \cdot z^{i-2} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \cdot \frac{1}{(1 - z/\alpha_1)} \\
 & \quad - \frac{z \cdot \alpha_2 \cdot (z^{i-1} + \alpha_2 \cdot z^{i-2} + \dots + \alpha_2^{i-2} \cdot z + \alpha_2^{i-1})}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \cdot \frac{1}{(1 - z/\alpha_1)} \\
 &= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z/\alpha_1}{1 - z/\alpha_1} \right)^m \\
 & \quad - \frac{\alpha_2 \cdot (z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^{i-2} \cdot z^2 + z \cdot \alpha_2^{i-1})}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z/\alpha_1}{1 - z/\alpha_1} \right)^m \\
 &= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z/\alpha_1}{1 - z/\alpha_1} \right)^m \\
 & \quad - \frac{\alpha_2 \cdot (z^i + \alpha_2 \cdot z^{i-1} + \dots + z \cdot \alpha_2^{i-1} + \alpha_2^i - \alpha_2^i)}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z/\alpha_1}{1 - z/\alpha_1} \right)^m \\
 &= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1 \cdot (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z/\alpha_1}{1 - z/\alpha_1} \right)^m \cdot [1 - \alpha_2]
 \end{aligned}$$

$$+ \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m$$

$$= \frac{(z^i + \alpha_2 \cdot z^{i-1} + \dots + \alpha_2^i)}{\lambda \cdot \alpha_1} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m + \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m$$

----- (34). The second term in (34) is

$$\frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1 (1 - \alpha_2)} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m = \sum_{m=0}^{\infty} \frac{\alpha_2^{i+1}}{\lambda \cdot (1 - \alpha_2) \alpha_1^{m+1}} \cdot z^m$$

Then $\alpha_2^{i+1} / [\lambda \cdot (1 - \alpha_2) \cdot \alpha_1^{m+1}]$ can be expanded as

$$\frac{\alpha_2^{i+1}}{\lambda \cdot (1 - \alpha_2) \cdot \alpha_1^{m+1}} = \frac{\alpha_2^{i+1}}{\lambda \cdot \alpha_1^{m+1}} \sum_{n=0}^{\infty} \alpha_2^n$$

since $\alpha_1 \cdot \alpha_2 = 2\mu/\lambda$ we have $\alpha_2 = 2\mu/(\lambda \alpha_1)$

$$= \frac{1}{\lambda} \left(\frac{2\mu}{\lambda \alpha_1} \right)^{i+1} \frac{1}{\alpha_1^{m+1}} \sum_{n=0}^{\infty} \left(\frac{2\mu}{\lambda \alpha_1} \right)^n$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{2\mu}{\lambda} \right)^{n+i+1} \frac{1}{\alpha_1^{n+i+1+m+1}}$$

The series of α_1 starts from $i+m+2$

$$= \frac{1}{\lambda} \sum_{k=i+m+2}^{\infty} \left(\frac{1}{\alpha_1^k} \right) \left(\frac{2\mu}{\lambda} \right)^k \cdot \left(\frac{2\mu}{\lambda} \right)^{-(m+1)}$$

$$= \frac{1}{\lambda} \left(\frac{\lambda}{2\mu} \right)^{m+1} \sum_{k=i+m+2}^{\infty} \frac{1}{\alpha_1^k} \cdot \left(\frac{2\mu}{\lambda} \right)^k$$

Therefore

$$(34) \text{ second} = \sum_{m=0}^{\infty} \frac{1}{\lambda} \left(\frac{\lambda}{2\mu} \right)^{m+1} \sum_{k=i+m+2}^{\infty} \frac{1}{\alpha_1^k} \cdot \left(\frac{2\mu}{\lambda} \right)^k \cdot z^m$$

The first term in the (34) is

$$= 1/(\lambda \alpha_1) \left[\sum_{m=0}^i z^m \cdot \sum_{l=0}^i + \sum_{m=i+1}^{\infty} z^m \cdot \sum_{l=0}^i \right] (\alpha_2^{i-l} / \alpha_1^{m-l})$$

$$- 1/(\lambda \alpha_1) \left[\sum_{m=0}^i z^m \cdot \sum_{l=0}^{i-m} \right] (\alpha_2^{i-l} / \alpha_1^{m-l})$$

The second term in (33) is

$$- \frac{(1-z) \alpha_2^{i+1} (s+\lambda)}{\lambda \cdot \alpha_1 \cdot (1-z/\alpha_1) (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]}$$

$$= - \frac{\alpha_2^{i+1} (s+\lambda) (1-z)}{\lambda \cdot \alpha_1 \cdot (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m$$

$1-z$ in numerator is separated to 1 and $1-z$

$$= \frac{\alpha_2^{i+1} \cdot (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} \left[\frac{z}{\alpha_1} \cdot \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m - \frac{1}{\alpha_1} \sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m \right]$$

The coefficient of z^m is

$$\sum_{m=0}^{\infty} \left(\frac{z}{\alpha_1} \right)^m - \sum_{m=0}^{\infty} \frac{z^m}{\alpha_1^{m+1}} = \sum_{m=0}^{\infty} \left(\frac{1}{\alpha_1^m} - \frac{1}{\alpha_1^{m+1}} \right) z^m - 1$$

Therefore expand the second term in (33) by z^m

$$= \frac{\alpha_2^{i+1} (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} \left[\sum_{m=0}^{\infty} \left(\frac{1}{\alpha_1^m} - \frac{1}{\alpha_1^{m+1}} \right) z^m - 1 \right]$$

Hence

$$\frac{\alpha_2^{i+1} (s+\lambda)}{\lambda \cdot (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} \left(\frac{1}{\alpha_1^m} - \frac{1}{\alpha_1^{m+1}} \right)$$

$$= - \frac{\alpha_2^{i+1} (s+\lambda) (1-\alpha_1)}{\lambda (1-\alpha_2) [2\mu + (s+\lambda) \alpha_2] \alpha_1^{m+1}}$$

$$\text{from (26)} \quad \frac{\alpha_2^{i+1}}{(1-\alpha_2) [2\mu + (s+\lambda) \alpha_2]} = P_0^*$$

$$= -P_0^* \frac{(s+\lambda) (1-\alpha_1)}{\lambda \cdot \alpha_1^{m+1}}$$

Thus the second term in (33) is

$$= -P_0^* \frac{(s+\lambda) (1-\alpha_1)}{\lambda} \sum_{m=0}^{\infty} \frac{z^m}{\alpha_1^{m+1}}$$

$$(33) \quad 1 = \frac{1}{\lambda \alpha_1} \left[\sum_{m=0}^i z^m \cdot \sum_{n=0}^m + \sum_{m=i+1}^{\infty} z^m \cdot \sum_{n=0}^i \right] (\alpha_2^{i-n} / \alpha_1^{m-n})$$

$$+ \frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{\lambda}{2\mu} \right)^{m+1} \sum_{k=i+m+2}^{\infty} \frac{1}{\alpha_1^k} \cdot \left(\frac{2\mu}{\lambda} \right)^k \cdot z^m$$

Then, we determine $P_0(t)$ from (26). Since

$$\left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k = \left(\frac{\alpha_2}{2} \right)^k \left(1 - \frac{\lambda \cdot \alpha_2^2}{4\mu \alpha_2} \right)^k$$

$$= \left(\frac{\alpha_2}{2} \right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m$$

Hence

$$\left[1 - \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right) \right]^{-1} = \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} \right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m$$

To expand $P_0^* 2\mu + (s+\lambda) \alpha_2$ in the denominator of (26)

$$P_0^*, \text{ we replace } s+\lambda \text{ by the eigen roots. Since}$$

$$s = -\lambda \cdot (1-\alpha_1) \cdot (1-\alpha_2) \quad \alpha_1 = 2\mu/(\lambda \alpha_2)$$

Thus

$$s+\lambda = \frac{1}{\alpha_2} [2\mu + \lambda \alpha_2^2 - 2\mu \alpha_2]$$

Hence, from (26)

$$P_0^* = \frac{\alpha_2^{i+1}}{(1-\alpha_2) [2\mu + 2\mu + \lambda \alpha_2^2 - 2\mu \alpha_2]}$$

$$= \frac{\alpha_2^{i+1}}{(1-\alpha_2) (\lambda \alpha_2^2 - 2\mu \alpha_2 + 4\mu)}$$

$$= \frac{\alpha_2^{i+1}}{(\lambda + 2\mu)} \left[\frac{1}{(1-\alpha_2)} + \frac{\lambda \cdot \alpha_2 + (\lambda - 2\mu)}{(\lambda \alpha_2^2 - 2\mu \alpha_2 + 4\mu)} \right]$$

The denominator of the second term is

$$= 4\mu [1 - (\alpha_2/2 - \lambda \alpha_2^2/4\mu)] \quad \text{Hence,}$$

$$P_0^* = \frac{\alpha_2^{i+1}}{(\lambda + 2\mu)} \left[\frac{1}{1-\alpha_2} + \frac{(\lambda \alpha_2 + \lambda - 2\mu)}{4\mu [1 - (\alpha_2/2 - \lambda \alpha_2^2/4\mu)]} \right]$$

$$= \frac{\alpha_2^{i+1}}{(\lambda + 2\mu)} \left[\sum_{k=0}^{\infty} (\alpha_2)^k + \frac{(\lambda \alpha_2 + \lambda - 2\mu)}{4\mu} \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} - \frac{\lambda \alpha_2^2}{4\mu} \right)^k \right]$$

$$= \frac{1}{(\lambda + 2\mu)} \left[\sum_{k=0}^{\infty} \alpha_2^{i+1+k} + \frac{\alpha_2^{i+1} (\lambda \alpha_2 + \lambda - 2\mu)}{4\mu} \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} \right)^k \cdot \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m \right]$$

$$= \frac{1}{(\lambda + 2\mu)} \left[\sum_{k=0}^{\infty} \alpha_2^{i+1+k} + \frac{\lambda \cdot \alpha_2^{i+2}}{4\mu} \sum_{k=0}^{\infty} \left(\frac{\alpha_2}{2} \right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda \alpha_2}{2\mu} \right)^m \right]$$

$$\begin{aligned}
& + \frac{(\lambda - 2\mu)\alpha_2^{i+1}}{4\mu} \sum_{k=0}^i \left(\frac{\alpha_2}{2}\right)^k \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda\alpha_2}{2\mu}\right)^m \Bigg] \\
& = \frac{1}{(\lambda + 2\mu)} \left[\sum_{k=0}^i \alpha_2^{k+i+1} + \frac{\lambda}{4\mu} \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{1}{2^k}\right) \left(\frac{\lambda}{2\mu}\right)^m \alpha_2^{i+2+k+m} \right. \\
& \quad \left. + \frac{(\lambda - 2\mu)}{4\mu} \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-1)^m \frac{1}{2^k} \left(\frac{\lambda}{2\mu}\right)^m \alpha_2^{i+1+k+m} \right]
\end{aligned}$$

The Laplace transform is

$$\begin{aligned}
L^{-1}[\alpha_2^n] &= L^{-1} \left[\lambda + 2\mu + s - [(\lambda + 2\mu + s)^2 - 4 \cdot 2\mu\lambda]^{1/2} \right]^{-n} / (2\lambda)^n \\
L^{-1} \left[\left(s - (s^2 - 8\mu\lambda)^{1/2} \right)^n \right] &= \frac{n \cdot (2\sqrt{2\mu\lambda})^n}{t} \cdot I_n(2\sqrt{2\mu\lambda} \cdot t) \quad \text{Hence} \\
L^{-1}[\alpha_2^{k+i+1}] &= \frac{(k+i+1)(2\sqrt{2\mu\lambda})^{k+i+1}}{t(2\lambda)^{k+i+1}} \cdot I_{k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \\
L^{-1}[\alpha_2^{m+k+i+2}] &= \frac{(m+k+i+2)(2\sqrt{2\mu\lambda})^{m+k+i+2}}{t(2\lambda)^{m+k+i+2}} \cdot I_{m+k+i+2}(2\sqrt{2\mu\lambda} \cdot t) \\
L^{-1}[\alpha_2^{m+k+i+1}] &= \frac{(m+k+i+1)(2\sqrt{2\mu\lambda})^{m+k+i+1}}{t(2\lambda)^{m+k+i+1}} \cdot I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t)
\end{aligned}$$

Thus, the invert Laplace transform of $P_0^*(s)$ is

$$\begin{aligned}
P_0(t) &= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu) \cdot t} \left[\sum_{k=0}^i \frac{(k+i+1) \cdot (2\sqrt{2\mu\lambda})^{k+i+1}}{(2\lambda)^{k+i+1}} \cdot I_{k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \right. \\
& \quad + \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda}{2\mu}\right)^m \frac{1}{2^k} \frac{(m+k+i+2)(2\sqrt{2\mu\lambda})^{m+k+i+2}}{(2\lambda)^{m+k+i+2}} \cdot I_{m+k+i+2}(2\sqrt{2\mu\lambda} \cdot t) \\
& \quad \left. + \frac{(\lambda-2\mu)}{4\mu} \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-1)^m \left(\frac{\lambda}{2\mu}\right)^m \frac{1}{2^k} \frac{(m+k+i+1)(2\sqrt{2\mu\lambda})^{m+k+i+1}}{(2\lambda)^{m+k+i+1}} \cdot I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \right]
\end{aligned}$$

Thereby, we replace $\lambda/2\mu$ by $\rho = \lambda/2\mu$

$$\begin{aligned}
P_0(t) &= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu) \cdot t} \left[\sum_{k=0}^i (\rho)^{-(k+i+1)F_2} \cdot (k+i+1) I_{k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \right. \\
& \quad + \frac{1}{4} \frac{\lambda}{\mu} \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot I_{m+k+i+2}(2\sqrt{2\mu\lambda} \cdot t) \\
& \quad \left. + \frac{1}{4\mu} (\lambda-2\mu) \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \right] \\
P_1^* &= \frac{(s+\lambda)\alpha_2^{i+1}}{\mu \cdot (1-\alpha_2)[2\mu + (s+\lambda) \cdot \alpha_2]} = \frac{(s+\lambda)}{\mu} P_0^* \\
&= \frac{1}{\mu} \cdot [s \cdot P_0^* + \lambda \cdot P_0^*] \quad \text{While} \\
L^{-1}[s \cdot \alpha_1^{-n}] &= \frac{\partial}{\partial t} L^{-1} \left[\frac{\lambda + 2\mu + s + [(\lambda + 2\mu + s)^2 - 8\mu\lambda]^{1/2}}{(2\lambda)^n} \right]^{-n} \\
&= (2\lambda)^n \frac{\partial}{\partial t} \left[L^{-1} \left[(8\mu\lambda)^{-n} \cdot \left(s - (s^2 - 8\mu\lambda)^{1/2} \right)^n \right] \cdot e^{-(\lambda+2\mu)t} \right] \\
L^{-1} \left[\left(s - (s^2 - 8\mu\lambda)^{1/2} \right)^n \right] &= \frac{n \cdot (2\sqrt{2\mu\lambda})^n}{t} \cdot I_n(2\sqrt{2\mu\lambda} \cdot t) \\
&= (2\lambda)^n (2\sqrt{2\mu\lambda})^n \cdot \frac{\partial}{\partial t} \left[\frac{n \cdot I_n(2\sqrt{2\mu\lambda} \cdot t)}{t} \cdot e^{-(\lambda+2\mu)t} \right]
\end{aligned}$$

Putting $a = 2\sqrt{2\mu\lambda}$

$$L^{-1}[s \cdot \alpha_1^{-n}] = (2\lambda)^n (2\sqrt{2\mu\lambda})^n$$

$$\begin{aligned}
& \left[n \left[2\sqrt{2\mu\lambda} t^{-1} \cdot I_{n-1}(2\sqrt{2\mu\lambda} \cdot t) - (n+1) \cdot t^{-2} \cdot I_n(2\sqrt{2\mu\lambda} \cdot t) \right] \right]^{-(\lambda+2\mu)t} \\
& + n \cdot I_n(2\sqrt{2\mu\lambda} \cdot t) \cdot (-\lambda - 2\mu) e^{-(\lambda+2\mu)t}
\end{aligned}$$

For the practical use, we set

$$P_1(t) = L^{-1} \left[\frac{(s \cdot P_0^* + \lambda P_0^*)}{\mu} \right] = \frac{1}{\mu} L^{-1}[s \cdot P_0^*] + \frac{\lambda}{\mu} L^{-1}[P_0^*]$$

The differentiation of the n-order modified Bessel function

$$\frac{\partial}{\partial t} \left[\frac{I_n(a \cdot t)}{t} \right] = [a \cdot t^{-1} \cdot I_{n-1}(at) - (n+1) \cdot t^{-2} \cdot I_n(at)]$$

Therefore we simply add the differential of $P_0(t)$

$$\frac{\partial I_n(at)}{\partial t} = a \cdot I_{n-1}(at) - n/t \cdot I_n(at)$$

By setting $n = k+i+1, m+k+i+2, m+k+i+1$

$$\begin{aligned}
1. \frac{\partial I_{k+i+1}(2\sqrt{2\mu\lambda} \cdot t)}{\partial t} &= 2\sqrt{2\mu\lambda} \cdot I_{k+i}(2\sqrt{2\mu\lambda} \cdot t) - (k+i+1)/t \cdot I_{k+i+1}(2\sqrt{2\mu\lambda} \cdot t) \\
& \quad F_1'(t) = f_1(t) \\
2. \frac{\partial I_{m+k+i+2}(2\sqrt{2\mu\lambda} \cdot t)}{\partial t} &= 2\sqrt{2\mu\lambda} \cdot I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t) - (m+k+i+2)/t \cdot I_{m+k+i+2}(2\sqrt{2\mu\lambda} \cdot t) \\
& \quad F_2'(t) = f_2(t) \\
3. \frac{\partial I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t)}{\partial t} &= 2\sqrt{2\mu\lambda} \cdot I_{m+k+i}(2\sqrt{2\mu\lambda} \cdot t) - (m+k+i+1)/t \cdot I_{m+k+i+1}(2\sqrt{2\mu\lambda} \cdot t)
\end{aligned}$$

Now putting $F_3'(t) = f_3(t)$ we have

$$\begin{aligned}
P_0(t) &= \frac{e^{-(\lambda+2\mu)t}}{(\lambda+2\mu) \cdot t} \left[\sum_{k=0}^i (\sqrt{\rho})^{-(k+i+1)} \cdot (k+i+1) \cdot F_1(t) \right. \\
& \quad + \frac{1}{4} \left(\frac{\lambda}{\mu} \right) \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot F_2(t) \\
& \quad \left. + \frac{1}{4\mu} (\lambda-2\mu) \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot F_3(t) \right] \\
\frac{\partial}{\partial t} \left[\frac{e^{-(\lambda+2\mu)t}}{t} \right] &= -e^{-(\lambda+2\mu)t} \left[(\lambda+2\mu) \cdot t^{-1} + t^{-2} \right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial P_0(t)}{\partial t} &= \frac{-(\lambda+2\mu)}{(\lambda+2\mu)} e^{-(\lambda+2\mu)t} \cdot [(\lambda+2\mu) + t^{-1}]^{(\lambda+2\mu)t} \cdot P_0(t) \\
& \quad + \frac{1}{(\lambda+2\mu)} \frac{e^{-(\lambda+2\mu)t}}{t} \cdot \frac{\partial}{\partial t} \left[\sum_{k=0}^i (\sqrt{\rho})^{-(k+i+1)} \cdot (k+i+1) \cdot f_1(t) \right. \\
& \quad + \frac{1}{4} \left(\frac{\lambda}{\mu} \right) \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+2)} \cdot (m+k+i+2) \cdot f_2(t) \\
& \quad \left. + \frac{1}{4\mu} (\lambda-2\mu) \sum_{k=0}^i \sum_{m=0}^k C_m \cdot (-\rho)^m \left(\frac{1}{2}\right)^k (\sqrt{\rho})^{-(m+k+i+1)} \cdot (m+k+i+1) \cdot f_3(t) \right]
\end{aligned}$$